Let \( \hat{n} \) be a unit vector in 3 dimensions with polar and azimuthal angles \((\theta, \phi)\) in a spherical coordinates (see figure 1.11 of the text). Consider the following vectors in the state space of a spin-\( \frac{1}{2} \) particle:

\[
|+\hat{n}\rangle := \cos \frac{\theta}{2} |+\hat{z}\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\hat{z}\rangle,
\]

\[
|-\hat{n}\rangle := \sin \frac{\theta}{2} |+\hat{z}\rangle - e^{i\phi} \cos \frac{\theta}{2} |-\hat{z}\rangle.
\]

**Problem 1:** Show that \(\{|+\hat{n}\rangle, |-\hat{n}\rangle\}\) form an orthonormal basis of the spin-\( \frac{1}{2} \) state space. **Solution:** To show they form an orthonormal basis, we need to show that

\[
\langle \pm \hat{n} | \pm \hat{n} \rangle = 1 \quad \text{and} \quad \langle \pm \hat{n} | \mp \hat{n} \rangle = 0.
\]

\[
\langle +\hat{n} | +\hat{n} \rangle = \left( \cos \frac{\theta}{2} |+\hat{z}\rangle + e^{-i\phi} \sin \frac{\theta}{2} |-\hat{z}\rangle \right) \left( \cos \frac{\theta}{2} |+\hat{z}\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\hat{z}\rangle \right)
\]

\[
= \cos \frac{\theta}{2} \cos \frac{\theta}{2} + e^{i(\phi - \phi')} \sin \frac{\theta}{2} \sin \frac{\theta}{2} = 1,
\]

using the orthonormality of the \(|\pm \hat{z}\rangle\) basis. An almost identical calculation shows that \(\langle -\hat{n} | -\hat{n} \rangle = 1\).

\[
\langle -\hat{n} | +\hat{n} \rangle = \left( \cos \frac{\theta}{2} |+\hat{z}\rangle + e^{-i\phi} \sin \frac{\theta}{2} |-\hat{z}\rangle \right) \left( \sin \frac{\theta}{2} |+\hat{z}\rangle - e^{i\phi} \cos \frac{\theta}{2} |-\hat{z}\rangle \right)
\]

\[
= \sin \frac{\theta}{2} \cos \frac{\theta}{2} |+\hat{z}\rangle - e^{i\phi} \sin \frac{\theta}{2} |-\hat{z}\rangle = 0,
\]

which also implies that \(\langle -\hat{n} | +\hat{n} \rangle = 0\).

**Problem 2:** Compute the probability \(P(|+\hat{n}\rangle \Rightarrow |+\hat{n}'\rangle)\) of measuring a particle prepared in state \(|+\hat{n}\rangle\) to be in state \(|+\hat{n}'\rangle\), where \(\hat{n}'\) is the unit vector with angles \((\theta', \phi')\). **Solution:** First calculate

\[
\langle +\hat{n}' | +\hat{n} \rangle = \left( \cos \frac{\theta'}{2} |+\hat{z}\rangle + e^{-i\phi'} \sin \frac{\theta'}{2} |-\hat{z}\rangle \right) \left( \cos \frac{\theta}{2} |+\hat{z}\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\hat{z}\rangle \right)
\]

\[
= \cos \frac{\theta}{2} \cos \frac{\theta'}{2} + e^{i(\phi - \phi')} \sin \frac{\theta}{2} \sin \frac{\theta'}{2}.
\]
Then
\[
P(| + \tilde{n} \rangle \Rightarrow | + \tilde{n}' \rangle) = |(+\tilde{n}') + \tilde{n}|^2
\]
\[
= | \cos \frac{\theta'}{2} \cos \frac{\theta}{2} + e^{i(\phi - \phi')} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} |^2
\]
\[
= \left( \cos \frac{\theta'}{2} \cos \frac{\theta}{2} + e^{i(\phi - \phi')} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right) \left( \cos \frac{\theta'}{2} \cos \frac{\theta}{2} + e^{-i(\phi - \phi')} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right)
\]
\[
= \frac{1}{4}[1 + \cos \theta][1 + \cos \theta'] + \frac{1}{4} e^{i(\phi - \phi')} \sin \theta \sin \theta' + \frac{1}{4} e^{-i(\phi - \phi')} \sin \theta \sin \theta' + \frac{1}{4}[1 - \cos \theta][1 - \cos \theta']
\]
\[
= \frac{1}{2}[1 + \cos \theta \cos \theta' + \cos(\phi - \phi') \sin \theta \sin \theta'],
\]
where in the 4th line I used the identities \(\sin \alpha \cos \alpha = \frac{1}{2} \sin(2\alpha)\), \(\cos^2 \alpha = \frac{1}{2}[1 + \cos(2\alpha)]\), \(\sin^2 \alpha = \frac{1}{2}[1 - \cos(2\alpha)]\), and in the last line I used \(\cos \alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})\).

A spin-\(\frac{3}{2}\) particle is described by a 4-dimensional state space with an orthonormal basis of states \(\{|m\rangle\}, m \in \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}\). These basis vectors are states with \(S_z = mh\). Consider the following two states,
\[
|\psi_1\rangle := N_1 \sum_{m \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{m} e^{im} |m\rangle, \quad \text{and} \quad |\psi_2\rangle := N_2 \sum_{m \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{m} |m\rangle,
\]
where \(N_1\) and \(N_2\) are some positive numbers.

