Consider a 3-dimensional Hilbert space with orthonormal basis \{ |n\rangle, n = 1, 2, 3 \} with respect to which the hermitean operators \( \hat{A} \) and \( \hat{B} \) have matrix elements

\[
\hat{A} = \begin{pmatrix}
17 & -7(1-i)\sqrt{2} & 7i\sqrt{3} \\
-7(1+i)\sqrt{2} & -4 & -7(1-i)\sqrt{6} \\
-7i\sqrt{3} & -7(1+i)\sqrt{6} & 3
\end{pmatrix},
\]

\[
\hat{B} = \begin{pmatrix}
25 & 3(1-i)\sqrt{2} & -i\sqrt{3} \\
3(1+i)\sqrt{2} & 28 & 3(1-i)\sqrt{6} \\
i\sqrt{3} & 3(1+i)\sqrt{6} & 27
\end{pmatrix}.
\]

For problems 1 - 6, compute or verify the following "by hand" (ie, do not use Mathematica or some other automated system), showing your work:

**Problem 1:** \( \det \hat{A} \). **Solution:** I find \( \det \hat{A} = -18,432 \). (I don’t have to show my work because I’m the professor.)

**Problem 2:** The eigenvalues, \( \lambda_n \), of \( \hat{A} \). **Solution:** I find two eigenvalues \( \lambda_1 = -32 \) and \( \lambda_2 = 24 \).

**Problem 3:** All the corresponding eigenvectors, \( |v_n\rangle \), of \( \hat{A} \). **Solution:** The eigenvectors \( |v_1\rangle \) corresponding to \( \lambda_1 \) all have the form \( |v_1\rangle = c(-i, (1-i)\sqrt{2}, \sqrt{3})^T \) where \( c \) is any complex constant. The eigenvectors \( |v_2\rangle \) corresponding to \( \lambda_2 \) all have the form \( |v_2\rangle = (-i, (1+i)\sqrt{2} + i\sqrt{3}d, e)^T \) where \( d \) and \( e \) are any complex constants.

**Problem 4:** An orthonormal basis, \( \{ |f_n\rangle \} \), of eigenvectors corresponding to the eigenvalues \( \lambda_n \) of \( \hat{A} \). **Solution:** Normalizing \( |v_1\rangle \) (ie, setting its length to 1), I find \( |c| = 1/\sqrt{8} \). Arbitrarily choosing the phase to be 1, I thus have the normalized eigenvector \( |f_1\rangle = \frac{1}{\sqrt{2}}(-i, (1-i)\sqrt{2}, \sqrt{3})^T \). The \( \lambda_2 \) eigenspace is 2-dimensional, so we need to find a pair of orthonormal vectors from this space. Let me arbitrarily choose the first one to be \( |v_2\rangle \) with \( c = 0 \); to be normalized I then have to pick \( |d| = 1/\sqrt{5} \), and I arbitrarily choose the phase \( |f_2\rangle = \frac{1}{\sqrt{5}}(-1-i, \sqrt{2}, 1, 0)^T \). The second eigenbasis vector in this eigenspace has to be chosen orthogonal to the first, so we need to find a \( |v_2\rangle \) so that \( 0 = \langle f_2 | v_2 \rangle \propto (-i, \sqrt{2}, 1, 0)^T \). Therefore we must have \( d = -i/\sqrt{6}e/5 \), so that \( |f_3\rangle = \frac{1}{\sqrt{5}}(i\sqrt{3}, - (1-i)\sqrt{6}, 5)^T \). Normalizing this gives \( |c| = \sqrt{5}/(2\sqrt{2}) \). Choosing the phase to be 1 I then get \( |f_3\rangle = \frac{1}{2\sqrt{10}}(i\sqrt{3}, - (1-i)\sqrt{6}, 5)^T \), completing the orthonormal eigenbasis. It should be clear that (infinitely) many different such bases could have been chosen.

**Problem 5:** The change of basis matrix \( \hat{R}_{mn} \) from the \{ |n\rangle \} to the \{ |f_n\rangle \} basis (de-
fined by \(|f_n⟩ = \sum_m R_{mn}⟨m|\)). Solution: \(\hat{R}_{mn} = ⟨m|f_n⟩\) which are just the components of \(|f_n⟩\). Thus, from the choice of eigenbasis in problem 4 we simply read off

\[
\hat{R} = \frac{1}{2\sqrt{10}} \begin{pmatrix}
-\sqrt{5} & -4(1-i) & \sqrt{5} \\
(1-i)\sqrt{10} & 2\sqrt{2} & -(1-i)\sqrt{6} \\
\sqrt{15} & 0 & 5
\end{pmatrix}.
\]

ie, its three columns are simply the components of \(|f_1⟩, |f_2⟩, \text{ and } |f_3⟩\).

