Problem Set 12

Problem 1: Use $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$ to verify that the angular momentum operators

$$\hat{L}_i = \sum_{jk} \epsilon_{ijk} \hat{x}_j \hat{p}_k$$

satisfy the commutation relations

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{L}_k,$$
$$[\hat{L}_i, \hat{x}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{x}_k,$$
$$[\hat{L}_i, \hat{p}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{p}_k.$$

Here $i, j, k \in \{1, 2, 3\}$ and sums run over these values.

Problem 2: A diatomic molecule made of two atoms of masses $m_1$ and $m_2$ with energy spectrum as in equation (9.111) of the text makes a purely rotational transition from an $\ell = \ell_0$ state to an $\ell = \ell_0 - 1$ state, emitting a photon of frequency $\omega_0$ (so of energy $\hbar \omega_0$). What is the interatomic distance of the two atoms in this molecule in terms of $m_1, m_2, \ell_0$, and $\omega_0$?

Problem 3: Consider a particle in a state with wave function $\psi = N(x + y + 2z)e^{-\alpha r}$ where $N$ is the normalization factor. Show, by rewriting the $Y_{1,0}^{\pm1.0}$ functions in terms of $x, y, z, r$, that

$$Y_{1,\pm1} = \mp \left( \frac{3}{8\pi} \right)^{1/2} \frac{x \pm iy}{r},$$
$$Y_{1,0} = \left( \frac{3}{4\pi} \right)^{1/2} \frac{z}{r}. \quad (1)$$

Using this, find the probabilities $\mathcal{P}(L_z = ?)$ of measuring the the possible values of $L_z$ for a particle in the state $\psi$ given above.

For problems 4-7, consider a rigid rotator immersed in a uniform magnetic field in the $z$ direction, with the Hamiltonian

$$\hat{H} = \frac{1}{2I} \hat{L}^2 + \omega_0 \hat{L}_z$$

where $I$ and $\omega_0$ are given positive constants. Suppose the wave function of the rotator at time $t = 0$ is given by

$$\langle \theta, \phi | \psi(0) \rangle = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi. \quad (2)$$

Problem 4: What values of $L_z$ will be obtained if a measurement is carried out at time $t = 0$, and with what probability will these values occur?

Problem 5: If a measurement of $\hat{L}_z$ is carried out at time $t = 0$, what results can be obtained and with what probabilities?
Problem 6: What is $\langle \theta, \phi | \psi(t) \rangle$?

Problem 7: What is $\langle L_x \rangle$ for this state at time $t$?