Differential Equations MATH 2073 Exam-2

Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y' - 2y = 4t$, $y(0) = 0$.
   This is linear equation with the integrating factor $\mu(t) = e^{-2t}$, so the general solution is $y(t) = \frac{1}{\mu(t)} \int 4t \mu(t) dt = e^{2t} \int 4te^{-2t} = e^{2t}(-2te^{-2t} - 2e^{-2t} + C) = Ce^{2t} - 1 - 2t$
   
   Answer: $y = e^{2t} - 1 - 2t$

2. A tank originally contains 120 gal of fresh water. Then water containing $\frac{1}{3}$ lb of salt per gallon is poured into the tank at a rate of 6 gal/min, and the mixture is allowed to leave at the same rate.
   (i) Setup the differential equation for the amount $A(t)$ of salt in the tank at time $t$.
      Answer: $A' = \frac{1}{3} \times 6 - \frac{A}{120} \times 6$ This simplifies to $A' = 2 - A/20$.

   (ii) What is the long run (equilibrium) salt contents $\lim_{t \to \infty} A(t)$?
      Answer: $A' = 0$ when $2 - A/20 = 0$ or when $A = 40$.

3. Find the general solution of the equation $y'' + 2y' - 3y = 0$.
   This is linear equation with constant coefficients. Characteristic equation $r^2 + 2r + 3$ has two roots: $r_1 = -3, r_2 = 1$.
   
   Answer: $y = C_1 e^{-3x} + C_2 e^{x}$
4. Solve \((x + y) + (x - y)y' = 0\). Leave your answer in implicit form.

\textbf{Solution I} Let \(M(x, y) = x + y, N(x, y) = x - y\). Then \(\frac{\partial M}{\partial y} = 1\) and \(\frac{\partial N}{\partial x} = 1\) are equal. so this is an exact equation. The solution is \(\psi(x, y) = C\), where \(\frac{\partial \psi}{\partial x} = x + y\) so \(\psi(x, y) = 1/2x^2 + xy + h(y)\). \(\frac{\partial \psi}{\partial y} = x + h'(y) = x - y\) so \(h(y) = -y^2/2\). Answer: \[\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = C\]

\textbf{Solution II} \(y' = \frac{x+y}{y-x}\) so this is a homogeneous equation \(y' = \frac{1+y/x}{y/x-1}\). Substituting \(v = y/x\) we get \(y = vx\) and \(y' = v + xv'\) so the equation is \(v + xv' = (1 + v)/(v - 1)\). This is now rewritten as a separable equation:

\[xv' = -\frac{1 + 2v - v^2}{1 - v}\]

\[
\int \frac{1 - v}{1 + 2v - v^2} dv = - \int \frac{1}{x} dx
\]

Substituting \(u = 1 + 2v - v^2\) so that \(du = 2 - 2v\) we get

\[
\frac{1}{2} \int \frac{1}{u} du = - \int \frac{1}{x} dx
\]

\[1/2 \ln u = c - \ln x\]

\[\ln u = \frac{C}{x^2}\]

\[1 + 2v - v^2 = \frac{C}{x^2}\]

\[1 + 2y/x - y^2/x^2 = C/x^2\]

\[x^2 + 2xy - y^2 = C\]

5. Solve \(xxy' = 5x^2 + y^2; y(1) = 1\) for \(x > 0\). \textit{Hint: Rewrite the equation into an appropriate form: it can be solved as homogenous, as Bernoulli, or by the integrating factor method.}

- **Solution 1** Rewrite equation as \(-(5x^2 + y^2) + xxy' = 0\). Check for integrating factors: \(M_y = -2y, N_x = y\), so \(M/N = \frac{-2}{2} = \frac{-1}{2}\). We find the integrating factor from (separable) equation \(d\mu/\mu = -3/xdx\). This gives \ln \mu = -3 \ln x\) or \(\mu(x) = 1/x^3\). Dividing both sides of equation by \(x^3\) we get the exact equation

\[-(\frac{5}{x} + \frac{y^2}{x^3}) + \frac{y}{x^2}y' = 0\]

