Problem Set 7

Each problem is worth 4 points.

Find $\vec{D}$, $\vec{E}$, $\vec{P}$, V, ρ_b, and σ_b *everywhere* in space (not just between the plates or inside the spheres) for the following configurations of linear dielectrics, charges, and conductors. The obvious coordinate systems are used. You can save yourself a great deal of effort by using the symmetries of the problems, and, where needed, adapting the appropriate series solutions from separation of variables that were derived in the last chapter.

Solution: As we went over in class, in a linear dielectric we have the constitutive relation $\vec{P} = \epsilon_0 \chi_e \vec{E}$ as well as the definitions $\vec{D} \doteq \epsilon_0 \vec{E} + \vec{P}$, $\epsilon \doteq \epsilon_0(1+\chi_e)$, $\rho_b \doteq -\vec{\nabla}\cdot\vec{P}$, and $\sigma_b \doteq (\vec{P}_{in} - \vec{P}_{out})\cdot\hat{n}$ (where, as always, the unit normal vector to the boundary points from "in" to "out"). Then, from the electrostatics equations $\vec{\nabla}\times\vec{E} = 0$ and $\epsilon_0 \vec{\nabla}\cdot\vec{E} = \rho_f + \rho_b$ it follows that there exists a continuous potential V such that $\vec{E} = -\vec{\nabla}V$, and V is determined up to one overall additive constant by the Poisson equation

$$\epsilon \nabla^2 V = -\rho_f. \tag{1}$$

This applies to bulk (interior) regions where ϵ is constant and ρ_f is smooth. At boundaries where ϵ jumps discontinuously and/or there is a free surface charge density σ_f, then V satisfies the boundary conditions

$$V_{out} = V_{in} \qquad \text{and} \qquad \epsilon_{out}\frac{\partial V_{out}}{\partial n} - \epsilon_{in}\frac{\partial V_{in}}{\partial n} = -\sigma_f. \tag{2}$$

Once V is known, all the other quantities follow,

$$\vec{E} = -\vec{\nabla}V, \qquad \vec{D} = \epsilon\vec{E}, \qquad \vec{P} = (\epsilon - \epsilon_0)\vec{E},$$
$$\rho_b = \frac{\epsilon_0 - \epsilon}{\epsilon}\rho_f, \qquad \sigma_b = \left(\vec{P}_{in} - \vec{P}_{out}\right)\cdot\hat{n}. \tag{3}$$

Problem 1. The region between two infinite parallel plates is filled with layers of dielectric material. The bottom plate is at $z = 0$ and carries a uniform surface charge $-\sigma_f$. A dielectric material with permittivity ϵ_1 fills the space $0 < z < a$, another with permittivity ϵ_2 fills $a < z < b$, and another with permittivity ϵ_3 fills $b < z < c$. A top plate at $z = c$ carries uniform surface charge $+\sigma_f$. ($z < 0$ and $z > c$ is vacuum.) Also assume the problem is translationally invariant along the x and y directions and that the potential at $z = -\infty$ is set to zero.

Solution: From the symmetries of the problem the electric field can only point in the z direction and is independent of x and y. So the potential is a function of z only. There is no bulk free charge ρ_f, so (1) implies $d^2V/dz^2 = 0$, or $V = \alpha + \beta z$ in each region, for some constants α and β, so

$$V = \begin{cases} \alpha_0 + \beta_0 z & z < 0 \\ \alpha_1 + \beta_1 z & 0 < z < a \\ \alpha_2 + \beta_2 z & a < z < b \\ \alpha_3 + \beta_3 z & b < z < c \\ \alpha_4 + \beta_4 z & c < z \end{cases}$$

Since $V = 0$ as $z \to -\infty$ we must have $\alpha_0 = \beta_0 = 0$. The boundary conditions (2) at $z = 0$ give $\alpha_1 = 0$ and $\beta_1 = \sigma_f/\epsilon_1$. Continuing similarly at the $z = a, b, c$ boundaries I find

$$V = \begin{cases} 0 & z < 0 \\ \sigma_f \frac{z}{\epsilon_1} & 0 < z < a \\ \sigma_f \left(\frac{a}{\epsilon_1} - \frac{a}{\epsilon_2} + \frac{z}{\epsilon_2} \right) & a < z < b \\ \sigma_f \left(\frac{a}{\epsilon_1} - \frac{a-b}{\epsilon_2} - \frac{b}{\epsilon_3} + \frac{z}{\epsilon_3} \right) & b < z < c \\ \sigma_f \left(\frac{a}{\epsilon_1} - \frac{a-b}{\epsilon_2} - \frac{b-c}{\epsilon_3} \right) & c < z \end{cases}$$

