Problem Set 6

Problem 1: Starting from equation (4.41) in the text, derive equation (4.45) in the approximation where you neglect the \( \exp\{\pm i(\omega + \omega_0) t\} \) terms in (4.41). Note that the initial conditions are assumed to be \( a(0) = 1 \) and \( b(0) = 0 \).

Solution: Eqn (4.41) with the \( (\omega + \omega_0) \) terms set to zero gives the two equations

\[
i c = \frac{\omega_1}{4} e^{-i(\omega - \omega_0) t} d, \quad \quad i d = \frac{\omega_1}{4} e^{i(\omega - \omega_0) t} c. \tag{1}\]

Multiplying the first through by \( e^{i(\omega - \omega_0) t} \), differentiating with respect to time, then using the second equation to eliminate \( \dot{d} \) in favor of \( c \) gives the single second-order equation

\[
0 = \left[ \frac{d^2}{dt^2} + i(\omega - \omega_0) \frac{d}{dt} + \left( \frac{\omega_1}{4} \right)^2 \right] c
\]

which has general solution (found, eg, by factorizing the second order differential operator above)

\[
c = A_+ \exp \left\{ \frac{it}{2} \left( \omega_0 - \omega + \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4} \right) \right\} + A_- \exp \left\{ \frac{it}{2} \left( \omega_0 - \omega - \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4} \right) \right\}.
\]

Then from the first equation in (1) we get that

\[
d = -\frac{2A_+}{\omega_1} \left[ \omega_0 - \omega + \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4} \right] \exp \left\{ \frac{it}{2} \left( \omega - \omega_0 + \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4} \right) \right\}

- \frac{2A_-}{\omega_1} \left[ \omega_0 - \omega - \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4} \right] \exp \left\{ \frac{it}{2} \left( \omega - \omega_0 - \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4} \right) \right\}. \tag{2}\]

Since \( a(t) = c(t)e^{-i\omega_0 t/2} \), \( b(t) = d(t)e^{i\omega_0 t/2} \), and \( a(0) = 1 \) and \( b(0) = 0 \), it follows that \( c(0) = 1 \) and \( d(0) = 0 \). Plugging these into the last two equations gives

\[
A_\pm = \frac{\pm(\omega - \omega_0) + \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4}}{2 \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4}}. \tag{3}\]

Plugging (3) into (2) gives

\[
d = \frac{-i \omega_1 e^{i t (\omega - \omega_0)/2}}{2 \sqrt{(\omega_0 - \omega)^2 + \omega_1^2/4}} \sin \left\{ \frac{t}{2} \sqrt{(\omega_0 - \omega)^2 + \frac{1}{4} \omega_1^2} \right\}.
\]

Now, we want to compute \( |\langle -\hat{z}\psi(t)\rangle|^2 = |b(t)|^2 = |d(t)|^2 \). So we get

\[
|\langle -\hat{z}\psi(t)\rangle|^2 = \left[ \frac{1}{4} \omega_1^2 \left( \frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega} \right) \right] \sin^2 \left\{ \frac{t}{2} \sqrt{(\omega_0 - \omega)^2 + \frac{1}{4} \omega_1^2} \right\},
\]

which is eqn (4.45) of the text.

Problems 2 through 9 consider the spin states of an electron in a uniform but increasing magnetic field in the \( \hat{x} \)-direction, \( \vec{B} = (t/\tau_0) B_0 \hat{x} \), where \( B_0 \) and \( \tau_0 \) are constants.

As usual, define the frequency \( \omega_0 = egB_0/2mc \) from the electron charge, magnetic moment \( g \)-factor, and mass, and express all your answers in terms of \( \omega_0 \) and \( \tau_0 \). At
time $t = 0$, the the electron is in its “spin down” state, $|\psi(0)\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$, in the usual $\hat{J}_z$ basis.

**Problem 2:** What is the Hamiltonian for this system, and what are its energy levels (ie, the energy eigenvalues) as a function of time? **Solution:** From the text, $\hat{H} = (eg/2mc)\vec{B} \cdot \vec{J} = (\omega_0 t/\tau_0)\hat{J}_z$. The eigenvalues of $\hat{J}_z$ are $\pm \hbar/2$, so the energy levels are $\pm \hbar\omega_0 t/2\tau_0$.