This means \(\frac{\partial \psi}{\partial y} = \frac{y}{x^2}\). Integrating, we get \(\psi(x, y) = \frac{1}{2}y^2 + h(x)\). So \(\frac{\partial \psi}{\partial x} = -y^2/x^3 + h'(x)\) and therefore we must have \(h'(x) = -\frac{5}{x}\). Thus \(h(x) = -5 \ln x\) and the general solution is \(\frac{1}{2}y^2 - 5 \ln x = C\). Put \(x = 1\) to compute \(C = \frac{1}{2}\) from the initial condition \(y(1) = 1\).

This gives the implicit solution \(\frac{y^2}{x^2} - 10 \ln x = 1\) which can be solved for \(y\). We get \(y = \pm x\sqrt{10 \ln x + 1}\)

- **Solution 2** Rewrite equation as \(\frac{dy}{dx} = \frac{5y}{x} + \frac{y}{x}\). This is a homogenous equation. Changing variable to \(v = xy\) we get separable equation \(xv' = \frac{5}{x}\). Integrating \(\int v dv = 5 \int \frac{1}{x} dx\) we get \(v^2/2 = 5 \ln x + C\). Thus \(v = \pm \sqrt{10 \ln x + 1}\) and \(y = \pm x\sqrt{10 \ln x + 1}\)

- **Solution 3** Rewrite the equation as \(y' - y/x = 5x/y\). This is a Bernoulli equation with \(n = -1\), so we substitute \(v = y^{-1-n} = y^2\). We have \(v' = 2yy'\) so we multiply the equation by \(2y\) to get \(2yy' - 2y^2/x = 10x\). We get the linear equation \(v' - 2v/x = 10x\) for the unknown function \(v = v(x)\).

The integrating factor for the latter satisfies \(d\mu/\mu = -2/xdx\). so \(\ln \mu = -2 \ln x\), and \(\mu = 1/x^2\). The equation is thus \((v/x^2)' = 10/x\) i.e. \(v/x^2 = 10 \ln x + C\). Since \(v(1) = y(1)^2 = 1\) we see that \(C = 1\). Thus \(v = x^2 + 10x^2 \ln x\) and \(y = \pm x \sqrt{10 \ln x + 1}\)
Differential Equations MATH 2073 Exam-2 Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y' - 3y = 9t$, $y(0) = 0$
   
   This is linear equation with the integrating factor $\mu(t) = e^{-3t}$. The general solution is $y = \frac{1}{\mu(t)} \int 9t \mu(t) dt = e^{3t} \int 9te^{-3t} dt = e^{3t} (-3te^{-3t} - e^{-3t} + C) = Ce^{3t} - 3t - 1$
   
   Answer: $y = e^{3t} - 3t - 1$

2. A tank originally contains 120 gal of fresh water. Then water containing $\frac{1}{4}$ lb of salt per gallon is poured into the tank at a rate of 12 gal/min, and the mixture is allowed to leave at the same rate.
   
   (i) Setup the differential equation for the amount $A(t)$ of salt in the tank at time $t$.
   
   Answer: $A' = \frac{1}{4} \times 12 - \frac{A}{120} \times 12$ This simplifies to $A' = 3 - A/10$.

   (ii) What is the long run (equilibrium) salt contents $\lim_{t\to\infty} A(t)$?
   
   Answer: $A' = 0$ when $3 - A/10 = 0$ or when $A = 30$.

3. Find the general solution of the equation $y'' - 3y' + 2y = 0$. This is a linear equation with general solution $y(x) = C_1e^x + C_2e^{2x}$.
4. Solve \((x + y) + (x - y)y' = 0\). Leave your answer in implicit form.

**Solution I** Let \(M(x, y) = x + y, N(x, y) = x - y\). Then \(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1\) are equal. So this is an exact equation. The solution is \(\psi(x, y) = C\), where \(\frac{\partial \psi}{\partial x} = x + y\) so \(\psi(x, y) = 1/2x^2 + xy + h(y)\). \(\frac{\partial \psi}{\partial y} = x + h'(y) = x - y\) so \(h(y) = -y^2/2\).

