Problem Set 8

Each problem is worth the number of points shown.

Problem 1. (2 points) A rigid square wire loop of side L is centered on the origin in the $z = 0$ plane with its sides parallel to the x or y axes. It carries a current I in the counterclockwise direction when looked at from the positive z-axis. It is in a magnetic field $\vec{B} = B_0\,(x/L)^3\,\hat{z}$. What is the net magnetic force on the loop?

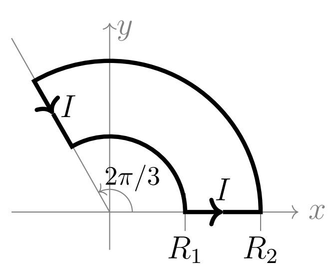
Solution: Use the Lorentz force law in the form for wires: $\vec{F} = I \oint d\vec{\ell} \times \vec{B}$. The sides parallel to the x axis (so at $y = \pm L/2$) contribute net zero force since the bottom side gives a term proportional to $\hat{x} \times \hat{z} = -\hat{y}$ at each point, while the top side gives a term proportional to $(-\hat{x}) \times \hat{z} = +\hat{y}$ at each point, and the rest of the integrand is the same since $\vec{B}$ is independent of y, so the force on the two sides cancel. The sides parallel to the y axis (at $x = \pm L/2$) contribute

$$\begin{aligned}\vec{F} &= I \int_{-L/2}^{L/2} dy (\hat{y} \times \hat{z}) B_0 (x/L)^3 \Big|_{x=+L/2} + I \int_{-L/2}^{L/2} dy (-\hat{y} \times \hat{z}) B_0 (x/L)^3 \Big|_{x=-L/2} \\ &= I\hat{x} B_0 (1/2)^3 \int_{-L/2}^{L/2} dy + I(-\hat{x}) B_0 (-1/2)^3 \int_{-L/2}^{L/2} dy = \hat{x}\, I\, B_0\, L/4.\end{aligned}$$

Problem 2. (2 points) A solid sphere of radius R with an imbedded charge density $\rho = \rho_0\,(r/R)^3$ spins counterclockwise around the z axis (as seen from the $+\hat{z}$ direction) with angular velocity ω. What is the current density $\vec{J}(r, \theta, \phi)$ everywhere in space?

Solution: The current density is $\vec{J}(\vec{r}) = \rho(\vec{r})\,\vec{v}(\vec{r})$ where $\vec{v}$ is the velocity of the charge at the point $\vec{r}$. Since the sphere is rotating at angular velocity ω around the z axis, the velocity of a point is $d\omega\hat{\phi}$ where d is the perpendicular distance of the point from the z axis. If the spherical coordinates of the point are (r, θ, ϕ), then $d = r \sin\theta$. The velocity points in the $+\hat{\phi}$ direction because counterclockwise around the z axis as seen from the $+\hat{z}$ direction corresponds to increasing azimuthal angle ϕ in spherical coordinates. So $\vec{v}(r, \theta, \phi) = \omega r \sin\theta\,\hat{\phi}$, and $\vec{J} = \rho\vec{v} = \omega\rho_0 (r^4/R^3) \sin\theta\,\hat{\phi}$.

Problem 3. (2 points) A loop of wire in the $z = 0$ plane consisting of radial segments connected by circular arcs of opening angle $2\pi/3$ with dimensions as shown in the figure carries counterclockwise current I. What is $\vec{B}$ at the origin?