**Problem 3:** Write down the Schrödinger equation for $|\psi(t)\rangle$. If $|\psi(t)\rangle$ in the $\hat{J}_z$ basis is

$$|\psi(t)\rangle = a(t)|\frac{1}{2}, \frac{1}{2}\rangle + b(t)|\frac{1}{2}, -\frac{1}{2}\rangle,$$  

(4)

write the Schrödinger equation as a coupled system of differential equations for $a(t)$ and $b(t)$. What are the initial conditions at $t = 0$ for these equations? **Solution:** Schrödinger’s equation is $\frac{d}{dt}|\psi(t)\rangle = -(i/\hbar)\hat{H}|\psi(t)\rangle = -(i\omega_0 t/\tau_0)\hat{J}_z|\psi(t)\rangle$. In the $\hat{J}_z$ basis $\hat{J}_z = (\hbar/2)(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, so SE becomes

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = -\frac{i\omega_0 t}{2\tau_0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{i\omega_0 t}{2\tau_0} \begin{pmatrix} b \\ a \end{pmatrix}.$$  

(5)

$|\psi(0)\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ implies $a(0) = 0$ and $b(0) = 1$.

**Problem 4:** Check that

$$a = iA \cos\left(\frac{\omega_0}{4\tau_0} t^2\right) - iB \sin\left(\frac{\omega_0}{4\tau_0} t^2\right),$$
$$b = A \sin\left(\frac{\omega_0}{4\tau_0} t^2\right) + B \cos\left(\frac{\omega_0}{4\tau_0} t^2\right),$$  

(6)

solve equation (6) for any values of the constants $A$ and $B$. The initial condition at $t = 0$ specifies what values for $A$ and $B$? **Solution:** This is just a matter of plugging into (5). $a(0) = 0$ implies $A = 0$, and $b(0) = 1$ implies $B = 1$.

**Problem 5:** What is the probability of measuring $J_z = -\hbar/2$ at time $t = 2t_0$? Approximate your answer for small $t_0$ to include the leading correction away from probability 1. **Solution:**

$$\mathcal{P}(J_z = -\hbar/2, t = 2t_0) = |\langle \frac{1}{2}, \frac{1}{2} | \psi(2t_0) \rangle|^2 = |b(2t_0)|^2 = \cos^2\left(\frac{\omega_0 t_0^4}{\tau_0}\right),$$

where we used (4), (6), and the initial conditions $A = 0$, $B = 1$. Thus

$$\mathcal{P}(-, 2t_0) \approx 1 - \frac{\omega_0^2 t_0^4}{\tau_0}.$$  

**Problem 6:** If $J_z = -\hbar/2$ is measured at time $t = t_0$, what is the state of the system immediately after the measurement? **Solution:** The state will be projected onto the eigenstate of the observed eigenvalue, so $\psi(t_0) \rightarrow |\frac{1}{2}, -\frac{1}{2}\rangle$. 
Problem 7: What is the probability of observing $J_z = -\hbar/2$ at time $t = 2t_0$ if $J_z = -\hbar/2$ is also measured at time $t = t_0$? How does this compare to the probability computed in problem 5 for small $t_0$? Solution: If $J_z = -\hbar/2$ is measured at $t = t_0$, then $\psi(t_0) \rightarrow \tilde{\psi}(t_0) = \frac{1}{\sqrt{2}} \{ \frac{1}{2}, -\frac{1}{2} \}$. After $t = t_0$, the state then evolves, according to (6), but with the initial condition $\tilde{b}(t_0) = 1$ and $\tilde{a}(t_0) = 0$. (I use $\tilde{a}$ and $\tilde{b}$ to denote the components of the state $\tilde{\psi}$ that results from the measurement at $t = t_0$.) The solution of (6) with these initial conditions has

$$A = \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right), \quad B = \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right),$$

so for $t > t_0$

$$\tilde{a} = i \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) - i \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) = -i \sin \left( \frac{\omega_0}{4\tau_0} (t^2 - t_0^2) \right),$$

$$\tilde{b} = \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) + \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) = \cos \left( \frac{\omega_0}{4\tau_0} (t^2 - t_0^2) \right).$$

So in this state, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is

$$\mathcal{P}(J_z = -\hbar/2, t = 2t_0) = \left| \left( \frac{1}{2}, -\frac{1}{2} \right) \tilde{\psi}(2t_0) \right|^2 = \left| \tilde{b}(2t_0) \right|^2 = \cos^2 \left( \frac{3\omega_0}{4\tau_0} t_0^2 \right). \quad (7)$$

For small $t_0$ this gives

$$\mathcal{P}(-, 2t_0|-, t_0) \approx 1 - \frac{9\omega_0^2}{16\tau_0^2} t_0^4,$$

so is greater than that found in problem 5.

