Problem Set 5

Problems are worth one point each.

Problem 1. Prove Earnshaw's theorem (described in section 3.1 of Griffiths).

Solution: For a positive charge q to be in stable equilibrium at a point $\vec{r}_0$, the electric field must vanish at that point (otherwise it would accelerate the charge away from the point) and furthermore at all nearby points must give a force that restores the particle to the equilibrium point. Since the force is $q\vec{E}$ and we assumed q positive, this means that $\vec{E}$ must point towards $\vec{r}_0$ at all points nearby. Since $\vec{E} = -\vec{\nabla}V$, that means that $\vec{\nabla}V(\vec{r}_0) = 0$ at the equilibrium point, and V must increase in all directions away form that point for the electric field to point back to $\vec{r}_0$ at all nearby points. Thus V must be a minimum at $\vec{r}_0$. But, by the no maximum or minimum property of Laplace's equation, no solution in electrostatics can have a local minimum for V. (If you changed the sign of q, the same argument would go through, but you would need a local maximum in V, which also does not exist.)

Problem 2. A thin conducting spherical shell of radius R is split into two parts separated by a very thin insulating strip: a circular cap consisting of the points with $0 \le \theta \le \theta_0$ (in spherical coordinates with origin at the center of the sphere) and the rest. The cap is kept at potential $V = V_0$ and the rest of the sphere is kept at $V = 0$. What is the potential at the center of the sphere?

Solution: By the average property of solutions of the Laplace equation we know that the value of the potential at the center of a sphere equals its average value over the surface of the sphere. So

$$\begin{aligned} V(0) &= \oint_{S_R} da V(\vec{r})/(4\pi R^2) = \frac{1}{4\pi R^2}\int_0^{2\pi} d\phi \int_0^{\pi} R^2 \sin\theta\, d\theta\, V(R,\theta) = \frac{1}{2}\int_0^{\theta_0} \sin\theta\, d\theta\, V_0 \\ &= \frac{V_0}{2}\int_1^{\cos^{-1}\theta_0} du = V_0 \frac{1-\cos^{-1}\theta_0}{2}, \end{aligned}$$

where in the first step S_R denotes the sphere of radius R centered on the origin; in the second step $V(R,\theta)$ denotes the value of the potential on that sphere; in the third step I have done the ϕ integral and inserted the value of $V(R,\theta)$ given in the problem; in the fourth step I changed variables to $u = \cos\theta$; and in the fifth step I did the (trivial) integral. Note that $\cos^{-1}$ means the functional inverse of cosine, also called ''arccos''.

Problem 3. Two metal circular cones both of whose tips are at the origin (but not quite touching there) and sharing a common axis are held at different potentials, V_1 and V_2. More specifically, the inner cone is described in spherical coordinates centered at the tip of the cones by the equation $\theta = \theta_1$ and is held at $V = V_1$, while the outer cone is described by $\theta = \theta_2$ and is held at $V = V_2$. Find the potential $V(r, \theta, \phi)$ for all points between the two cones.

Solution: By the rotational symmetry of the problem around the z axis, V must be independent of ϕ in spherical coordinates. By the radial symmetry of the problem, V must be independent of r. Thus $V = V(\theta)$ and Laplace's equation reduces to

$$0 = \nabla^2 V = \frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right).$$

Multiplying through by $r^2 \sin\theta$ and integrating once with respect to theta gives

$$A = \sin\theta \frac{dV}{d\theta},$$

for A a constant. Dividing through by $\sin\theta$ and integrating a second time with respect to θ then gives

$$V = A \ln\tan(\theta/2) + B, \tag{1}$$

for B another constant. Now plugging in the boundary conditions $V(\theta_1) = V_1$ and $V(\theta_2) = V_2$ implies

$$V_1 = A \ln\tan(\theta_1/2) + B, \qquad V_2 = A \ln\tan(\theta_2/2) + B,$$

which implies

$$A = \frac{V_1 - V_2}{\ln\tan(\theta_1/2) - \ln\tan(\theta_2/2)}, \qquad B = \frac{1}{2}\left(V_1 + V_2 - (V_1 - V_2)\frac{\ln\tan(\theta_1/2) + \ln\tan(\theta_2/2)}{\ln\tan(\theta_1/2) - \ln\tan(\theta_2/2)}\right).$$

Plugging these values for A and B in (1) gives the desired answer.

Problem 4. The six faces of a hollow cube of side L are made up of square metal plates. If five of the faces are held at potential $V = 0$ and the sixth is held at potential $V = V_1$, prove that the potential at the center of the cube is $V = V_1/6$. [*Hint:* Use the symmetries of the cube.]