Solution: Use the Biot-Savart law in the form for wires: $\vec{B}(\vec{r}) = (\mu_0/4\pi) I \oint d\vec{\ell}' \times \hat{\mathscr{r}} / \mathscr{r}^2$. The radial segments of the loop have $\vec{\mathscr{r}}$ parallel (or anti-parallel) to $d\vec{\ell}'$, so the cross product in the integrand vanishes, and these segments give 0. The angular segments give, using that

$\vec{\mathscr{r}} \doteq \vec{r} - \vec{r}' = -\vec{r}'$, and using polar coordinates (s, ϕ) in the x–y plane so that $\vec{r}' = s\hat{s}$,

$$\vec{B}(0) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi/3} R_2 d\phi \hat{\phi} \times (-\hat{s})(R_2)^{-2} + \frac{\mu_0 I}{4\pi} \int_0^{2\pi/3} R_1 d\phi (-\hat{\phi}) \times (-\hat{s})(R_1)^{-2}$$
$$= \frac{\mu_0 I \hat{z}}{4\pi R_2} \int_0^{2\pi/3} d\phi - \frac{\mu_0 I \hat{z}}{4\pi R_1} \int_0^{2\pi/3} d\phi = \frac{\mu_0 I \hat{z}}{6} \left(\frac{1}{R_2} - \frac{1}{R_1} \right).$$

Problem 4. (2 points) Two infinite parallel straight dielectric lines each carrying constant linear charge density λ point in the $+\hat{z}$ direction, are separated by a distance d, and are moving together with velocity $v\hat{z}$. At what velocity v does the total Lorentz force per unit length between the two lines vanish? Write this critical v in terms of c, the speed of light (as well as d and λ). If $\lambda = 1$ C/m and $d = 1$m, what is v/c?

Solution: Place line one at the origin and line two at $d\hat{x}$ in the x–y plane. By Gauss's law --- $\oint_S \delta\vec{a} \cdot \vec{E} = Q_{enc}/\epsilon_0$ --- the electric field due to line one at the position of line two is $\vec{E} = \hat{x}\lambda/(2\pi\epsilon_0 d)$. So the electric Lorentz force per unit length, $\vec{f}_E \doteq \lambda\vec{E}$, this line exerts on the other is $\vec{f}_E = \hat{x}\lambda^2/(2\pi\epsilon_0 d)$, a repulsion. A moving wire carries current $I = \lambda\vec{v} = \hat{x}\lambda v$, so by Ampere's law --- $\oint_C d\vec{\ell} \cdot \vec{B} = \mu_0 I_{enc}$ --- creates a magnetic field at the position of line two is $\vec{B} = \hat{y}\mu_0\lambda v/(2\pi d)$. So the magnetic Lorentz force per unit length, $\vec{f}_B \doteq \lambda\vec{v}\times\vec{B}$, this line exerts on the other is $\vec{f}_B = -\hat{x}\lambda^2 v^2 \mu_0/(2\pi d)$, an attraction. Thus these two forces are opposite, and cancel if $0 = \vec{f}_E + \vec{f}_B = \hat{x}\lambda^2/(2\pi\epsilon_0 d) - \hat{x}\lambda^2 v^2\mu_0/(2\pi d) = \hat{x}\lambda^2/(2\pi\epsilon_0 d)\,(1 - v^2\epsilon_0\mu_0)$. So the forces cancel when $v^2 = (\epsilon_0\mu_0)^{-1}$, independent of λ and d! Recall that (by definition) $(\epsilon_0\mu_0)^{-1} \doteq c^2$. Thus the forces balance only when $v = c$. Since, by special relativity, velocities of massive objects are always smaller than the speed of light, we see that the magnetic attraction is always weaker than the electric repulsion. When $\lambda = 1$ C/m and $d = 1$ m, the critical velocity is still $v/c = 1$, since it is independent of the values of λ and d.

Problem 5. (2 points) An infinite cable of circular cross section with radius R is aligned along the z axis. It carries a current density $\vec{J} = J_0\,(s/R)^2\hat{z}$ (expressed in cylindrical coordinates). What is $\vec{B}$ everywhere inside and outside the cable?