**Problem 3:** Compute \(\langle m|n\rangle\) for all \(m,n \in \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}\). (Ie, write a simple formula that gives the answer for all values of \(m\) and \(n\).) **Solution:** Since the problem stated that \(|m\rangle\) form an orthonormal basis, \(\langle m|n\rangle = \delta_{m,n}\) by the definition of ‘‘orthonormal’’.

**Problem 4:** Compute \(N_1\) and \(N_2\) so that \(|\psi_1\rangle\) and \(|\psi_2\rangle\) are normalized. **Solution:** \(|\psi_1\rangle\) normalized means that
\[
1 = \langle \psi_1|\psi_1 \rangle = \left( N_1 \sum_{n \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{n} e^{-in} \langle n| \right) \left( N_1 \sum_{m \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{m} e^{im} |m\rangle \right)
\]
\[
= N_1^2 \sum_{m,n \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{nm} e^{i(m-n)} \langle n|m \rangle = N_1^2 \sum_{m,n \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{nm} e^{i(m-n)} \delta_{n,m}
\]
\[
= N_1^2 \sum_{m \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{m^2} = N_1^2 \frac{80}{9} \Rightarrow N_1 = \frac{3}{4\sqrt{5}}.
\]
The computation for \(N_2\) gives the same result since \(|\psi_1\rangle\) and \(|\psi_2\rangle\) only differ in the phases of their terms and those cancelled in the above computation. So we have
\[
N_1 = N_2 = \frac{3}{4\sqrt{5}}.
\]
Problem 5: Compute \( \langle S_z \rangle \) in the state \( |\psi_1\rangle \) and in the state \( |\psi_2\rangle \). Solution: From the text, in a state \( |\psi\rangle = \sum_m c_m |m\rangle \), \( \langle S_z \rangle = \sum_m |c_m|^2 m \hbar \). Applying this to \( |\psi_1\rangle \) gives
\[
\langle S_z \rangle = N_1^2 \sum_m \left| \frac{1}{m} e^{im} \right|^2 \hbar N_1^2 \sum_{m \in \{\pm \frac{3}{2}, \pm \frac{1}{2}\}} \frac{1}{m} = \hbar N_1^2 \left( \frac{-2}{3} - \frac{2}{1} + \frac{2}{1} + \frac{3}{2} \right) = 0.
\]
The answer is the same for the expectation value in the state \( |\psi_2\rangle \) since, again, the phases all cancelled.

Problem 6: Compute \( \Delta S_z \) in the state \( |\psi_1\rangle \) and in the state \( |\psi_2\rangle \). Solution: From the text, \( (\Delta S_z)^2 = \langle (S_z)^2 \rangle - \langle S_z \rangle^2 \). From the last problem \( \langle S_z \rangle = 0 \), so we need only compute \( \langle (S_z)^2 \rangle = \sum_m |c_m|^2 (m \hbar)^2 \), giving in state \( |\psi_1\rangle \)
\[
(\Delta S_z)^2 = N_1^2 \sum_m \left| \frac{1}{m} e^{im} \right|^2 (m \hbar)^2 = \frac{9}{80} \hbar^2 \sum_m 1 = \frac{9}{20} \hbar^2 \quad \Rightarrow \quad \Delta S_z = \frac{3\hbar}{2\sqrt{5}}.
\]
Again, the answer is the same for the state \( |\psi_2\rangle \), for the same reason as in the last two problems.

Problem 7: Compute the probabilities that a measurement gives \( S_z = -\frac{\hbar}{2} \) in the state \( |\psi_1\rangle \) and in the state \( |\psi_2\rangle \). Solution:
\[
P(|\psi_1 \Rightarrow -\frac{1}{2}) = |\langle -\frac{1}{2} |\psi_1\rangle|^2 = \left| -\frac{1}{2} |N_1 \left( \sum_m \frac{1}{m} e^{im} |m\rangle \right) \right|^2 = N_1^2 \left| \sum_m \frac{1}{m} e^{im} \delta_{m,-1/2} \right|^2 = N_1^2 \left| -\frac{2}{1} e^{-i/2} \right|^2 = \frac{9}{80} \cdot 4 = \frac{9}{20}.
\]
The answer is the same for the state \( |\psi_2\rangle \), for the same reason as in the last three problems.

Problem 8: Compute the probability \( P(|\psi_1 \Rightarrow |\psi_2\rangle) \) of measuring a particle prepared in state \( |\psi_1\rangle \) to be in state \( |\psi_2\rangle \). Solution:
\[
P(|\psi_1 \Rightarrow |\psi_2\rangle) = |\langle \psi_2 |\psi_1\rangle|^2 = N_2^2 \left( \sum_m \frac{1}{n} |n\rangle \right) N_1 \left( \sum_m \frac{1}{m} e^{im} |m\rangle \right) = N_1^2 N_2^2 \left| \sum_{n,m} \frac{e^{im}}{nm} |n\rangle |m\rangle \right|^2 = N_1^2 N_2^2 \left| \sum_{n,m} \frac{e^{im}}{nm} \delta_{n,m} \right|^2 = N_1^2 N_2^2 \left| \sum_m \frac{e^{im}}{m^2} \right|^2 = \frac{81}{6400} \left| 4e^{-3i/2}/9 + 4e^{-i/2} + 4e^{+i/2} + 4e^{+3i/2}/9 \right|^2
\]
\[
= \frac{81}{6400} \left( 8 \cos\left(\frac{3}{2}\right) + 8 \cos\left(\frac{1}{2}\right) \right)^2 = \frac{1}{100} \left[ \cos\left(\frac{3}{2}\right) + 9 \cos\left(\frac{1}{2}\right) \right]^2 \approx 0.64.
\]