Then, using the first line of (3), we have

$$\vec{E} = \begin{cases} 0 & z < 0 \\ -\frac{\sigma_f}{\epsilon_1}\hat{z} & 0 < z < a \\ -\frac{\sigma_f}{\epsilon_2}\hat{z} & a < z < b \\ -\frac{\sigma_f}{\epsilon_3}\hat{z} & b < z < c \\ 0 & c < z \end{cases} \qquad \vec{D} = \begin{cases} 0 & z < 0 \\ -\sigma_f \hat{z} & 0 < z < c \\ 0 & c < z \end{cases} \qquad \vec{P} = \begin{cases} 0 & z < 0 \\ (\epsilon_0 - \epsilon_1)\frac{\sigma_f}{\epsilon_1}\hat{z} & 0 < z < a \\ (\epsilon_0 - \epsilon_2)\frac{\sigma_f}{\epsilon_2}\hat{z} & a < z < b \\ (\epsilon_0 - \epsilon_3)\frac{\sigma_f}{\epsilon_3}\hat{z} & b < z < c \\ 0 & c < z \end{cases}.$$

Since $\rho_f = 0$ in this problem, we have $\rho_b = 0$, and, from the second line in (3) we find

$$\sigma_b = \begin{cases} \sigma_f \left(1 - \frac{\epsilon_0}{\epsilon_1}\right) & \text{at } z = 0 \\ \sigma_f \left(\frac{\epsilon_0}{\epsilon_1} - \frac{\epsilon_0}{\epsilon_2}\right) & \text{at } z = a \\ \sigma_f \left(\frac{\epsilon_0}{\epsilon_2} - \frac{\epsilon_0}{\epsilon_3}\right) & \text{at } z = b \\ \sigma_f \left(\frac{\epsilon_0}{\epsilon_3} - 1\right) & \text{at } z = c. \end{cases}$$

Problem 2. A solid dielectric sphere of permittivity ϵ and radius R with an imbedded free charge density $\rho_f = \rho_0 (r/R)^4$. ($r > R$ is vacuum.)

Solution: Here it is easiest to solve for $\vec{D}$ using Gauss's law $\vec{\nabla}\cdot\vec{D} = \rho_f$. Spherical symmetry implies $\vec{D} = D(r)\hat{r}$ and integrating Gauss's law on a sphere of radius r and using the divergence theorem gives $\oint_{S_r} d\vec{a}\cdot\vec{D} = Q_{f\,enc}$. The left hand side is just $4\pi r^2 D(r)$ while the right is $Q_{f\,enc} = \int_{B_r} d\tau' \, \rho_f = 4\pi \int_0^r (r')^2 dr \, \rho_0 (r'/R)^4 = (4\pi\rho_0/7)(r^7/R^4)$. So we have

$$\vec{D} = \begin{cases} \frac{\rho_0}{7}\frac{r^5}{R^4}\hat{r} & r < R \\ \frac{\rho_0}{7}\frac{R^3}{r^2}\hat{r} & R < r \end{cases}$$

It then follows from (3) that

$$\vec{E} = \begin{cases} \frac{\rho_0}{7\epsilon}\frac{r^5}{R^4}\hat{r} & r < R \\ \frac{\rho_0}{7\epsilon_0}\frac{R^3}{r^2}\hat{r} & R < r \end{cases}, \quad \vec{P} = \begin{cases} \rho_0 \frac{\epsilon - \epsilon_0}{7\epsilon}\frac{r^5}{R^4}\hat{r} & r < R \\ 0 & R < r \end{cases}, \quad \rho_b = \begin{cases} \frac{\epsilon_0 - \epsilon}{\epsilon}\rho_0 \frac{r^4}{R^4} & r < R \\ 0 & R < r \end{cases}, \quad \sigma_b = \rho_0 \frac{\epsilon - \epsilon_0}{7\epsilon} R \text{ at r=R}.$$

Finally, get V by integrating $-\vec{\nabla}V = \vec{E}$ to get

$$V(r) = \int_r^\infty dr\,\hat{r}\cdot\vec{E} = \frac{\rho_0 R^2}{7\epsilon_0}\begin{cases}\left(1+\frac{\epsilon_0}{6\epsilon}\right) - \frac{\epsilon_0}{6\epsilon}\frac{r^6}{R^6} & r<R \\ \frac{R}{r} & R<r\end{cases}.$$

I have chosen $V=0$ at infinity.