Solution: Say the value of V at the center of the cube is V_0. Now rotate the cube by $\pi/2$ around the origin to get the same problem but with $V = V_1$ on a different face. Since the rotation was a symmetry of the cube keeping the origin fixed, this new problem also has $V = V_0$ at the origin. Now rotate the problem 4 more times, so that you have 6 versions of the problem all with $V = V_1$ on a different face (and $V = 0$ on the other 5 faces) and $V = V_0$ in the center. Now add (superpose) all 6 problems to get a problem with $V = V_1$ on all six faces and $V = 6V_0$ in the center. But the solution of this problem is obvious: if $V = V_1$ everywhere on the boundary, then $V = V_1$ (i.e., constant) everywhere in the interior. Therefore, at the center $V = V_1 = 6V_0$, giving the answer.

Problem 5. If the faces of the cube of the last problem are all held at different potentials, V_1, V_2, $\ldots$, V_6, what is the potential at the center of the cube and why?

Solution: This problem can be decomposed into the sum of 6 versions of problem 4, each with $V = V_i$ $(i = 1, \ldots, 6)$ on one face and zero on the remaining 5 faces. The solution to each of these problems is that $V = V_i/6$ at the center (by problem 4), so summing these problems we get $V = \sum_{i=1}^{6} V_i/6$ at the center.

Problem 6. Consider a metal sphere of radius R, and place the origin of our coordinates at the center of the sphere. A point charge q is placed at position $\vec{r} = d\,\hat{z}$ with $d > R$ (so outside the sphere). The sphere is kept at potential $V = 0$, which is also the potential at infinity. Show that the potential outside the sphere is given by the solution for the potential of the point charge in empty space (with $V = 0$ at infinity) plus the same solution for an image point charge q' at position $\vec{r}' = d'\,\hat{z}$ with $d' < R$ (so inside the sphere), where

$$q' = -q\frac{R}{d}, \qquad\qquad d' = \frac{R^2}{d}. \tag{2}$$

Solution: The proposed potential is the sum of the potentials due to the charge and the image charge, so

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0|\vec{r}-d\hat{z}|} + \frac{q'}{4\pi\epsilon_0|\vec{r}-d'\hat{z}|} = \frac{q}{4\pi\epsilon_0}\left(|\vec{r}-d\hat{z}|^{-1} - \frac{R}{d}|\vec{r}-(R^2/d)\hat{z}|^{-1}\right).$$

This solves Poisson's equation outside the sphere and clearly obeys the $V = 0$ at infinity boundary condition. So we only have to check that is also satisfies the $V = 0$ at $r = R$ (i.e., on the sphere) boundary condition:

$$\begin{aligned}
V(R\hat{r}) &= \frac{q}{4\pi\epsilon_0}\left([R^2+d^2-2Rd\cos\theta]^{-1/2} - \frac{R}{d}[R^2+(R^4/d^2)-2(R^3/d)\cos\theta]^{-1/2}\right)\\
&= \frac{q}{4\pi\epsilon_0}\left([R^2+d^2-2Rd\cos\theta]^{-1/2} - [(d^2/R^2)\{R^2+(R^4/d^2)-2(R^3/d)\cos\theta\}]^{-1/2}\right)\\
&= \frac{q}{4\pi\epsilon_0}\left([R^2+d^2-2Rd\cos\theta]^{-1/2} - [d^2+R^2-2Rd\cos\theta]^{-1/2}\right) = 0,
\end{aligned}$$

where in the first line we used that $\hat{r}\cdot\hat{z} = \cos\theta$, and in the second line we moved the (R/d) factor in the numerator of the second term into the denominator and pulled inside the square root.

Problem 7. Use the solution found in problem 6 to compute the surface charge density, $\sigma(\theta,\phi)$ induced on the sphere (in spherical coordinates).

Solution: σ is given by the normal derivative of V at the surface,

$$\begin{aligned}
\sigma &= -\epsilon_0\frac{\partial V}{\partial n} = -\epsilon_0\frac{\partial V}{\partial r}\Big|_{r=R}\\
&= -\epsilon_0\frac{\partial}{\partial r}\left\{\frac{q}{4\pi\epsilon_0}\left([r^2+d^2-2rd\cos\theta]^{-1/2} - \frac{R}{d}[r^2+(R^4/d^2)-2r(R^2/d)\cos\theta]^{-1/2}\right)\right\}\Big|_{r=R}\\
&= -\frac{q}{4\pi}\left\{-\frac{1}{2}(2r-2d\cos\theta)[r^2+d^2-2rd\cos\theta]^{-3/2} + \frac{R}{d}\frac{1}{2}(2r-2(R^2/d)\cos\theta)[r^2+(R^4/d^2)-2r(R^2/d)\cos\theta]^{-3/2}\right\}\Big|_{r=R}\\
&= \frac{q}{4\pi}\left\{(R-d\cos\theta)[R^2+d^2-2Rd\cos\theta]^{-3/2} - \frac{d^2}{R^2}(R-(R^2/d)\cos\theta)[d^2+R^2-2Rd\cos\theta]^{-3/2}\right\}\\
&= \frac{q}{4\pi}\left\{(R-d\cos\theta) - ((d^2/R)-d\cos\theta)\right\}[d^2+R^2-2Rd\cos\theta]^{-3/2}\\
&= -\frac{q}{4\pi R}\frac{d^2-R^2}{(d^2+R^2-2dR\cos\theta)^{3/2}}.
\end{aligned}$$

