Intro to QM

Problem Set 2

Let \{ |n\rangle, n = 1, \ldots, d \} be an orthonormal basis of a Hilbert space, and consider the two operators

\[ \hat{P}_\ell := \sum_{n=1}^{d} |n\rangle \langle n|, \quad \hat{P}_{d-\ell} := \sum_{n=\ell+1}^{d} |n\rangle \langle n|, \]

where \( 1 \leq \ell \leq d \) is some given integer. The following three problems, combined, ask you to show that \( \hat{P}_\ell \) and \( \hat{P}_{d-\ell} \) are orthogonal projection operators which sum to the identity. Please use only the properties of kets, bras, and the inner product to show the following:

**Problem 1:** \((\hat{P}_\ell)^\dagger = \hat{P}_\ell\) and \((\hat{P}_{d-\ell})^\dagger = \hat{P}_{d-\ell}\). **Solution:** \((\hat{P}_\ell)^\dagger = (\sum_{n=1}^{\ell} |n\rangle \langle n|)^\dagger = \sum_{n=1}^{\ell} (|n\rangle \langle n|)^\dagger = \sum_{n=1}^{\ell} |n\rangle \langle n| = \hat{P}_\ell\), and similarly for \( \hat{P}_{d-\ell} \).

**Problem 2:** \((\hat{P}_\ell)^2 = \hat{P}_\ell\), \((\hat{P}_{d-\ell})^2 = \hat{P}_{d-\ell}\), and \(\hat{P}_\ell \hat{P}_{d-\ell} = \hat{P}_{d-\ell} \hat{P}_\ell = 0\). **Solution:** \((\hat{P}_\ell)^2 = (\sum_{n=1}^{\ell} |n\rangle \langle n|)^2 = (\sum_{n=1}^{\ell} |n\rangle \langle n|)(\sum_{m=1}^{\ell} |m\rangle \langle m|) = \sum_{n,m=1}^{\ell} |n\rangle \langle m| |m\rangle \langle n| = \hat{P}_\ell\), and similarly for \( \hat{P}_{d-\ell} \). \(\hat{P}_\ell \hat{P}_{d-\ell} = (\sum_{n=1}^{\ell} |n\rangle \langle n|)(\sum_{m=\ell+1}^{d} |m\rangle \langle m|) = \sum_{n,m=1}^{\ell} |n\rangle \langle m| \delta_{n,m} (|m\rangle \langle n| = \sum_{n=1}^{\ell} \sum_{m=\ell+1}^{d} |n\rangle \langle m| \delta_{n,m} = 0\), where in the last step we used that since \( n \) never equals \( m \) in the sum so \( \delta_{n,m} = 0 \) for every term. A similar calculation shows \( \hat{P}_{d-\ell} \hat{P}_\ell = 0\).

**Problem 3:** \( \hat{P}_\ell + \hat{P}_{d-\ell} = 1 \). **Solution:** \( \hat{P}_\ell + \hat{P}_{d-\ell} = (\sum_{n=1}^{\ell} |n\rangle \langle n|) + (\sum_{n=\ell+1}^{d} |n\rangle \langle n|) = \sum_{n=1}^{d} |n\rangle \langle n| = 1\), where in the last step we used the completeness relation for the orthonormal basis.

The 2-dimensional Hilbert space for a spin-\( \frac{1}{2} \) particle has an orthonormal basis of states \{ \( |\pm z\rangle \) \}. We define another orthonormal basis, \{ \( |\pm x\rangle \) \}, for this Hilbert space by

\[ |\pm x\rangle := \frac{1}{\sqrt{2}} (|+z\rangle \pm |-z\rangle) \]  

(signs correlated).

The (spin) angular momentum operators \( \hat{S}_z \) and \( \hat{S}_x \) are defined by their actions on these orthonormal bases:

\[ \hat{S}_z |\pm z\rangle := \pm \frac{\hbar}{2} |\pm z\rangle, \quad \hat{S}_x |\pm x\rangle := \pm \frac{\hbar}{2} |\pm x\rangle. \]

Finally, the rotation operators \( \hat{R}(\theta \hat{z}) \) and \( \hat{R}(\theta \hat{x}) \) implementing the effects of rotations in space by an angle \( \theta \) around the \( \hat{z} \) and \( \hat{x} \) axes, respectively, on states in the Hilbert space are defined by

\[ \hat{R}(\theta \hat{z}) := \exp \left\{ -i \frac{\hbar}{\theta} \hat{S}_z \right\}, \quad \hat{R}(\theta \hat{x}) := \exp \left\{ -i \frac{\hbar}{\theta} \hat{S}_x \right\}. \]
Problem 4: Compute $\hat{R}(\theta \hat{z})|+z\rangle$, writing your answer in the $|\pm z\rangle$ basis. Solution:

Since $\hat{S}_z + z = (\hbar/2)|+z\rangle$, it follows by successive application that $(\hat{S}_z)^n|+z\rangle = (\hbar/2)^n|+z\rangle$.

