Differential Equations MATH 2073 Exam-3A Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Find the solution of the initial value problem $y'' - 3y' + 2y = 0$, $y(0) = 2$, $y'(0) = 3$. This is a linear equation with general solution $y(x) = C_1 e^x + C_2 e^{2x}$.

   Answer $y = e^x + e^{2x}$

2. Find the general solution of the nonhomogeneous equation $y'' - 3y' + 2y = 10 \cos t$.

   The homogeneous equation was solved in the previous problem. Using method of undetermined parameters, we seek the particular solution $y_p = A \cos t + B \sin t$.

   Answer $y(x) = C_1 e^t \cos t + C_2 e^t \sin t + \cos t - 3 \sin t$

3. Find the solution of the initial value problem $y'' - 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = 2$.

   This is linear equation with constant coefficients. Characteristic equation $r^2 - 2r + 2$ has two roots: $r = 1 \pm i$. The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t$.

   Answer $y = e^t (\cos t + \sin t)$
4. Find the general solution of nonhomogeneous equation \( y'' - 4y' + 4y = 8 + 4t \).

The general solution of homogeneous equation is \( y = C_1 e^{2t} + C_2 te^{2t} \). By undetermined coefficients method we seek particular solution of the form \( y_p = A + Bt \). We get \( A = 3, B = 1 \) so the general solution is \( y(t) = C_1 e^{2t} + C_2 te^{2t} + 3 + t \).

5. Find the general solution of the equation \( y'' - 4y = 9e^t \)

The homogeneous equation has characteristic equation \( r^2 - 4r = 1 \) with roots \( r = \pm 2 \) that give \( y_1 = e^{2t}, y_2 = e^{-2t} \). Using method of undetermined parameters, we seek the particular solution \( y_p = Ae^t \). Answer: \( y(t) = C_1 e^{2t} + C_2 e^{-2t} - 3e^t \)

6. Use the reduction of order method to find the general solution of differential equation \( t^2 y'' - ty' + y = 0 \). \textit{Hint:} Verify that \( y_1(t) = t \) solves the equation.

\textbf{Solution:} Let \( y = tv(t) \). Then \( y' = tv' + v, y'' = tv'' + 2v' \). The equation becomes

\[ t^3v'' + t^2v' = 0 \]

Substitute \( u = v' \). This gives separable equation \( tu' + u = 0 \) which has the solution \( u = C_1/t \). Therefore \( v = \int u dt = C_2 + C_1 \ln t \). The answer is \( y = tv = C_1 t \ln t + C_2 t \).
Differential Equations MATH 2073 Exam-3 Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Find the solution of the initial value problem \( y'' - 4y' + 3y = 0 \). \( y(0) = 2, \ y'(0) = 4 \)

   Characteristic equation is \( r^2 - 4r + 3 = 0 \).

   Answer: \( \left\{ y = e^x + e^{3x} \right\} \)

2. Find the general solution of the nonhomogeneous equation \( y'' - 4y' + 3y = 10 \cos t \). The homogeneous equation was solved in the previous problem. Using method of undetermined parameters, we seek the particular solution \( y_* = A \cos t + B \sin t \).

   Answer: \( \left\{ y(x) = C_1 e^x + C_2 e^{3x} + \cos t - 2 \sin t \right\} \)

3. Find the solution of the initial value problem \( y'' - 2y' + 2y = 0 \). \( y(0) = 1, \ y'(0) = 0 \).

   This is linear equation with constant coefficients. Characteristic equation \( r^2 - 2r + 2 \) has two roots: \( r = 1 \pm i \). The general solution is \( y = C_1 e^t \cos t + C_2 e^t \sin t \).

   Answer: \( \left\{ y = e^t (\cos t - \sin t) \right\} \)
4. Find the general solution of nonhomogeneous equation \( y'' - 4y' + 4y = 4 + 4t \).

The general solution of homogeneous equation is \( y = C_1 e^{2t} + C_2 t e^{2t} \). By undetermined coefficients method we seek particular solution of the form \( y_p = A + Bt \). We get \( A = 2 \), \( B = 1 \) so the general solution is \( y = C_1 e^{2t} + C_2 t e^{2t} + 2 + t \).

5. Find the general solution of the equation \( y'' + 4y = 10e^t \).

The homogeneous equation has characteristic equation \( r^2 + 4r = 1 \) with roots \( r = \pm 2i \) that give \( y_1 = \cos 2t \), \( y_2 = \sin 2t \). Using method of undetermined parameters, we seek the particular solution \( y_p = Ae^t \).

Answer: \( y(t) = C_1 \cos 2t + C_2 \sin 2t + 2e^t \).

6. Use the reduction of order method to find the general solution of differential equation \( t^2 y'' - ty' + y = 0 \). Hint: Verify that \( y(t) = t \) solves the equation.

**Solution:** Let \( y = tv(t) \). Then \( y' = tv' + v, y'' = tv'' + 2v' \). The equation becomes
\[
t^3 v'' + t^2 v' = 0
\]
Substitute \( u = v' \). This gives separable equation \( tv' + u = 0 \) which has the solution \( u = C_1 / t \). Therefore \( v = \int u \, dt = C_2 + C_1 \ln t \). The answer is \( y = tv = C_1 t \ln t + C_2 t \).