Intro to QM

Problem Set 4

All problems in this problem set will be worth 3 points instead of the usual 1 point.

For all these problems we are considering the spin states of a spin-1 particle. The Hilbert space of these states form an \( J^2 \) eigenspace with eigenvalue \( 2\hbar^2 \), and in the \( \hat{J}_z \) eigenbasis it is spanned by the 3 orthonormal states \(|j,m\rangle\) with \( j = 1 \) and \( m \in \{-1,0,+1\} \).

Problem 1: Starting from the formulas (3.54b), (3.61), and (3.62) in the text for the matrix elements of the \( \hat{J}_x \) and \( \hat{J}_\pm \) operators, show that in the \( \hat{J}_z \) eigenbasis that the matrix representation of the angular momentum operators is

\[
\hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{J}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

where we order the \( \hat{J}_z \) eigenbasis so that \(|1,1\rangle\) is the first row/column, \(|1,0\rangle\) is the second, and \(|1,-1\rangle\) is the third. Solution: From the definitions of \( \hat{J}_\pm \), \( \hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-) \) and \( \hat{J}_y = \frac{-i}{2}(\hat{J}_+ - \hat{J}_-) \). From the text, \( \hat{J}_\pm |1,m\rangle = \hbar\sqrt{2 - m^2 \mp m} |1,m \pm 1\rangle \), so the \( \hat{J}_x \) matrix elements follow:

\[
\langle 1,n|\hat{J}_x|1,m\rangle = \langle 1,n\rangle\frac{1}{2} \left( \hbar\sqrt{2 - m^2 - m} \langle 1,m + 1\rangle + \hbar\sqrt{2 - m^2 + m} \langle 1,m - 1\rangle \right)
= \frac{\hbar}{2} \left( \hbar\sqrt{2 - m^2 - m} \delta_{n,m+1} + \hbar\sqrt{2 - m^2 + m} \delta_{n,m-1} \right).
\]

Plugging in \( n,m = 1,0,-1 \) then gives the nine matrix elements of the \( \hat{J}_x \). A similar calculation goes for \( \hat{J}_y \). \( \hat{J}_z \) is even easier since \(|1,m\rangle\) is, by definition, the eigenbasis of \( \hat{J}_z \), so in this basis \( \hat{J}_z \) is diagonal with its eigenvalues on the diagonal.

Problem 2: What are the possible values you could find if you measured \( \hat{J}_z \)? Solution: The possible outcomes are the eigenvalues of \( \hat{J}_z \) which are \( J_z = \{ \hbar,0,-\hbar \} \).

Problem 3: Consider the state in which \( J_z = \hbar \). In this state what are \( \langle J_x\rangle, \langle J_x^2\rangle \), and \( \Delta J_x \)? Solution: \( \hat{J}_z |\psi\rangle = \hbar \cdot |\psi\rangle \) implies

\[
|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]
in the $\hat{J}_z$ eigenbasis. (Note that I have normalized $|\psi\rangle$!) Then

$$
\langle J_x \rangle = \langle \psi | \hat{J}_x | \psi \rangle = (1 \ 0 \ 0) \begin{pmatrix} \frac{\hbar}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\hbar}{\sqrt{2}} (1 \ 0 \ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.
$$

$$
\langle J_x^2 \rangle = \langle \psi | \hat{J}_x^2 | \psi \rangle = (1 \ 0 \ 0) \begin{pmatrix} \frac{\hbar^2}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\hbar^2}{2} (0 \ 1 \ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{2}.
$$

$$
\Delta J_x = \sqrt{\langle J_x^2 \rangle - \langle J_x \rangle^2} = \sqrt{\left(\frac{\hbar^2}{2}\right) - 0^2} = \frac{\hbar}{\sqrt{2}}.
$$

**Problem 4:** Find the normalized eigenstates and the eigenvalues of $\hat{J}_x$ in the $\hat{J}_z$ eigenbasis. **Solution:** The characteristic equation for $\hat{J}_x$ is

$$
0 = \det(\hat{J}_x - \lambda) = \det \begin{pmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{pmatrix} = \hbar^2 \lambda - \lambda^3, \quad \Rightarrow \lambda \in \{\hbar, 0, -\hbar\}.
$$