Problem 3. A solid dielectric sphere of permittivity ϵ and radius R with an imbedded point dipole $\vec{p} = p\hat{z}$ at its center. ($r > R$ is vacuum.)

Solution: Inside the dielectric sphere the electric field satisfies Gauss's law but with ϵ_0 replaced by ϵ. So the field and potential of a point dipole inside the sphere would be the same as those in vacuum, but with the substitution $\epsilon_0 \to \epsilon$ if the dielectric sphere filled all of space. Thus, from our results on point dipoles from a previous chapter we have that in spherical coordinates

$$V(r,\theta) = \frac{p}{4\pi\epsilon}\frac{\cos\theta}{r^2} \qquad \text{(not correct!)}. \tag{4}$$

But this cannot be the whole answer since there is a boundary condition at $r=R$ where the dielectric ends, which will modify the fields. The expression (4) would be exact if we took $R\to\infty$. Equivalently, it is valid in the limit that $r \ll R$. What this means is that the boundary condition on V at $r=0$ is that

$$V(r,\theta) = \frac{p}{4\pi\epsilon}\frac{\cos\theta}{r^2}(1+\mathcal{O}(r^2)) \qquad \text{as } r\to 0. \tag{5}$$

Note that the first allowed sub-leading term occurs at order r^2, i.e., the possible corrections to V can only have non-negative powers of r. The reason there can be no $\mathcal{O}(r)$ term is because such a term would give a $1/r$ contribution to V, which is the potential of an electric monopole, i.e., a net non-zero charge. But the problem specified only a point dipole, and no monopole at the origin, so this term must be absent.

The problem has axial symmetry and the boundary condition is at the sphere $r=R$, so we can write a general solution of the Laplace equation for V inside and outside of the sphere using our spherical separation of variables result,

$$V(r,\theta) = \begin{cases}\sum_{\ell=0}^\infty (A_\ell r^\ell + B_\ell r^{-\ell-1})P_\ell(\cos\theta) & \text{for } r<R \\ \sum_{\ell=0}^\infty (\widetilde{A}_\ell r^\ell + \widetilde{B}_\ell r^{-\ell-1})P_\ell(\cos\theta) & \text{for } r>R.\end{cases}$$

The boundary condition at $r=\infty$ is that $V=0$ (or any constant, but I'm choosing 0), which immediately implies that $\widetilde{A}_\ell = 0$ for all ℓ. The boundary condition (5) at $r=0$ implies that $B_\ell = 0$ for all $\ell \geq 2$ and for $\ell = 0$, and, since $P_1(\cos\theta) = \cos\theta$, that $B_1 = p/(4\pi\epsilon)$. Thus we have that

$$V(r,\theta) = \begin{cases}A_0P_0 + (A_1 r + \frac{p}{4\pi\epsilon}r^{-2})P_1 + \sum_{\ell=2}^\infty A_\ell r^\ell P_\ell & \text{for } r<R \\ \sum_{\ell=0}^\infty \widetilde{B}_\ell r^{-\ell-1}P_\ell & \text{for } r>R.\end{cases}$$

I've stopped writing the $\cos\theta$ dependence of the P_ℓ's out of laziness. We determine the coefficients using the boundary conditions (2) at $r=R$. By orthogonality of the Legendre polynomials, these conditions apply for each P_ℓ separately. Continuity of V (1st column) and discontinuity of the normal derivative of V (2nd column) gives

$$\begin{aligned} A_0 R &= \widetilde{B}_0, & 0 &= -\epsilon_0 \widetilde{B}_0, \\ A_1 R^3 + \frac{p}{4\pi\epsilon} &= \widetilde{B}_1, & \epsilon A_1 R^3 - \frac{2p}{4\pi} &= -\epsilon_0 2\widetilde{B}_1, \\ A_\ell R^{2\ell+1} &= \widetilde{B}_\ell, & \epsilon \ell A_\ell R^{2\ell+1} &= -\epsilon_0(\ell+1)\widetilde{B}_\ell, \end{aligned}$$