Solution: Since the problem has rotational symmetry around the z axis and translational symmetry along the z direction, $\vec{B}$ must be of the form $\vec{B} = B_s(s)\hat{s} + B_\phi(s)\hat{\phi} + B_z(s)\hat{z}$. But from the Biot-Savart law, the magnetic field at any point must be perpendicular to $\vec{J} \propto \hat{z}$, so can only have components in the $\hat{s}$ and $\hat{\phi}$ directions. Thus $B_z(s) = 0$. Also, the problem is symmetric under the reflection $z \to -z$. By the Biot-Savart law, $\vec{B} \propto \vec{J}$ (i.e., depends linearly on $\vec{J}$) so if you reverse the direction of $\vec{J}$ the $\hat{s}$ component of $\vec{B}$ would change sign. Thus $B_s(s) = 0$, and we learn that $\vec{B} = B(s)\hat{\phi}$. Now use Ampere's law in integral form --- $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$ --- with C a circle of radius constant s at $z = 0$. Then $\oint_C \vec{B} \cdot d\vec{\ell} = 2\pi s B(s)$ and

$$I_{enc} = \int_0^s s'ds' \int_0^{2\pi} d\phi' \vec{J} \cdot \hat{z} = \begin{cases} \int_0^s s'ds' \int_0^{2\pi} d\phi' J_0 (s'/R)^2 = \frac{\pi J_0}{2R^2} s^4 & s < R \\ \int_0^R s'ds' \int_0^{2\pi} d\phi' J_0 (s'/R)^2 = \frac{\pi J_0}{2} R^2 & s > R \end{cases}.$$

So

$$\vec{B} = \frac{\mu_0 J_0 \hat{\phi}}{4} \begin{cases} s^3/R^2 & s < R \\ R^2/s & s > R \end{cases}.$$

Problem 6. (3 points) Two infinite concentric solenoids of radii $R_1 < R_2$ are aligned along the z axis. The inner solenoid consists of a wire wrapped n_1 times per unit length carrying current I_1 counterclockwise (as seen from the $+\hat{z}$ direction), and the outer solenoid is wrapped n_2 times per unit length with clockwise current I_2. What are the surface current densities, $\vec{K}_1$ and $\vec{K}_2$, of the two solenoids in cylindrical coordinates? What is $\vec{B}$ everywhere?

Solution: If the wire is wrapped on the inner solenoid n_1 times per unit length, the total current flowing around the cylinder per unit length is $I_1 n_1$. Since it is flowing counterclockwise as seen from the $+\hat{z}$ direction, it is flowing in the $\hat{\phi}$ direction. Thus $\vec{K}_1 = I_1 n_1 \hat{\phi}$. A similar argument gives $\vec{K}_2 = -I_2 n_2 \hat{\phi}$. The total magnetic field is just the sum of the magnetic fields due to each solenoid separately. Griffiths (example 5.9) shows us, using Ampere's law, that the magnetic field of an infinite solenoid aligned along the z axis wrapped by n turns per unit length of a wire carrying current I counterclockwise is $\mu_0 n I \hat{z}$ inside the solenoid and 0 outside. So in our case we have

$$\vec{B} = \mu_0 \hat{z} \begin{cases} n_1 I_1 - n_2 I_2 & s < R_1 \\ n_2 I_2 & R_1 < s < R_2 \\ 0 & R_2 < s \end{cases} .$$

Problem 7. (2 points) What is the most general possible vector potential in Coulomb gauge that describes a given uniform magnetic field $\vec{B}$? In other words, what is the most general $\vec{A}(\vec{r})$ satisfying $\vec{\nabla} \cdot \vec{A} = 0$ and $\vec{\nabla} \times \vec{A} = \vec{B}$? [*Hint:* These are linear first-order PDEs with constant coefficients, so you can assume that $\vec{A}(\vec{r})$ is linear in $\vec{r}$. The general solution will have 8 undetermined coefficients.]

