Consider a 3-dimensional Hilbert space with orthonormal basis \{|n\}, n = 1, 2, 3\} with respect to which the hermitean operators \(\hat{A}\) and \(\hat{B}\) have matrix elements
\[
\hat{A} = \begin{pmatrix}
17 & -7(1-i)\sqrt{2} & 7i\sqrt{3} \\
-7(1+i)\sqrt{2} & -4 & -7(1-i)\sqrt{6} \\
-7i\sqrt{3} & -7(1+i)\sqrt{6} & 3
\end{pmatrix},
\]
\[
\hat{B} = \begin{pmatrix}
25 & 3(1-i)\sqrt{2} & -i\sqrt{3} \\
3(1+i)\sqrt{2} & 28 & 3(1-i)\sqrt{6} \\
i\sqrt{3} & 3(1+i)\sqrt{6} & 27
\end{pmatrix}.
\]

For problems 1 - 6, compute or verify the following “by hand” (ie, do not use Mathematica or some other automated system), showing your work:

**Problem 1:** \(\det \hat{A}\). **Solution:** I find \(\det \hat{A} = -18,432\). (I don’t have to show my work because I’m the professor.)

**Problem 2:** The eigenvalues, \(\lambda_n\), of \(\hat{A}\). **Solution:** I find two eigenvalues \(\lambda_1 = -32\) and \(\lambda_2 = 24\).

**Problem 3:** All the corresponding eigenvectors, \(|v_n\rangle\), of \(\hat{A}\). **Solution:** The eigenvectors \(|v_1\rangle\) corresponding to \(\lambda_1\) all have the form \(|v_1\rangle = c(-i,(1-i)\sqrt{2},\sqrt{3})^T\) where \(c\) is any complex constant. The eigenvectors \(|v_2\rangle\) corresponding to \(\lambda_2\) all have the form \(|v_2\rangle = (-1-i)\sqrt{2}d + i\sqrt{3}e,d,e)^T\) where \(d\) and \(e\) are any complex constants.

**Problem 4:** An orthonormal basis, \(|f_n\rangle\}, of eigenvectors corresponding to the eigenvalues \(\lambda_n\) of \(\hat{A}\). **Solution:** Normalizing \(|v_1\rangle\) (ie, setting its length to 1), I find \(|c| = 1/\sqrt{8}\). Arbitrarily choosing the phase to be 1, I thus have the normalized eigenvector \(|f_1\rangle = \frac{1}{ \sqrt{2}}(-i,(1-i)\sqrt{2},\sqrt{3})^T\). The \(\lambda_2\) eigenspace is 2-dimensional, so we need to find a pair of orthonormal vectors from this space. Let me arbitrarily choose the first one to be \(|v_2\rangle\) with \(e = 0\); to be normalized I then have to pick \(|d| = 1/\sqrt{5}\), and I arbitrarily choose the phase \(|f_2\rangle = \frac{1}{ \sqrt{5}}(-(1-i)\sqrt{2},1,0)^T\). The second eigenbasis vector in this eigenspace has to be chosen orthogonal to the first, so we need to find a \(|v_2\rangle\) so that \(0 = \langle f_2 | v_2 \rangle \propto (-1-i)\sqrt{2},1,0)^*(-(1-i)\sqrt{2}d+i\sqrt{3}e,d,e)^T = -(1+i)\sqrt{2}(-1-i)\sqrt{2}d+i\sqrt{3}e+1d+0e = 5d+(1-i)\sqrt{6}e\). Therefore we must have \(d = -(1-i)\sqrt{6}e/5\), so that \(|f_3\rangle = \frac{\xi}{ \sqrt{5}}(i\sqrt{3},-(1-i)\sqrt{6},5)^T\). Normalizing this gives \(|c| = \sqrt{5}/(2\sqrt{2})\). Choosing the phase to be 1 I then get \(|f_3\rangle = \frac{1}{ \sqrt{10}}(i\sqrt{3},-(1-i)\sqrt{6},5)^T\), completing the orthonormal eigenbasis. It should be clear that (infinitely) many different such bases could have been chosen.

**Problem 5:** The change of basis matrix \(\hat{R}_{mn}\) from the \(|\{n\}\rangle\) to the \(|\{f_n\}\rangle\) basis (de-
fined by $|f_n⟩ = \sum_m \hat{R}_{mn} |m⟩$. Solution: $\hat{R}_{mn} = \langle m|f_n⟩$ which are just the components of $|f_n⟩$. Thus, from the choice of eigenbasis in problem 4 we simply read off

$$\hat{R} = \frac{1}{2\sqrt{10}} \begin{pmatrix} -i\sqrt{5} & -4(1-i) & i\sqrt{3} \\ (1-i)\sqrt{10} & 2\sqrt{2} & -(1-i)\sqrt{6} \\ \sqrt{15} & 0 & 5 \end{pmatrix}.$$ 

ie, its three columns are simply the components of $|f_1⟩$, $|f_2⟩$, and $|f_3⟩$.