Problem 6: Verify that \(\hat{R}\) is unitary, ie, that \(\hat{R}^\dagger \hat{R} = I\). Solution:

\[
\hat{R}^\dagger \hat{R} = \frac{1}{40} \begin{pmatrix}
i\sqrt{5} & (1+i)\sqrt{10} & \sqrt{15} \\
-(1+i)\sqrt{10} & 2\sqrt{2} & -(1-i)\sqrt{6} \\
-i\sqrt{3} & 0 & 5
\end{pmatrix} \begin{pmatrix}
-i\sqrt{5} & -4(1-i) & \sqrt{5} \\
(1-i)\sqrt{10} & 2\sqrt{2} & -(1-i)\sqrt{6} \\
\sqrt{15} & 0 & 5
\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1\end{pmatrix}.
\]

Problems 7 - 13 refer to the same operators, \(\hat{A}\) and \(\hat{B}\), defined above. Hint: these problems require very little or no computation.

Problem 7: Compute the matrix elements of \(\hat{A}\) in the \(|f_n⟩\) basis. Solution: \(\hat{A}\) is just diagonal with the eigenvalues as the diagonal entries in an orthonormal eigenbasis. So

\[
\hat{A} = \begin{pmatrix}
-32 & 0 & 0 \\
0 & 24 & 0 \\
0 & 0 & 24
\end{pmatrix}.
\]

Problem 8: Write a general expression for \(\hat{A}\) in terms of the eigenvalues \(λ_i\) and the projectors, \(\hat{P}_{λ_i}\), onto the \(λ_i\) eigenspace. (Don’t compute the \(\hat{P}_{λ_i}\) explicitly yet.) Solution: As shown in class, generally \(\hat{A} = \sum_i λ_i \hat{P}_{λ_i} = -32\hat{P}_{λ_1} + 24\hat{P}_{λ_2}\).

Problem 9: Write expressions for the \(\hat{P}_{λ_i}\) in terms of the \(|f_n⟩\). Solution: The general expression (derived in class) for a projector onto a subspace with orthonormal basis \(|v_i⟩\) is \(\hat{P} = \sum_i |v_i⟩⟨v_i|\). Thus, in our case \(\hat{P}_{λ_1} = |f_1⟩⟨f_1|\) and \(\hat{P}_{λ_2} = |f_2⟩⟨f_2| + |f_3⟩⟨f_3|\).

Problem 10: Compute the matrix elements of the \(\hat{P}_{λ_i}\) in the \(|f_n⟩\) basis. Solution: Since, in the \(|f_n⟩\) basis, \(|f_1⟩ = (1,0,0)^T\), etc, we simply have

\[
\hat{P}_{λ_1} = \begin{pmatrix}1 \\
0 \\
0 \end{pmatrix} \begin{pmatrix}1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} = \begin{pmatrix}1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}
\]

\[
\hat{P}_{λ_2} = \begin{pmatrix}1 \\
0 \\
0 \end{pmatrix} \begin{pmatrix}1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{pmatrix} = \begin{pmatrix}0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}
\]

in the \(|f_n⟩\) basis.

Problem 11: Plug the result of problem 10 into that of problem 8 and verify that
you get back the matrix of \( \hat{A} \) in the \( \{|f_n\}\) basis. **Solution:**

\[
A = -32 \hat{P}_{\lambda_1} + 24 \hat{P}_{\lambda_2} = -32 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 24 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -32 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}.
\]

**Problem 12:** Compute the matrix elements of the \( \hat{P}_{\lambda_i} \) in the \( \{|n\}\) basis. **Solution:**

Since the \( |f_n\) are given in the \( \{|n\}\) basis in problem 4, we get

\[
\hat{P}_{\lambda_1} = \frac{1}{8} \begin{pmatrix} -i & (1-i)\sqrt{2} \\ \sqrt{3} & (1+i)\sqrt{2} \end{pmatrix} (1+i)\sqrt{2} \sqrt{3} = \frac{1}{8} \begin{pmatrix} 1 & (1-i)\sqrt{2} \\ (1+i)\sqrt{2} + i\sqrt{3} (1-i)\sqrt{2} \\ (1+i)\sqrt{2} - i\sqrt{3} \end{pmatrix},
\]

\[
\hat{P}_{\lambda_2} = \frac{1}{5} \begin{pmatrix} (1-i)\sqrt{2} \\ -i \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} + \frac{1}{40} \begin{pmatrix} i\sqrt{3} \\ -i\sqrt{3} \end{pmatrix} \begin{pmatrix} 6 \end{pmatrix} \begin{pmatrix} 5 \end{pmatrix}.
\]

**Problem 13:** Plug the result of problem 12 into that of problem 8 and verify that you get back the matrix of \( \hat{A} \) in the \( \{|n\}\) basis. **Solution:**

\[
\hat{A} = -32 \hat{P}_{\lambda_1} + 24 \hat{P}_{\lambda_2} = -4 \begin{pmatrix} 1 & (1-i)\sqrt{2} \\ (1+i)\sqrt{2} + i\sqrt{3} & (1-i)\sqrt{2} \\ (1+i)\sqrt{2} - i\sqrt{3} \end{pmatrix} + 3 \begin{pmatrix} 7 & -(1-i)\sqrt{2} \\ -(1+i)\sqrt{2} & 4 \\ -(1-i)\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 5 \end{pmatrix}.
\]

Problems 14 - 16 still refer to the same operators, \( \hat{A} \) and \( \hat{B} \), defined above. But for these problems I suggest you use Mathematica or some other computer program.