where the last equations are for all $\ell \geq 2$. It is easy to see that the only solutions of the $\ell = 0$ and $\ell \geq 2$ equations are that $A_\ell = \widetilde{B}_\ell = 0$. This just leaves the 2 equations for A_1 and $\widetilde{B}_1$, whose solution is

$$A_1 = \frac{1}{R^3}\frac{p}{2\pi\epsilon}\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right), \qquad \widetilde{B}_1 = \frac{p}{4\pi\epsilon}\left(\frac{3\epsilon}{\epsilon + 2\epsilon_0}\right).$$

Putting this all together, we have the solution

$$V(r,\theta) = \begin{cases} \frac{p}{4\pi\epsilon}\left[\frac{r}{R^3}\left(\frac{2\epsilon-2\epsilon_0}{\epsilon+2\epsilon_0}\right) + \frac{1}{r^2}\right]\cos\theta & \text{for } r < R \\ \frac{p}{4\pi\epsilon}\frac{1}{r^2}\left(\frac{3\epsilon}{\epsilon+2\epsilon_0}\right)\cos\theta & \text{for } r > R. \end{cases}$$

To evaluate $\vec{E} = -\vec{\nabla}V$, we recognize $-\vec{\nabla}(p\cos\theta/r^2) = [3(\vec{p}\cdot\widehat{r})\widehat{r} - \vec{p}]/r^3$ from the dipole results of last chapter, and $-\vec{\nabla}(pr\cos\theta) = -\vec{\nabla}(pz) = -p\widehat{z} = -\vec{p}$. So, using (3),

$$\vec{E} = \begin{cases} \frac{-\vec{p}}{2\pi\epsilon R^3}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right) + \frac{[3(\vec{p}\cdot\widehat{r})\widehat{r}-\vec{p}]}{4\pi\epsilon r^3} & (r < R) \\ \frac{[3(\vec{p}\cdot\widehat{r})\widehat{r}-\vec{p}]}{4\pi r^3}\left(\frac{3}{\epsilon+2\epsilon_0}\right) & (r > R) \end{cases}, \qquad \vec{D} = \begin{cases} \frac{-\vec{p}}{2\pi R^3}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right) + \frac{[3(\vec{p}\cdot\widehat{r})\widehat{r}-\vec{p}]}{4\pi\epsilon r^3} & (r < R) \\ \frac{[3(\vec{p}\cdot\widehat{r})\widehat{r}-\vec{p}]}{4\pi r^3}\left(\frac{3\epsilon_0}{\epsilon+2\epsilon_0}\right) & (r > R) \end{cases},$$

$$\vec{P} = \begin{cases} \frac{-\vec{p}}{2\pi\epsilon R^3}\frac{(\epsilon-\epsilon_0)^2}{\epsilon+2\epsilon_0} + \frac{[3(\vec{p}\cdot\widehat{r})\widehat{r}-\vec{p}]}{4\pi\epsilon r^3}(\epsilon - \epsilon_0) & (r < R) \\ 0 & (r > R) \end{cases}, \qquad \sigma_b = \frac{\vec{p}\cdot\widehat{r}}{2\pi R^3}\cdot\frac{3\epsilon_0(\epsilon-\epsilon_0)}{\epsilon(\epsilon+2\epsilon_0)} \quad \text{at } r = R.$$

And since there is no bulk free charge, $\rho_b = 0$.

Problem 4. A solid metal sphere of radius R_1 carrying no net charge, surrounded by a solid dielectric spherical shell of permittivity ϵ extending from $R_1 < r < R_2$, all immersed in a uniform applied electric field $\vec{E}_0 = E_0\widehat{z}$. (In other words, $\vec{E}_0$ is the electric field at infinity, and $r > R$ is vacuum.)