Problem 8. What is the solution for the potential outside a metal sphere of radius R kept at potential $V = 0$ (which is also the potential at infinity) in the presence of two charges q_1 and q_2 at positions $\vec{r}_1$ and $\vec{r}_2$ outside the sphere, respectively?

Solution: This is just the superposition (sum) of two copies of problem 6. The image charge for q_1 is $-q_1(R/r_1)$ at $\vec{r}_1' = (R^2/r_1)\hat{r}_1$ where $\hat{r}_1$ is the unit vector in the direction of $\vec{r}_1$. Similarly, the image charge for q_2 is $-q_2(R/r_2)$ at $\vec{r}_2' = (R^2/r_2)\hat{r}_2$. Thus the total potential outside the sphere is

$$V(\vec{r}) = \frac{q_1}{4\pi\epsilon_0}\left(|\vec{r} - r_1\hat{r}_1|^{-1} - \frac{R}{r_1}|\vec{r} - (R^2/r_1)\hat{r}_1|^{-1}\right) + \frac{q_2}{4\pi\epsilon_0}\left(|\vec{r} - r_2\hat{r}_2|^{-1} - \frac{R}{r_2}|\vec{r} - (R^2/r_2)\hat{r}_2|^{-1}\right).$$

Problem 9. Consider first two charges $q_1 = -q_2 = q$ at positions $\vec{r}_1 = -\vec{r}_2 = d\,\hat{z}$ in empty space. In the limit that $d \to \infty$ and $q \to \infty$, what combination of q and d should be kept fixed to keep the electric field at the origin fixed at $\vec{E} = -E_0\,\hat{z}$? Why does this limit of the solution of problem 8 (which has a conducting sphere in addition to the two charges) describe the potential outside a conducting sphere at potential $V = 0$ in the presence of a constant applied electric field? Take this limit of the solution of problem 8 to determine the potential outside the sphere and the surface charge density on the sphere in terms of R and E_0 only. [*Hint:* This problem involves very little calculation if you use the solutions of the previous two problems appropriately.]

Solution: The electric field at the origin due to the two charges is, by Coulomb's law,

$$\vec{E}(0) = \frac{q_1(-\hat{r}_1)}{4\pi\epsilon_0 r_1^2} + \frac{q_2(-\hat{r}_2)}{4\pi\epsilon_0 r_2^2}.$$

Putting in $q_1 = -q_2 = q$ and $\vec{r}_1 = -\vec{r}_2 = d\,\hat{z}$ gives

$$\vec{E}(0) = \frac{q(-\hat{z})}{4\pi\epsilon_0 d^2} - \frac{q(\hat{z})}{4\pi\epsilon_0 d^2} = -\frac{q\hat{z}}{2\pi\epsilon_0 d^2}.$$

To keep this constant at $-E_0\hat{z}$, we must take the limit keeping

$$q = 2\pi\epsilon_0 E_0 d^2 \tag{3}$$

fixed.

As you take $d \to \infty$ the electric field due to the two charges becomes more and more uniform in the vicinity of the origin. The reason is simply that the only length scale in the problem is the distance d, so as it goes to infinity, the resulting electric field can only vary over that scale. Since we are taking the limit in such a way to keep the electric field constant at the origin, the result of the limit is to make the electric field constant everywhere.

Now plug in $q_1 = -q_2 = q$ and $\vec{r}_1 = -\vec{r}_2 = d\,\hat{z}$ with q given by (3) into the solution of problem 8 to find

$$\begin{aligned} V(\vec{r}) &= \frac{2\pi\epsilon_0 E_0 d^2}{4\pi\epsilon_0}\left(|\vec{r} - d\hat{z}|^{-1} - \frac{R}{d}|\vec{r} - (R^2/d)\hat{z}|^{-1}\right) - \frac{2\pi\epsilon_0 E_0 d^2}{4\pi\epsilon_0}\left(|\vec{r} - d(-\hat{z})|^{-1} - \frac{R}{d}|\vec{r} - (R^2/d)(-\hat{z})|^{-1}\right). \\ &= \frac{2\pi\epsilon_0 E_0 d^2}{4\pi\epsilon_0}\left(|\vec{r} - d\hat{z}|^{-1} - |\vec{r} + d\hat{z}|^{-1} - |(d/R)\vec{r} - R\hat{z}|^{-1} + |(d/R)\vec{r} + R\hat{z}|^{-1}\right). \end{aligned}$$