Answer: \(\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = C\)

**Solution II** \(y' = \frac{x+y}{y-x}\) so this is a homogeneous equation \(y' = \frac{1+y/x}{y/x-1}\). Substituting \(v = y/x\) we get \(y = vx\) and \(y' = v + xv'\) so the equation is \(v + xv' = (1 + v)/(v - 1)\). This is now rewritten as a separable equation:

\[
xv' = -\frac{1 + 2v - v^2}{1 - v}
\]

\[
\int \frac{1 - v}{1 + 2v - v^2} dv = - \int \frac{1}{x} dx
\]

Substituting \(u = 1 + 2v - v^2\) so that \(du = 2 - 2v\) we get

\[
\frac{1}{2} \int \frac{1}{u} du = - \int \frac{1}{x} dx
\]

\[
\frac{1}{2} \ln u = - \ln x
\]

\[
u = \frac{C}{x^2}
\]

\[
1 + 2v - v^2 = \frac{C}{x^2}
\]

\[
x^2 + 2xy - y^2 = C
\]

5. Solve \(xy' = 5x^2 + y^2\); \(y(1) = 1\) for \(x > 0\). Hint: Rewrite the equation into an appropriate form: it can be solved as homogenous, as Bernoulli, or by the integrating factor method.

- **Solution 1** Rewrite equation as \(xy' - (5x^2 + y^2) = 0\). Check for integrating factors: \(M_y = -2y, N_x = y\), so \(\frac{M_x - N_y}{N} = -\frac{3}{2}\). We find an integrating factor \(\mu(x) = 1/x^3\). Dividing both sides of equation by \(x^3\) we get the exact equation \(\frac{y}{x}y' - (\frac{5}{2} + \frac{v^2}{x^2}) = 0\). This means \(\psi(x, y) = \frac{1}{2}y^2 + h(x)\) and \(h'(x) = -\frac{5}{2}\). Thus \(h(x) = -5\ln x\) and the general solution is \(\frac{1}{2}y^2 = 5\ln x = C\). Put \(x = 1\) to compute \(C = \frac{1}{2}\). The implicit solution \(\frac{y^2}{2x^2} - 5\ln x = 1\) can be solved for \(y\). We get \(y = \pm x\sqrt{10\ln x + 1}\).

- **Solution 2** Rewrite equation as \(\frac{dy}{dx} = 5\frac{y}{x} + \frac{y^2}{x}\). This is a homogenous equation. Changing variable to \(v = y/x\) we get separable equation \(xv' = \frac{5}{v}\). Integrating \(\int v dv = 5 \int \frac{1}{x} dx\) we get \(v^2/2 = 5 \ln x + C\). Thus \(v = \pm \sqrt{10 \ln x + 1}\) and \(y = \pm \frac{x}{\sqrt{10 \ln x + 1}}\).

- **Solution 3** Rewrite the equation as \(y' - y/x = 5x/y\). This is a Bernoulli equation with \(n = -1\), so we substitute \(v = y^{1-n} = y^2\). We have \(v' = 2yy'\) so we multiply the equation by \(2y\) to get \(2yy' - 2y^2/x = 10x\). We get a linear equation \(v' - 2v/x = 10x\) for \(v\).

The integrating factor for the linear equation that we found satisfies \(d\mu/\mu = -2/x dx\) so \(\ln \mu = -2\ln x\) and \(\mu = 1/x^2\). The equation is thus \((v/x^2) = 10/x\) i.e. \(v/x^2 = 10 \ln x + C\). Since \(v(1) = y(1)^2 = 1\) we see that \(C = 1\). Thus \(v = x^2 + 10x^3 \ln x\) and \(y = \pm \sqrt{x^2(1 + 10 \ln x)} = x \pm \sqrt{1 + 10 \ln x}\)