For **problem 1** consider a system whose 2-dimensional Hilbert space has an orthonormal basis \{\ket{0}, \ket{1}\}. In this basis the Hamiltonian for the system is given by the matrix

\[
\hat{H} = \hbar \begin{pmatrix} \omega_0 & \omega_1 \\ \omega_1 & \omega_0 \end{pmatrix}.
\]

The energy eigenvalues, \(E_{\pm}\), and associated normalized eigenstates, \(\ket{\pm}\), are

\[
E_{\pm} = \hbar (\omega_0 \pm \omega_1), \quad \ket{\pm} = \frac{1}{\sqrt{2}} \left( \ket{0} \pm \ket{1} \right).
\]

The state of the system is

\[
\ket{\psi(t)} = c_0(t) \ket{0} + c_1(t) \ket{1}.
\]

**Problem 1:** (20 points) Write down the solution of Schrödinger’s equation for \(c_0(t)\) and \(c_1(t)\) in terms of their initial values, \(c_0(0)\) and \(c_1(0)\), at time \(t = 0\).

*Hint:* Since this is a problem with a time-independent Hamiltonian, the general solution of Schrödinger's equation was derived in class in terms of the energy eigenvalues and eigenvectors.

**Solution:** The general solution of Schrödinger’s equation for a time-independent Hamiltonian is \(\ket{\psi(t)} = \sum_n e^{-iE_n t/\hbar} \ket{E_n} \langle E_n | \psi(0) \rangle\). Substituting in this expression from (2) and (3) gives

\[
c_0(t) \ket{0} + c_1(t) \ket{1} = \langle \psi(t) \rangle = e^{-iE_{\pm} t/\hbar} \langle + \ket{\psi(0)} + e^{-iE_{\pm} t/\hbar} \langle - \ket{\psi(0)}
\]

\[
= \frac{1}{\sqrt{2}} e^{-i(\omega_0 + \omega_1) t} \left( \ket{0} + \langle 1 \rangle \right) \left( c_0(0) \ket{0} + c_1(0) \ket{1} \right)
\]

\[
+ \frac{1}{\sqrt{2}} e^{-i(\omega_0 - \omega_1) t} \left( \ket{0} - \langle 1 \rangle \right) \left( c_0(0) \ket{0} + c_1(0) \ket{1} \right)
\]

\[
= \frac{1}{2} e^{-i(\omega_0 + \omega_1) t} \left( \ket{0} + \langle 1 \rangle \right) \left( c_0(0) + c_1(0) \right) + \frac{1}{2} e^{-i(\omega_0 - \omega_1) t} \left( \ket{0} - \langle 1 \rangle \right) \left( c_0(0) - c_1(0) \right)
\]

\[
= e^{-i\omega t} \left( c_0(0) \cos(\omega_1 t) - ic_1(0) \sin(\omega_1 t) \right) \ket{0} + \left( c_1(0) \cos(\omega_1 t) - ic_0(0) \sin(\omega_1 t) \right) \ket{1}.
\]

Comparing the left and right sides then gives

\[
c_0(t) = e^{-i\omega_0 t} \left[ c_0(0) \cos(\omega_1 t) - ic_1(0) \sin(\omega_1 t) \right]
\]

\[
c_1(t) = e^{-i\omega_0 t} \left[ c_1(0) \cos(\omega_1 t) - ic_0(0) \sin(\omega_1 t) \right].
\]
For problem 2, consider the same 2-state system with general state given by eqn. (3), but with a time-dependent Hamiltonian given by

$$\hat{H}(t) = \hbar \begin{pmatrix} \omega_0 & \omega_1 \cos(\omega t) \\ \omega_1 \cos(\omega t) & \omega_0 \end{pmatrix}. \quad (4)$$

in the \{\ket{0}, \ket{1}\} basis.

Problem 2: (10 points) Write down Schrödinger’s equation as a coupled system of differential equations for \(c_0(t)\) and \(c_1(t)\). (I am not asking you to solve the equation!)

Solution: Schrödinger’s equation is

$$\left(\frac{d}{dt}\right)\ket{\psi(t)} = -\frac{i}{\hbar} \hat{H}\ket{\psi(t)}.$$  

Since in the \{\ket{0}, \ket{1}\} basis we have from (3) that \(\ket{\psi(t)} = \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix}\), we get from (4)

$$\begin{align*}
\frac{d}{dt} \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} &= -\frac{i}{\hbar} \hat{H} \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = -i \begin{pmatrix} \omega_0 & \omega_1 \cos(\omega t) \\ \omega_1 \cos(\omega t) & \omega_0 \end{pmatrix} \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = -i \begin{pmatrix} \omega_0 c_0(t) + \omega_1 \cos(\omega t) c_1(t) \\ \omega_1 \cos(\omega t) c_0(t) + \omega_0 c_1(t) \end{pmatrix},
\end{align*}$$

implying

$$\begin{align*}
\frac{d}{dt} c_0(t) &= -i\omega_0 c_0(t) - i\omega_1 \cos(\omega t) c_1(t) \\
\frac{d}{dt} c_1(t) &= -i\omega_1 \cos(\omega t) c_0(t) - i\omega_0 c_1(t).
\end{align*}$$

Problem 3: (10 points) Suppose you have two particles of spins \(j_1\) and \(j_2\). What is the dimension of the Hilbert space describing the combined spin states of the particles?