Solution: The problem has axial symmetry with spherical boundaries so we can use our spherical separation of variables result to write the general form of the potential

$$V(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty}(\widetilde{A}_\ell r^\ell + \widetilde{B}_\ell r^{-\ell-1})P_\ell & \text{for } r < R_1 \\ \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})P_\ell & \text{for } R_1 < r < R_2 \\ \sum_{\ell=0}^{\infty}(D_\ell r^\ell + C_\ell r^{-\ell-1})P_\ell & \text{for } R_2 < r. \end{cases}$$

(The θ-dependence of the P_ℓ's is understood.) Now start applying boundary conditions. First, we know that the potential is constant on (and inside) a conductor, so we must have $\widetilde{A}_\ell = \widetilde{B}_\ell = 0$ for all ℓ except for $\widetilde{A}_0$. But we have one overall additive constant that we can choose arbitrarily for V, so let's choose $\widetilde{A}_0 = 0$ as well. Thus we have chosen $V = 0$ for $r < R_1$. Next, since $\vec{E} = E_0\widehat{z}$ at infinity, we must have $V \to -E_0 z$ at infinity. Since $z = r\cos\theta$, this means that $V = -E_0 r\cos\theta(1 + \mathcal{O}(r^{-1}))$. Thus all the terms growing as r^ℓ with $\ell \geq 2$ must vanish for $r > R_2$, so we find that $D_\ell = 0$ for $\ell \geq 2$ and $D_1 = -E_0$. Finally, the conductor and dielectric are electrically neutral (uncharged), so the electric monopole moment of this configuration must vanish. This means that $C_0 = 0$, since it is the coefficient of the monopole term in V. Putting this together we now have

$$V(r,\theta) = \begin{cases} 0 & \text{for } r < R_1 \\ \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})P_\ell & \text{for } R_1 < r < R_2 \\ -E_0 r P_1 + D_0 P_0 + \sum_{\ell=1}^{\infty} C_\ell r^{-\ell-1} P_\ell & \text{for } R_2 < r. \end{cases}$$

The boundary conditions (3) at $r = R_2$ give

$$\sum_{\ell=0}^{\infty}(A_\ell R_2^\ell + B_\ell R_2^{-\ell-1})P_\ell = -E_0 R_2 P_1 + D_0 P_0 + \sum_{\ell=1}^{\infty} C_\ell R_2^{-\ell-1} P_\ell,$$

$$\sum_{\ell=0}^{\infty}(\ell\epsilon A_\ell R_2^{\ell-1} - (\ell+1)\epsilon B_\ell R_2^{-\ell-2})P_\ell = -\epsilon_0 E_0 P_1 - \sum_{\ell=1}^{\infty}(\ell+1)\epsilon_0 C_\ell R_2^{-\ell-2} P_\ell.$$

By orthogonality of the P_ℓ's it must be satisfied for each ℓ separately, giving

$$\begin{aligned} A_0 R_2 + B_0 &= D_0 R_2, & \epsilon B_0 &= 0, \\ A_1 R_2^3 + B_1 &= -E_0 R_2^3 + C_1, & \epsilon A_1 R_2^3 - 2\epsilon B_1 &= -\epsilon_0 E_0 R_2^3 - 2\epsilon_0 C_1, \\ A_\ell R_2^{2\ell+1} + B_\ell &= C_\ell, & \ell\epsilon A_\ell R_2^{2\ell+1} - (\ell+1)\epsilon B_\ell &= -(\ell+1)\epsilon_0 C_\ell, \end{aligned}$$

where the last line is for $\ell \geq 2$. There is one more boundary condition at $r = R_1$, where continuity of V gives simply

$$0 = A_\ell R_1^{2\ell+1} + B_\ell$$

for all $\ell \geq 0$. Combined with the last equation these can be easily solved to find $A_\ell = B_\ell = C_\ell = 0$ for $\ell \geq 2$, that $A_0 = B_0 = D_0 = 0$, and that

$$\begin{aligned} &A_1 = -E_0\frac{\alpha}{\delta}, \quad B_1 = E_0\frac{\beta}{\delta}, \qquad C_1 = E_0\frac{\gamma}{\delta}, \\ &\alpha \doteq 3\epsilon_0 R_2^3, \qquad \beta \doteq 3\epsilon_0 R_1^3 R_2^3, \qquad \gamma \doteq [(\epsilon-\epsilon_0)R_2^3 + (2\epsilon+\epsilon_0)R_1^3]R_2^3, \quad \delta \doteq (\epsilon+2\epsilon_0)R_2^3 + 2(\epsilon-\epsilon_0)R_1^3. \end{aligned} \tag{6}$$