The corresponding eigenvectors, $|\lambda\rangle$, then satisfy

$$
0 = (\hat{J}_x - \lambda)|\lambda\rangle = \begin{pmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\lambda a + \frac{\hbar}{\sqrt{2}} b \\ \frac{\hbar}{\sqrt{2}} a - \lambda b + \frac{\hbar}{\sqrt{2}} c \\ \frac{\hbar}{\sqrt{2}} b - \lambda a \end{pmatrix},
$$

where we have parameterized the components of $|\lambda\rangle$ by $(a \ b \ c)$. For $\lambda = \hbar$, we can solve for $b$ and $c$ in terms of $a$, giving $b = \sqrt{2}a$ and $c = a$. We then determine $a$ by normalizing $|\lambda = \hbar\rangle$:

$$
|\lambda = \hbar\rangle = \begin{pmatrix} a \\ \sqrt{2}a \\ a \end{pmatrix}, \quad \Rightarrow \quad 1 = \langle \lambda = \hbar | \lambda = \hbar \rangle = (a^* \sqrt{2}a^* \ a) \begin{pmatrix} a \\ \sqrt{2}a \\ a \end{pmatrix} = 4|a|^2, \quad \Rightarrow \quad a = \frac{1}{2}
$$

where I have chosen the arbitrary phase to be 1). Thus, and doing the same thing for $\lambda = 0$ and $\lambda = -\hbar$, gives

$$
|J_x = \hbar\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad |J_x = 0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad |J_x = -\hbar\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.
$$

**Problem 5:** If the particle is in the state with $J_z = -\hbar$, and $\hat{J}_x$ is measured, what are the possible outcomes and their probabilities? **Solution:** The possible outcomes are $J_x = \{\hbar, 0, -\hbar\}$, which are the eigenvalues of $\hat{J}_x$. $|\psi\rangle$ is the normalized eigenstate of $\hat{J}_z$ with eigenvalue $J_z = -\hbar$, which is

$$
|\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$
in the $\hat{J}_z$ eigenbasis. So (here $\mathcal{P}$ stands for "probability of"):

\[
\mathcal{P}(J_x = \hbar) = |\langle J_x = \hbar | \psi \rangle|^2 = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{4},
\]

\[
\mathcal{P}(J_x = 0) = |\langle J_x = 0 | \psi \rangle|^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2},
\]

\[
\mathcal{P}(J_x = -\hbar) = |\langle J_x = -\hbar | \psi \rangle|^2 = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{4}.
\]

**Problem 6:** Consider the state whose components are the column vector

\[
|\psi\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \sqrt{2} \end{pmatrix}
\]

in the $\hat{J}_z$ eigenbasis (ordered as in problem 1). If $\hat{J}_z^2$ is measured in this state and the result $+\hbar^2$ is obtained, what is the state after the measurement? How probable was this result? If, instead, we had measured $\hat{J}_z$, what are the outcomes and their probabilities?

**Solution:**

\[
\hat{J}_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Rightarrow \quad \text{the possible outcomes are } J_z^2 = \{0, \hbar^2\},
\]

since those are its eigenvalues. An eigenbasis of the $J_z^2 = \hbar^2$ eigenspace is $\{|a\rangle, |b\rangle\}$ with

\[
|a\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |b\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Therefore, upon measuring $\hat{J}_z^2 = \hbar^2$, the state collapses to

\[
|\psi\rangle \longrightarrow |\psi'\rangle = \frac{|\langle a\rangle + |b\rangle \langle b| |\psi\rangle}{|\langle a\rangle + |b\rangle \langle b| |\psi\rangle|}.
\]

But

\[
|\langle a| + |b\rangle \langle b)| |\psi\rangle = \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right] \frac{1}{2} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{2} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix},
\]

has norm

\[
\sqrt{\frac{1}{2} \begin{pmatrix} 1 & 0 & \sqrt{2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \frac{\sqrt{3}}{2}.
\]
\[
|\psi\rangle = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}.
\]