**Problem 6:** Verify that $\hat{R}$ is unitary, ie, that $\hat{R}^\dagger \hat{R} = \hat{I}$. Solution:

$$\hat{R}^\dagger \hat{R} = \frac{1}{40} \begin{pmatrix} i\sqrt{5} & (1+i)\sqrt{10} & \sqrt{15} \\ -4(1+i) & 2\sqrt{2} & 0 \\ -i\sqrt{3} & -(1+i)\sqrt{6} & 5 \end{pmatrix} \begin{pmatrix} -i\sqrt{5} & -4(1-i) & i\sqrt{3} \\ (1-i)\sqrt{10} & 2\sqrt{2} & -(1-i)\sqrt{6} \\ \sqrt{15} & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problems 7 - 13 refer to the same operators, $\hat{A}$ and $\hat{B}$, defined above. Hint: these problems require very little or no computation.

**Problem 7:** Compute the matrix elements of $\hat{A}$ in the $\{|f_n⟩\}$ basis. Solution: $\hat{A}$ is just diagonal with the eigenvalues as the diagonal entries in an orthonormal eigenbasis. So

$$\hat{A} = \begin{pmatrix} -32 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}.$$

**Problem 8:** Write a general expression for $\hat{A}$ in terms of the eigenvalues $λ_i$ and the projectors, $\hat{P}_{λ_i}$, onto the $λ_i$ eigenspace. (Don’t compute the $\hat{P}_{λ_i}$ explicitly yet.) Solution: As shown in class, generally $\hat{A} = \sum_i λ_i \hat{P}_{λ_i} = -32\hat{P}_{λ_1} + 24\hat{P}_{λ_2}$.

**Problem 9:** Write expressions for the $\hat{P}_{λ_i}$ in terms of the $|f_n⟩$. Solution: The general expression (derived in class) for a projector onto a subspace with orthonormal basis $\{|v_i⟩\}$ is $\hat{P} = \sum_i |v_i⟩⟨v_i|$. Thus, in our case $\hat{P}_{λ_1} = |f_1⟩⟨f_1|$ and $\hat{P}_{λ_2} = |f_2⟩⟨f_2| + |f_3⟩⟨f_3|$.

**Problem 10:** Compute the matrix elements of the $\hat{P}_{λ_i}$ in the $\{|f_n⟩\}$ basis. Solution: Since, in the $\{|f_n⟩\}$ basis, $|f_1⟩ = (1,0,0)^T$, etc, we simply have

$$\hat{P}_{λ_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 0 \ 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{P}_{λ_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0 \ 1 \ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0 \ 0 \ 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the $\{|f_n⟩\}$ basis.

**Problem 11:** Plug the result of problem 10 into that of problem 8 and verify that
you get back the matrix of \( \hat{A} \) in the \( \{|f_n\} \) basis. Solution:

\[
A = -32\hat{P}_{\lambda_1} + 24\hat{P}_{\lambda_2} = -32 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 24 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -32 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}.
\]

Problem 12: Compute the matrix elements of the \( \hat{P}_{\lambda_i} \) in the \( \{|n\} \) basis. Solution:

Since the \( |f_n\rangle \) are given in the \( \{|n\} \) basis in problem 4, we get

\[
\hat{P}_{\lambda_1} = \frac{1}{8} \begin{pmatrix} i \sqrt{2} \\ -i \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ (1+i)\sqrt{2} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 \\ (1+i)\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 \\ (1-i)\sqrt{6} \end{pmatrix},
\]

\[
\hat{P}_{\lambda_2} = \frac{1}{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -(1-i)\sqrt{2} \\ -(1+i)\sqrt{2} \end{pmatrix} + \frac{1}{40} \begin{pmatrix} i\sqrt{3} \\ 5 \end{pmatrix} \begin{pmatrix} -i\sqrt{3} \\ -(1+i)\sqrt{6} \end{pmatrix} + \frac{1}{8} \begin{pmatrix} -7 \\ -i\sqrt{3} \end{pmatrix} \begin{pmatrix} -(1-i)\sqrt{2} \\ -(1+i)\sqrt{6} \end{pmatrix} = \begin{pmatrix} 7 \\ -i\sqrt{3} \\ -7(1+i)\sqrt{6} \end{pmatrix}.
\]

Problem 13: Plug the result of problem 12 into that of problem 8 and verify that you get back the matrix of \( \hat{A} \) in the \( \{|n\} \) basis. Solution:

\[
\hat{A} = -32\hat{P}_{\lambda_1} + 24\hat{P}_{\lambda_2} = -4 \begin{pmatrix} 1 \\ (1+i)\sqrt{2} \end{pmatrix} \begin{pmatrix} 7 \\ -(1-i)\sqrt{2} \end{pmatrix} + 3 \begin{pmatrix} 4 \\ (1+i)\sqrt{6} \end{pmatrix} \begin{pmatrix} -i\sqrt{3} \\ -(1+i)\sqrt{6} \end{pmatrix} = \begin{pmatrix} 17 \\ -7(1-i)\sqrt{2} \\ -7(1+i)\sqrt{6} \end{pmatrix}.
\]

Problems 14 - 16 still refer to the same operators, \( \hat{A} \) and \( \hat{B} \), defined above. But for these problems I suggest you use Mathematica or some other computer program.