Putting this all together, our solution is

$$V(r,\theta) = \frac{E_0}{\delta}\begin{cases} 0 & \text{for } r < R_1 \\ (-\alpha r + \beta r^{-2})\cos\theta & \text{for } R_1 < r < R_2 \\ (-\delta r + \gamma r^{-2})\cos\theta & \text{for } R_2 < r \end{cases}$$

with $\alpha, \beta, \gamma, \delta$ given in (6). Using, as in the last problem, that $-\vec{\nabla}(E_0 r\cos\theta) = -\vec{E}_0$ and $-\vec{\nabla}(E_0 r^{-2}\cos\theta) = [3(\vec{E}_0\cdot\hat{r})\hat{r} - \vec{E}_0]/r^3$, we get from $\vec{E} = -\vec{\nabla}V$ that

$$\vec{E} = \frac{1}{\delta}\begin{cases} 0 & \text{for } r < R_1 \\ \alpha\vec{E}_0 + \beta\,[3(\vec{E}_0\cdot\hat{r})\hat{r} - \vec{E}_0]\ r^{-3} & \text{for } R_1 < r < R_2 \\ \delta\vec{E}_0 + \gamma\,[3(\vec{E}_0\cdot\hat{r})\hat{r} - \vec{E}_0]\ r^{-3} & \text{for } R_2 < r \end{cases}$$

This solution might seem a bit odd since we only used the boundary condition that V is continuous at $r = R_1$, and didn't use the second condition in (2) determining the discontinuity of the normal derivative of V at the boundary. The reason is that a conductor is not a dielectric, and this boundary condition does not apply: there is no definition of "ϵ" for a conductor. (Formally you can try to think of it as the limit $\epsilon \to \infty$, but this is a very difficult and sometimes misleading way to treat conductors.) Instead, as we saw in chapter 2 of Griffiths, the discontinuity of the normal derivative of $\epsilon_0 V$ is proportional to the total surface charge density. Recall that in conductors charge distributes itself on the surface in order to impose that V is constant inside the conductor. The uniqueness theorems of chapter 3 implied that a solution for V with conductors is determined only by specifying the total charge of the conductor (together with the continuity of V). Indeed, this is the only information we used to uniquely determine the solution given above. But this also means that there is no meaning to $\vec{D}$, $\vec{P}$, ρ_b, and σ_b in or on a conductor. In place of σ_b there is the total surface charge on the conductor, which is given by

$$\epsilon_0 \partial V_{in}/\partial n - \epsilon_0 \partial V_{out}/\partial n = \sigma_b.$$

Note that it is ϵ_0 and not ϵ_{out} that appears here. Also note that $\partial V_{in}/\partial n = 0$ since V is constant in a conductor. So, using (3), we get

$$\vec{D} = \frac{1}{\delta}\begin{cases} \text{undefined} \\ \epsilon\alpha\vec{E}_0 + \epsilon\beta\,[3(\vec{E}_0\cdot\hat{r})\hat{r} - \vec{E}_0]\ \ r^{-3} \\ \epsilon_0\delta\vec{E}_0 + \epsilon_0\gamma\,[3(\vec{E}_0\cdot\hat{r})\hat{r} - \vec{E}_0]\ \ r^{-3} \end{cases} , \quad \vec{P} = \frac{\epsilon-\epsilon_0}{\delta}\begin{cases} \text{undefined} & \text{for } r < R_1 \\ \alpha\vec{E}_0 + \beta\,[3(\vec{E}_0\cdot\hat{r})\hat{r} - \vec{E}_0]\ \ r^{-3} & \text{for } R_1 < r < R_2 \\ 0 & \text{for } R_2 < r \end{cases} ,$$

$$\rho_b = \begin{cases} \text{undefined} & \text{for } r < R_1 \\ 0 & \text{for } R_1 < r \end{cases} , \qquad \sigma_b = \frac{E_0}{\delta}\begin{cases} \epsilon_0(\alpha + 2\beta R_1^{-3})\cos\theta & \text{at } r = R_1 \\ (\epsilon-\epsilon_0)(\alpha + 2\beta\, R_2^{-3})\cos\theta & \text{at } r = R_2 \end{cases} ,$$

where I used that $\vec{E}_0\cdot\hat{r} = E_0\cos\theta$ in computing σ_b.