Problem 8: What is the probability of observing $J_z = -\hbar/2$ at time $t = 2t_0$ if instead $J_z = +\hbar/2$ is measured at time $t = t_0$? How does this compare to the probability computed in problem 5 for small $t_0$? Solution: If $J_z = +\hbar/2$ is measured at $t = t_0$, then $\psi(t_0) \rightarrow \tilde{\psi}(t_0) = \frac{1}{\sqrt{2}} \{ \frac{1}{2}, \frac{1}{2} \}$. After $t = t_0$, the state then evolves, according to (6), but with the initial condition $\tilde{b}(t_0) = 0$ and $\tilde{a}(t_0) = 1$. The solution of (6) with these initial conditions has

$$A = -i \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right), \quad B = +i \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right),$$

so for $t > t_0$

$$\tilde{a} = \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) + \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) = \cos \left( \frac{\omega_0}{4\tau_0} (t^2 - t_0^2) \right),$$

$$\tilde{b} = -i \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) + i \sin \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) \cos \left( \frac{\omega_0 t_0^2}{4\tau_0^2} \right) = -i \sin \left( \frac{\omega_0}{4\tau_0} (t^2 - t_0^2) \right).$$

So in this state, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is

$$\mathcal{P}(J_z = -\hbar/2, t = 2t_0) = \left| \left( \frac{1}{2}, -\frac{1}{2} \right) \tilde{\psi}(2t_0) \right|^2 = \left| \tilde{b}(2t_0) \right|^2 = \sin^2 \left( \frac{3\omega_0}{4\tau_0} t_0^2 \right). \quad (8)$$
For small $t_0$ this gives

$$P(-,2t_0|+,t_0) \approx \frac{9\omega_0^2}{16\tau_0^2 t_0^4},$$

so is much smaller than that found in problem 5.

**Problem 9:** (This is a bit tricky!) What is the probability of observing $J_z = -\hbar/2$ at time $t = 2t_0$ if $J_z$ is measured at time $t = t_0$? (I’m just telling you that $J_z$ is measured at time $t = t_0$, but not specifying which outcome is actually found.) How does this compare to the probability computed in problem 5 at small $t_0$? **Solution:** At time $t_0$ the probability of measuring $J_z = -\hbar/2$ is $P(-, t_0) = \cos^2 \left( \frac{\omega_0 t_0^2}{4\tau_0} \right)$ by problem 5. So the probability of measuring $J_z = +\hbar/2$ at the same time is $P(+, t_0) = 1 - P(-, t_0)$. If $J_z = -\hbar/2$ is measured at $t = t_0$, then from problem 7, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is $P(-, 2t_0|-, t_0)$ given in (7). Likewise, if $J_z = +\hbar/2$ is measured at $t = t_0$, then from problem 8, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is $P(-, 2t_0|+, t_0)$ given in (8). Thus the total probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ given a measurement of $\hat{J}_z$ at $t = t_0$ is

$$P(-,2t_0 \mid J_z \text{ at } t_0) = P(-, t_0)P(-, 2t_0|-, t_0) + P(+, t_0)P(-, 2t_0|+, t_0)$$

$$= \cos^2 \left( \frac{\omega_0 t_0^2}{4\tau_0} \right) \cos^2 \left( \frac{3\omega_0 t_0^2}{4\tau_0} \right) + \sin^2 \left( \frac{\omega_0 t_0^2}{4\tau_0} \right) \sin^2 \left( \frac{3\omega_0 t_0^2}{4\tau_0} \right)$$

$$\approx \left( 1 - \frac{\omega_0^2}{16\tau_0^2 t_0^4} \right) \left( 1 - \frac{9\omega_0^2}{16\tau_0^2 t_0^4} \right) + \frac{\omega_0^2}{16\tau_0^2 t_0^4} \frac{9\omega_0^2}{16\tau_0^2 t_0^4}$$

$$\approx 1 - \frac{5\omega_0^2}{8\tau_0^2 t_0^4}.$$

This is greater than the result found in problem 5 when no measurement is made at an intermediate time.