So, using the series definition of the exponential, it follows that $\hat{R}(\theta \hat{z})|+z\rangle = \exp(-i\theta \hat{S}_z/\hbar)|+z\rangle = \exp(-i\theta \hat{S}_z/\hbar)|+z\rangle = e^{-i\theta/2}|+z\rangle = (\cos \theta/2 - i \sin \theta/2)|+z\rangle$.

Problem 5: Compute $\hat{R}(\theta \hat{z})|+z\rangle$, writing your answer in the $|\pm z\rangle$ basis. Solution:

From the definition of $|\pm x\rangle$ given above, it follows that $|+z\rangle = (1/\sqrt{2})(|+x\rangle + |-x\rangle)$.

Then, by the same kind of calculation as in the last problem, we get $\hat{R}(\theta \hat{z})|+z\rangle = \exp(-i\theta \hat{S}_z/\hbar)(|+x\rangle + |-x\rangle)/\sqrt{2} = (\exp(-i\theta \hat{S}_x/\hbar)|+x\rangle + \exp(-i\theta \hat{S}_x/\hbar)|-x\rangle)/\sqrt{2} = (\exp(-i\theta (+\hbar/2)/\hbar)|+x\rangle + \exp(-i\theta (-\hbar/2)/\hbar)|-x\rangle)/\sqrt{2} = (e^{-i\theta/2}|+x\rangle + e^{+i\theta/2}|-x\rangle)/\sqrt{2}$. Now use the definition of $|\pm x\rangle$ again to rewrite this back in terms of the $|\pm z\rangle$ basis: $\hat{R}(\theta \hat{z})|+z\rangle = (e^{-i\theta/2}|+x\rangle + e^{+i\theta/2}|-x\rangle)/\sqrt{2} = 1/2 e^{-i\theta/2}(|+z\rangle + |-z\rangle) + 1/2 e^{+i\theta/2}(|+z\rangle - |-z\rangle) = 1/2 (e^{-i\theta/2} + e^{+i\theta/2})|+z\rangle + 1/2 (e^{-i\theta/2} - e^{+i\theta/2})|-z\rangle = \cos \theta/2|+z\rangle - i \sin \theta/2|-z\rangle$.

Problem 6: Check that your answers to the previous 2 problems give $\hat{R}(2\pi \hat{z})|+z\rangle = \hat{R}(2\pi \hat{z})|+z\rangle = -|+z\rangle$. This shows that $2\pi$ rotations in space, which in classical mechanics are the same as identity transformations, give, instead, in quantum mechanics for spin-$1/2$ particles minus the identity — ie, all states get multiplied by $-1$.

Does this give any observable effect on measurements (eg, in the outcome of Stern-Gerlach experiments)? Solution: Plugging $\theta = 2\pi$ into the solutions to the last two problems gives $\hat{R}(2\pi \hat{z})|+z\rangle = (\cos \pi - i \sin \pi)|+z\rangle = -|+z\rangle$ and $\hat{R}(2\pi \hat{z})|+z\rangle = \cos \pi|+z\rangle - i \sin \pi|-z\rangle = -|+z\rangle$. No, this ‘extra’ minus sign upon rotating a spin-$1/2$ system by $2\pi$ does not affect the outcome of any measurement, since, as discussed in chapter 1, the overall phase factor of a state vector is unobservable.

However, if, instead, one considers a system made up of many spin-$1/2$ particles and one rotates, say, only one of the particles by $2\pi$ while doing nothing to the other particles, then the state vector of the system instead of getting an overall phase of $-1$ will acquire a relative phase of minus one between the rotated particle and the rest. This phase is observable, and, indeed, is responsible for important effects, notably the Pauli exclusion principle and Fermi-Dirac statistics of identical half-odd spin particles. We will discuss these effects next semester.