Now take the $d \to \infty$ limit and keep just the leading terms using

$$|\vec{r} \pm d\hat{z}|^{-1} = d^{-1}\left(1 \pm 2(r/d)\cos\theta + (r/d)^2\right)^{-1/2} \approx d^{-1}\left(1 \mp (r/d)\cos\theta + \mathcal{O}(d^{-2})\right)$$
$$|(d/R)\vec{r} \pm R\hat{z}|^{-1} = (rd/R)^{-1}\left(1 \pm 2(R^2/rd)\cos\theta + (R^2/rd)^2\right)^{-1/2} \approx (rd/R)^{-1}\left(1 \mp (R^2/rd)\cos\theta + \mathcal{O}(d^{-2})\right),$$

where in each case I pulled out the largest term at $d \to \infty$ in the square root, then expanded using the binomial expansion (or Taylor expansion), keeping only the leading terms in an expansion in inverse powers of d. Plugging these expansions into the above expression for $V(\vec{r})$ gives

$$\begin{aligned} V(\vec{r}) = \frac{2\pi\epsilon_0 E_0 d^2}{4\pi\epsilon_0}\Bigg[&\frac{1}{d}\left(1 + (r/d)\cos\theta + \mathcal{O}(d^{-2})\right) - \frac{1}{d}\left(1 - (r/d)\cos\theta + \mathcal{O}(d^{-2})\right) \\ &- \frac{R}{rd}\left(1 + (R^2/rd)\cos\theta + \mathcal{O}(d^{-2})\right) + \frac{R}{rd}\left(1 - (R^2/rd)\cos\theta + \mathcal{O}(d^{-2})\right)\Bigg] \\ = E_0\frac{r^3 - R^3}{r^2}\cos\theta + \mathcal{O}(d^{-1}). \end{aligned}$$

Therefore, in the $d \to \infty$ limit we find that the potential is simply

$$V(\vec{r}) = E_0\frac{r^3 - R^3}{r^2}\cos\theta.$$

For the surface charge density, take the radial derivative at $r = R$ as in problem 7 to find

$$\sigma = -\epsilon_0\frac{\partial V}{\partial n} = -\epsilon_0\frac{\partial V}{\partial r}\Big|_{r=R} = -\epsilon_0 E_0\cos\theta\frac{d}{dr}\left[\frac{r^3 - R^3}{r^2}\right]_{r=R} = -3\epsilon_0 E_0\cos\theta.$$

Problem 10. How much energy does it take to remove an electron of charge $-e$ from a metal surface to a distance d above the surface, if the only force keeping the electron bound to the surface of the metal is its electrostatic attraction? Assume the metal surface is an infinite plane. (This is asking you to calculate the "classical" contribution to the work function, Φ, of a metal introduced in the last problem set.) [*Hint:* Compute the work required to move the electron against the electrostatic force of the electron's attraction to the surface charge it induces on the metal surface.]

Solution: From the text (or lectures), the force on a charge q a distance z above a conducting surface is $\vec{F}(z) = -(4\pi\epsilon_0)^{-1}q^2(2z)^{-2}\hat{z}$, so towards the surface (because the induced charge is of the opposite sign, so there is an attractive force). Therefore the total work needed to move an electron $(q = -e)$ a distance d above the surface is

$$W = -\int_0^d dz\vec{F}(z)\cdot\hat{z} = \frac{(-e)^2}{4\pi\epsilon_0}\int_0^d \frac{dz}{(2z)^2} = \frac{e^2}{16\pi\epsilon_0}\left[-\frac{1}{z}\right]_0^d = \infty.$$

!? What's going wrong?

Well, the infinity is coming from the lower limit of the integration, where the electron is at 0 distance from the metal. In real materials the conduction electrons are not at zero distance from the positive ions (nuclei), but are at average distances of about an angstrom (10^{-10}m) due to quantum effects. Thus the actual energy needed to pull an electron from the metal --- the work function Φ --- cannot be calculated within classical electrodynamics because classical electrodynamics cannot give a model of stable atoms (the electrons spiral in to the nucleus on very short time scales, classically). But if you put in the above classical formula the lower distance at an angstrom and for d put any much larger distance, then you get a reasonable value of about 5eV for the work required to remove the electron, similar to actually measured values of the work function for metals.