Exam 1

Problem 1: (15 points) Consider a two-dimensional Hilbert space with some orthonormal basis in which three operators \( \hat{A}, \hat{B}, \hat{C} \) have matrix elements

\[
\hat{A} = \begin{pmatrix} 2 & i \\ -i & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 & 2i \\ -2i & 1 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 1 & 2i \\ -2i & i \end{pmatrix}.
\]

It may be useful to recall that \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \).

(a) Which of these operators is hermitian?

Solution: \( \hat{A} \) and \( \hat{B} \).

(b) What are the eigenvalues of \( \hat{A} \)?

Solution: The characteristic equation is \( 0 = \det(\hat{A} - \lambda) = (2-\lambda)(-\lambda) - (i)(-i) = \lambda^2 - 2\lambda - 1 \), giving \( \lambda = 1 \pm \sqrt{2} \).

(c) Are \( \hat{A} \) and \( \hat{B} \) simultaneously diagonalizable?

Solution: No, because \( [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = \begin{pmatrix} 2 & 5i \\ 0 & 2i \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -5i & 2i \end{pmatrix} = \begin{pmatrix} 0 & 5i \\ 5i & 0 \end{pmatrix} \neq 0 \).

Problem 2: (20 points) Consider a three-dimensional Hilbert space with some orthonormal basis in which a hermitian operator \( \hat{A} \) has the following matrix elements, and a (normalized) state \( |\psi\rangle \) has components

\[
\hat{A} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad |\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix}.
\]

(a) What is the probability for measuring \( A = \sqrt{2} \) in the state \( |\psi\rangle \)?

Solution:

\[
P(A=\sqrt{2}) = \langle \psi | \hat{P}_{A=\sqrt{2}} | \psi \rangle = |\langle A=\sqrt{2} | \psi \rangle|^2 = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & i \end{pmatrix} \right| \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2}.
\]

(b) What is the probability for measuring \( A = 3 \) in the state \( |\psi\rangle \)?

Solution:

\[
P(A=3) = \sum_{j=1,2} |\langle A=3_j | \psi \rangle|^2 = \left| \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right| \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix} \right|^2 + \left| \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right| \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2}.
\]

(c) What is the probability for measuring \( A = \frac{1}{2}(3 + \sqrt{2}) \) in the state \( |\psi\rangle \)?

Solution: The probability is 0, since \( (3 + \sqrt{2})/2 \) is not an eigenvalue of \( \hat{A} \).
(d) If $A$ is measured on $|\psi\rangle$ and the result $A = 3$ is found, what is the state of the system immediately after the measurement?

**Solution:** Upon measuring $A = 3$, $|\psi\rangle$ is projected onto the $A = 3$ eigenspace and normalized:

$$|\psi\rangle \rightarrow |\psi'\rangle = \frac{\hat{P}_{A=3}|\psi\rangle}{\|\hat{P}_{A=3}|\psi\rangle\|},$$

where

$$\hat{P}_{A=3} = \sum_{j=1,2} |A=3_j\rangle\langle A=3_j| = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\hat{P}_{A=3}|\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ 1 \\ i\sqrt{2} \end{pmatrix}$$

and $\|\hat{P}_{A=3}|\psi\rangle\| = 1/\sqrt{2}$, so

$$|\psi'\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ i\sqrt{2} \end{pmatrix}.$$

**Problem 3:** (15 points) A deuteron is a kind of hydrogen nucleus which is a bound state of a neutron and a proton. It turns out that it can occur\(^1\) in states with total spin quantum numbers $j = 0$ or $j = 1$. Thus it is described by a 4-dimensional Hilbert space with orthonormal basis given by the simultaneous eigenvectors $|j,m\rangle$ of the $\hat{J}_2$ and $\hat{J}_z$ operators: $\{ |1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle \}$. In this basis in this order (ie, so that $|1,1\rangle$ corresponds to the first row/column, $|1,0\rangle$ corresponds to the second row/column, etc.) the matrix elements of the $\hat{J}_x$ operator are:

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Recall also that $\hat{J}_2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$ and $\hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$. Say we have prepared a deuteron in the (normalized) state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left( i\sqrt{2} |1,0\rangle + |0,0\rangle \right).$$

(a) What are the possible values you can find if you measure $\hat{J}_z$ on this state and what are their probabilities? **Solution:** The possible outcomes are $J_z = 0$ with probability $P(J_z=0) = 1$.

\(^1\)Actually, the $j = 0$ state is very short-lived compared to the $j = 1$ states; but we will ignore this fact.
(b) What are the possible values you can find if you measure $\hat{J}^2$ on this state and what are their probabilities? **Solution:** The possible outcomes are $J^2 = \hbar^2$ with probability $P(J^2=\hbar^2) = 2/3$ and $J^2 = 0$ with probability $P(J^2=0) = 1/3$.

(c) What is the expectation value of $\hat{J}_x$ in the state $|\psi\rangle$? **Solution:**

$$
\langle J_x \rangle = \langle \psi | \hat{J}_x | \psi \rangle = \frac{\hbar}{3\sqrt{2}} \begin{pmatrix} 0 & -i\sqrt{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ i\sqrt{2} \\ 0 \\ 1 \end{pmatrix} = 0.
$$