Problem 14: Compute \([\hat{A}, \hat{B}]\). Solution: \([\hat{A}, \hat{B}] = 0\).

Problem 15: Compute the eigenvalues, \( \kappa_i \), and a corresponding orthonormal basis of eigenvectors, \( |w_n\rangle \), of \( \hat{B} \). Solution: Mathematica gives the eigenvalues \( \kappa_1 = 40 \), \( \kappa_2 = 24 \), and \( \kappa_3 = 16 \), with corresponding (non-normalized) eigenvectors (I cleared the denominators) \( |\tilde{w}_1\rangle = -(i, \sqrt{2}, \sqrt{3})^T \), \( |\tilde{w}_2\rangle = (i\sqrt{3}, 0, 1)^T \), and \( |\tilde{w}_3\rangle = -(i, -(1-i)\sqrt{2}, \sqrt{3})^T \). To make them orthonormal, we only have to normalize them (divide by their lengths). I get \( |w_1\rangle = |\tilde{w}_1\rangle/\sqrt{3} \), \( |w_2\rangle = |\tilde{w}_2\rangle/\sqrt{3} \), and \( |w_3\rangle = |\tilde{w}_3\rangle/\sqrt{3} \).

Problem 16: Do \( \hat{A} \) and \( \hat{B} \) have a common orthonormal basis of eigenvectors? If not, why not? If they do, what is it? Solution: Yes, they do. It is simply \( \{|w_n\} \),
the eigenbasis of $\hat{B}$. You can check that $\hat{A}|w_1\rangle = -32|w_1\rangle$, $\hat{A}|w_2\rangle = 24|w_2\rangle$, and $\hat{A}|w_3\rangle = 24|w_3\rangle$. Or, we know this must be true since $[\hat{A}, \hat{B}] = 0$ (see problem 19 below).

The remaining three problems refer to arbitrary operators in an unspecified Hilbert space.

**Problem 17:** Show that any eigenvalue, $\lambda$, of a hermitean operator, $\hat{A}$, must be real. **Solution:** Say an eigenvector corresponding to eigenvalue $\lambda$ is $|v\rangle$, so $\hat{A}|v\rangle = \lambda|v\rangle$. Consider $\langle v|\hat{A}|v\rangle$. On the one hand, $\langle v|\hat{A}|v\rangle = \langle v|\lambda|v\rangle = \lambda\langle v|v\rangle$. On the other hand, $(\langle v|\hat{A}|v\rangle)^* = \langle v|A^\dagger v\rangle = \langle v|A|v\rangle = \lambda\langle v|v\rangle$, where we have used that $A$ is hermitean. Comparing the two we learn that $\lambda\langle v|v\rangle = \lambda^*\langle v|v\rangle = \lambda^*(v|v)$, and so $\lambda = \lambda^*$.

**Problem 18:** Show that if a hermitean operator $\hat{A}$ has eigenvectors $|v\rangle$ and $|w\rangle$ corresponding to two different eigenvalues $\lambda$ and $\mu$, $\lambda \neq \mu$, then $\langle v|w\rangle = 0$. **Solution:** Consider $\langle v|\hat{A}|w\rangle$. On the one hand, $\langle v|\hat{A}|w\rangle = \langle v|\mu|w\rangle = \mu\langle v|w\rangle$. On the other hand, $(\langle v|\hat{A}|w\rangle)^* = \langle w|A^\dagger v\rangle = \langle w|A|v\rangle = \lambda\langle w|v\rangle$, where we have used that $A$ is hermitean. So, we have shown that $\mu^*\langle v|w\rangle = \lambda\langle w|v\rangle$, or, since $\langle v|w\rangle^* = \langle w|v\rangle$, that $(\mu^* - \lambda)\langle w|v\rangle = 0$. But from the previous problem we know that $\mu^* = \mu$, so this becomes $0 = (\mu - \lambda)\langle w|v\rangle$. Since by assumption $\mu - \lambda \neq 0$, this implies $\langle v|w\rangle = 0$.

**Problem 19:** If an operator, $\hat{A}$, has an eigenvalue $\lambda$ such that all its corresponding eigenvectors are of the form $c|v\rangle$ for some $|v\rangle$ where $c$ is any complex number, and $\hat{A}$ commutes with another operator, $\hat{B}$, then show that $|v\rangle$ is an eigenvector of $\hat{B}$. **Solution:** Consider $\hat{A}(\hat{B}|v\rangle) = \hat{B}\hat{A}|v\rangle = \hat{B}(\lambda|v\rangle) = \lambda(\hat{B}|v\rangle)$. This shows that $\hat{B}|v\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue $\lambda$. By assumption all such eigenvectors have the form $c|v\rangle$ for some $c$. Therefore, we must have $\hat{B}|v\rangle = c|v\rangle$ for some $c$, showing that $|v\rangle$ is also an eigenvector of $\hat{B}$.