Solution: The Hilbert space, \(V_j\), of spin states of a particle of spin \(j\) has basis \{\ket{j, m}\} where \(m \in \{-j, -j+1, \ldots, j\}\). Since there are \(2j+1\) values for \(m\) in this set, the dimension of \(V_j\) is \(\dim(V_j) = 2j+1\). The Hilbert space, \(V\), of two particles is the tensor product \(V = V_{j_1} \otimes V_{j_2}\). The dimension of the tensor product of two spaces is the product of the dimensions of the spaces, so

$$\dim(V) = \dim(V_{j_1}) \cdot \dim(V_{j_2}) = (2j_1 + 1)(2j_2 + 1).$$

Problem 4: (20 points) Suppose you have a system of two spin \(j = \frac{1}{2}\) particles with hamiltonian

$$\hat{H} = \frac{2\omega}{\hbar} \vec{J}_1 \cdot \vec{J}_2 + 2\omega \left( \hat{J}_{1z} + \hat{J}_{2z} \right).$$  

(5)
What are the energy eigenvalues of this system?

**Hint:** Recall that $J^2 = J_1^2 + 2 \vec{J}_1 \cdot \vec{J}_2 + J_2^2$.

**Solution:** The hint implies $2 \vec{J}_1 \cdot \vec{J}_2 = J^2 - J_1^2 - J_2^2$. Also $J_z = J_{1z} + J_{2z}$. Use these to rewrite (5) as

$$\hat{H} = \frac{\omega}{\hbar} \left( \hat{J}^2 - \hat{J}_1^2 - \hat{J}_2^2 \right) + 2 \omega \hat{J}_z.$$

Since particles 1 and 2 are both spin-$\frac{1}{2}$, all their spin states are in eigenstates of $\hat{J}_1^2$ and $\hat{J}_2^2$ with $J_i^2 = \hbar^2 j_i (j_i + 1) = \hbar^2 \frac{1}{2} (\frac{1}{2} + 1) = \frac{3}{4} \hbar^2$. Thus

$$\hat{H} = \frac{\omega}{\hbar} \left( \hat{J}^2 - \hbar^2 \frac{3}{2} \right) + 2 \omega \hat{J}_z.$$

This Hamiltonian is written in terms of $\hat{J}^2$ and $\hat{J}_z$ which are commuting observables with simultaneous eigenstates $|j, m\rangle$, as usual. Therefore, the $|j, m\rangle$ are an eigenbasis for $\hat{H}$, and we can find the eigenvalues of $\hat{H}$ simply by acting on them:

$$\hat{H}|j, m\rangle = \frac{\omega}{\hbar} \hat{J}^2 |j, m\rangle - \hbar \omega \frac{3}{2} |j, m\rangle + 2 \omega \hat{J}_z |j, m\rangle$$

$$= \frac{\omega}{\hbar} \hbar^2 j(j+1) |j, m\rangle - \hbar \omega \frac{3}{2} |j, m\rangle + 2 \omega m |j, m\rangle$$

$$= \hbar \omega \left[ j(j+1) + 2m - \frac{3}{2} \right] |j, m\rangle,$$

so the eigenvalues are $E_{jm} = \hbar \omega \left[ j(j+1) + 2m - \frac{3}{2} \right]$. It remains only to determine what values of $j$ and $m$ are realized in this system. But we know from the addition of angular momentum of two spin-$\frac{1}{2}$'s that only the $j = 1$ triplet of states with $m \in \{-1, 0, 1\}$ and the $j = 0$ single state with $m = 0$ occur. So, putting in these values we find the energy eigenvalues

$$E_{1,1} = \hbar \omega \left[ 1(1+1) + 2 \cdot 1 - \frac{3}{2} \right] = \frac{5}{2} \hbar \omega,$$

$$E_{1,0} = \hbar \omega \left[ 1(1+1) + 2 \cdot 0 - \frac{3}{2} \right] = \frac{1}{2} \hbar \omega,$$

$$E_{1,-1} = \hbar \omega \left[ 1(1+1) + 2 \cdot (-1) - \frac{3}{2} \right] = -\frac{3}{2} \hbar \omega,$$

$$E_{0,0} = \hbar \omega \left[ 0(0+1) + 2 \cdot 0 - \frac{3}{2} \right] = -\frac{3}{2} \hbar \omega.$$