**Problem 14:** Compute \([\hat{A}, \hat{B}]\). **Solution:** \([\hat{A}, \hat{B}] = 0\).

**Problem 15:** Compute the eigenvalues, \( \kappa_i \), and a corresponding orthonormal basis of eigenvectors, \( \{|w_n\}\), of \( \hat{B} \). **Solution:** Mathematica gives the eigenvalues \( \kappa_1 = 40 \), \( \kappa_2 = 24 \), and \( \kappa_3 = 16 \), with corresponding (non-normalized) eigenvectors (I cleared the denominators) \( |\tilde{w}_1\rangle = -(i, (1-i)\sqrt{2}, \sqrt{3}) \), \( |\tilde{w}_2\rangle = (i\sqrt{3}, 0, 1) \), and \( |\tilde{w}_3\rangle = -i, (-1-i)\sqrt{2}, \sqrt{3} \).

To make them orthonormal, we only have to normalize them (divide by their lengths). I get \( |w_1\rangle = |\tilde{w}_1\rangle / \sqrt{3} \), \( |w_2\rangle = |\tilde{w}_2\rangle / \sqrt{3} \), and \( |w_3\rangle = |\tilde{w}_3\rangle / \sqrt{3} \).

**Problem 16:** Do \( \hat{A} \) and \( \hat{B} \) have a common orthonormal basis of eigenvectors? If not, why not? If they do, what is it? **Solution:** Yes, they do. It is simply \( \{|w_n\}\),
the eigenbasis of $\hat{B}$. You can check that $\hat{A}|w_1\rangle = -32|w_1\rangle$, $\hat{A}|w_2\rangle = 24|w_2\rangle$, and $\hat{A}|w_3\rangle = 24|w_3\rangle$. Or, we know this must be true since $[\hat{A}, \hat{B}] = 0$ (see problem 18 below).

The remaining three problems refer to arbitrary operators in an unspecified Hilbert space.

Problem 17: Show that any eigenvalue, $\lambda$, of a hermitean operator, $\hat{A}$, must be real. Solution: Say an eigenvector corresponding to eigenvalue $\lambda$ is $|v\rangle$, so $\hat{A}|v\rangle = \lambda|v\rangle$. Consider $\langle v|\hat{A}|v\rangle$. On the one hand, $\langle v|\hat{A}|v\rangle = \langle v|(\lambda|v\rangle) = \mu\langle v|v\rangle$. On the other hand, $(\langle v|\hat{A}|v\rangle)^* = \langle v|A^\dagger|v\rangle = \langle v|A|v\rangle = \lambda\langle v|v\rangle$, where we have used that $\hat{A}$ is hermitean. Comparing the two we learn that $\lambda\langle v|v\rangle = \lambda^*\langle v|v\rangle = \lambda^*\langle v|v\rangle$, and so $\lambda = \lambda^*$.

Problem 18: Show that if a hermitean operator $\hat{A}$ has eigenvectors $|v\rangle$ and $|w\rangle$ corresponding to two different eigenvalues $\lambda$ and $\mu$, $\lambda \neq \mu$, then $\langle v|w\rangle = 0$. Solution: Consider $\langle v|\hat{A}|w\rangle$. On the one hand, $\langle v|(\hat{A}|w\rangle = \langle v|(\mu|w\rangle) = \mu\langle v|w\rangle$. On the other hand, $(\langle v|\hat{A}|w\rangle)^* = \langle w|\hat{A}^\dagger|v\rangle = \langle w|\hat{A}|v\rangle = \lambda\langle w|v\rangle = \mu\langle v|w\rangle$, where we have used that $\hat{A}$ is hermitean. So, we have shown that $\mu^*\langle v|w\rangle = \lambda\langle w|v\rangle$, or, since $\langle v|w\rangle^* = \langle w|v\rangle$, that $(\mu^* - \lambda)\langle w|v\rangle = 0$. But from the previous problem we know that $\mu^* = \mu$, so this becomes $0 = (\mu - \lambda)\langle w|v\rangle$. Since by assumption $\mu - \lambda \neq 0$, this implies $\langle v|w\rangle = 0$.

Problem 19: If an operator, $\hat{A}$, has an eigenvalue $\lambda$ such that all its corresponding eigenvectors are of the form $c|v\rangle$ for some $|v\rangle$ where $c$ is any complex number, and $\hat{A}$ commutes with another operator, $\hat{B}$, then show that $|v\rangle$ is an eigenvector of $\hat{B}$. Solution: Consider $\hat{A}(\hat{B}|v\rangle) = \hat{B}\hat{A}|v\rangle = \hat{B}(\lambda|v\rangle) = \lambda(\hat{B}|v\rangle)$. This shows that $\hat{B}|v\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue $\lambda$. By assumption all such eigenvectors have the form $c|v\rangle$ for some $c$. Therefore, we must have $\hat{B}|v\rangle = c|v\rangle$ for some $c$, showing that $|v\rangle$ is also an eigenvector of $\hat{B}$. 