The probability of \(J_z^2 = +\hbar^2\) is
\[
\mathcal{P}(J_z^2 = \hbar^2) = \langle \psi | (|a\rangle\langle a| + |b\rangle\langle b|) |\psi\rangle = |\langle a|\psi\rangle|^2 + |\langle b|\psi\rangle|^2
\]
\[
= \left| \frac{1}{2} (1 \ 0 \ 0) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right|^2 + \left| \frac{1}{2} (0 \ 0 \ 1) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.
\]

If instead we measured \(\hat{J}_z\) the possible outcomes are the eigenvalues of \(\hat{J}_z\), \(\{0, \pm \hbar\}\), with probabilities
\[
\mathcal{P}(J_z = \hbar) = \left| \langle 1 \ 0 \ 0 | \psi' \rangle \right|^2 = \left| \frac{1}{2} (1 \ 0 \ 0) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{4},
\]
\[
\mathcal{P}(J_z = 0) = \left| \langle 0 \ 1 \ 0 | \psi' \rangle \right|^2 = \left| \frac{1}{2} (0 \ 1 \ 0) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{4},
\]
\[
\mathcal{P}(J_z = -\hbar) = \left| \langle 0 \ 0 \ 1 | \psi' \rangle \right|^2 = \left| \frac{1}{2} (0 \ 0 \ 1) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2}.
\]

**Problem 7:** A particle is in a state for which the probabilities \(\mathcal{P}(J_z = \hbar) = 1/4\), \(\mathcal{P}(J_z = 0) = 1/2\), and \(\mathcal{P}(J_z = -\hbar) = 1/4\). Show that the most general normalized state with this property is
\[
|\psi\rangle = e^{i\delta_1} \frac{1}{2} |1, 1\rangle + e^{i\delta_2} \frac{1}{\sqrt{2}} |1, 0\rangle + e^{i\delta_3} \frac{1}{2} |1, -1\rangle. \tag{1}
\]

It was stated in the text (and by me in class) that if \(|\psi\rangle\) is a normalized state then the state \(e^{i\theta}|\psi\rangle\) is a physically equivalent normalized state. Does this mean that the factors of \(e^{i\delta_j}\) multiplying the \(\hat{J}_z\) eigenstates are irrelevant? To test this, calculate, for example, \(\mathcal{P}(J_x = 0)\). **Solution:** In the \(\hat{J}_z\) eigenbasis,
\[
|J_z = +\hbar\rangle = |1, 1\rangle, \quad |J_z = 0\rangle = |1, 0\rangle, \quad |J_z = -\hbar\rangle = |1, -1\rangle,
\]
write the unknown state as
\[
|\psi\rangle = a|1, 1\rangle + b|1, 0\rangle + c|1, -1\rangle.
\]
Then
\[
\mathcal{P}(J_z = +\hbar) = \frac{1}{4} = |\langle 1, 1 |\psi\rangle|^2 = |a|^2,
\]
\[
\mathcal{P}(J_z = 0) = \frac{1}{2} = |\langle 1, 0 |\psi\rangle|^2 = |b|^2,
\]
\[
\mathcal{P}(J_z = -\hbar) = \frac{1}{4} = |\langle 1, -1 |\psi\rangle|^2 = |c|^2.
\]
The most general solution to these three equations is then

\[ a = \frac{1}{2} e^{i\delta_1}, \quad b = \frac{1}{\sqrt{2}} e^{i\delta_2}, \quad c = \frac{1}{2} e^{i\delta_3}, \]

for some arbitrary phases \(\delta_i\), which gives the desired answer.

The \(\delta_i\) phase factors are not irrelevant. For example

\[
\mathcal{P}(J_x = 0) = |\langle J_x = 0 | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \left( \langle 1,1 | - \langle 1,-1 | \right) | \psi \rangle \right|^2 = \left| \frac{e^{i\delta_1}}{2\sqrt{2}} - \frac{e^{i\delta_3}}{2\sqrt{2}} \right|^2
\]

\[= \frac{1}{8} (e^{i\delta_1} - e^{i\delta_3})(e^{-i\delta_1} - e^{-i\delta_3}) = \frac{1}{8} \left( 1 - e^{i(\delta_3 - \delta_1)} - e^{-i(\delta_3 - \delta_1)} + 1 \right)\]

\[= \frac{1}{4} (1 - \cos(\delta_3 - \delta_1)),\]

so something measurable (a probability) depends on the difference of the phases.