Solution: Write in cartesian coordinates, $\vec{r} = \sum_i x_i \hat{x}_i$, $\vec{B} = \sum_i B_i \hat{x}_i$, and $\vec{A}(\vec{r}) = \sum_i A_i(\vec{r}) \hat{x}_i$, where B_i are constants and sums over i, j, k, etc are understood to run from 1 to 3. Taking the hint we write $A_i(\vec{r}) = \alpha_i + \sum_j \beta_{ij} x_j$ in terms of 12 undetermined coefficients α_i, β_{ij}. Impose the Lorentz gauge condition,

$$0 = \vec{\nabla} \cdot \vec{A} = \sum_k \frac{\partial A_k}{\partial x_k} = \sum_{j,k} \beta_{kj} \frac{\partial x_j}{\partial x_k} = \sum_{j,k} \beta_{kj} \delta_{j,k} = \sum_k \beta_{kk}. \tag{1}$$

If we think of β_{ij} as a 3×3 matrix, this says that it is traceless. Next impose the curl equation

$$B_i = (\vec{\nabla} \times \vec{A})_i = \sum_{j,k} \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} = \sum_{j,k} \epsilon_{ijk} \sum_\ell \beta_{k\ell} \delta_{\ell,j} = \sum_{j,k} \epsilon_{ijk} \beta_{kj}, \tag{2}$$

where ϵ_{ijk} is the usual rank-3 unit antisymmetric tensor which appears in the definition of the cross product. Any 3×3 matrix can be written uniquely as a sum of a symmetric part and an antisymmetric part, $\beta_{ij} = \beta^A_{ij} + \beta^S_{ij}$ where $\beta^A_{ij} \doteq \frac{1}{2}(\beta_{ij} - \beta_{ji})$ and $\beta^S_{ij} = \frac{1}{2}(\beta_{ij} + \beta_{ji})$. Note that β^A_{ij} has 3 independent coefficients and β^S_{ij} has 6. Write them as

$$\beta^S = \begin{pmatrix} \beta^S_{11} & \beta^S_{12} & \beta^S_{13} \\ \beta^S_{12} & \beta^S_{22} & \beta^S_{23} \\ \beta^S_{13} & \beta^S_{23} & \beta^S_{33} \end{pmatrix}, \qquad \beta^A = \begin{pmatrix} 0 & \beta^A_{12} & \beta^A_{13} \\ -\beta^A_{12} & 0 & \beta^A_{23} \\ -\beta^A_{13} & -\beta^A_{23} & 0 \end{pmatrix}.$$

Now $\sum_{j,k} \epsilon_{ijk}\beta^S_{kj} = 0$ since ϵ_{ijk} is antisymmetric on jk while β^S_{jk} is symmetric on the pair, so the sum automatically vanishes. This means that the curl equation (2) puts no restriction on the 6 β^S coefficients. But the traceless condition (1) does, implying that β^S has only 5 independent coefficients:

$$\beta^S = \begin{pmatrix} \beta^S_{11} & \beta^S_{12} & \beta^S_{13} \\ \beta^S_{12} & \beta^S_{22} & \beta^S_{23} \\ \beta^S_{13} & \beta^S_{23} & -\beta^S_{11} - \beta^S_{22} \end{pmatrix}. \tag{3}$$

By contrast, the 3 independent coefficients of β^A are fixed by the curl equation (2). For example

$$B_1 = \sum_{jk} \epsilon_{1jk}\beta^A_{kj} = \beta^A_{32} - \beta^A_{23} = -2\beta^A_{23},$$

and similarly for the other two. This implies

$$\beta^A = \frac{1}{2}\begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}. \tag{4}$$

Putting this all together, the most general solution for $\vec{A}$ is

$$A_i = \alpha_i + \sum_j (\beta^S_{ij} + \beta^A_{ij})x_j$$

with the α_i arbitrary and β^S and β^A given in (3) and (4). Finally, note that $\sum_j \beta^A_{ij}x_j = -\frac{1}{2}\sum_{jk} \epsilon_{ijk}x_j B_k = -\frac{1}{2}(\vec{r}\times\vec{B})_i$ so we can also write the general solution for $\vec{A}$ as

$$\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B} + \vec{\alpha} + \sum_{ij} x_i\beta^S_{ij}\widehat{x}_j$$

where $\vec{\alpha} \doteq \sum_i \alpha_i\widehat{x}_i$ and β^S is given in (3).