Problem Set 4

Problems are worth one point each.

Problem 1. If you are given the electric potential only along the z-axis — i.e., your are given only $V(x{=}0,\ y{=}0\,,z)$ — what can you deduce about the electric field from $\vec{E} = -\vec{\nabla} V$?

Solution: Since $\vec{E} = -\vec{\nabla} V = -\hat{z}\partial_z V - \hat{y}\partial_y V - \hat{x}\partial_x V$ and we do not have enough information to compute the ∂_y and ∂_x derivatives of V, we can only deduce the $\hat{z}$-component of $\vec{E}$ along the z-axis.

Problem 2. An insulating sphere of radius R has charge density $\rho = \rho_0 (r/R)^2$ where ρ_0 is a constant and r is the distance from the center of the sphere. What is the total electrostatic energy of this charge distribution?

Solution: From spherical symmetry and Gauss's law,

$$\vec{E}(\vec{r}) = E(r)\hat{r} = \frac{\hat{r}}{\epsilon_0 r^2}\int_0^r dr'\,(r')^2 \rho(r') = \frac{\rho_0 \hat{r}}{5\epsilon_0}\begin{cases} r^3/R^2 & \texttt{for } r < R, \\ R^3/r^2 & \texttt{for } r > R. \end{cases}$$

Then total energy is

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int d\tau E^2 = \frac{\epsilon_0}{2} 4\pi \int_0^\infty dr\, r^2 \frac{\rho_0^2}{(5\epsilon_0)^2}\begin{cases} r^6/R^4 & r < R, \\ R^6/r^4 & r > R, \end{cases} \\ &= \frac{2\pi\rho_0^2}{25\epsilon_0}\left(\int_0^R dr\,\frac{r^8}{R^4} + \int_R^\infty dr\,\frac{R^6}{r^2}\right) = \frac{2\pi\rho_0^2}{25\epsilon_0}\left(\frac{1}{9}R^5 + R^5\right) = \frac{4\pi\rho_0^2 R^5}{45\epsilon_0}. \end{aligned}$$

Alternatively, you can solve for V inside the sphere,

$$\begin{aligned} V(r) &= \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(r')}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0} 2\pi \int_0^R dr'\,(r')^2 \rho_0 \frac{(r')^2}{R^2}\int_0^\pi d\theta'\,\sin\theta'\,\frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\theta}} \\ &= \frac{\rho_0}{2\epsilon_0 R^2}\int_0^R dr'\,(r')^4 \frac{r + r' - |r - r'|}{rr'} = \frac{\rho_0}{2\epsilon_0 r R^2}\left(\int_0^r dr'\,(r')^3 2r' + \int_r^R dr'\,(r')^3 2r\right) \\ &= \frac{\rho_0}{\epsilon_0 r R^2}\left(\frac{r^5}{5} + r\frac{R^4 - r^4}{4}\right) = \frac{\rho_0}{20\epsilon_0 R^2}(5R^4 - r^4). \end{aligned}$$

Then

$$\begin{aligned} W &= \frac{1}{2}\int d\tau\,\rho V = \frac{1}{2}4\pi\int_0^R dr\, r^2 \rho_0 \frac{r^2}{R^2}\frac{\rho_0}{20\epsilon_0 R^2}(5R^4 - r^4) = \frac{\pi\rho_0^2}{10\epsilon_0 R^4}\int_0^R dr (5R^4 r^4 - r^8) \\ &= \frac{\pi\rho_0^2}{10\epsilon_0 R^4}\left(5R^4\frac{R^5}{5} - \frac{R^9}{9}\right) = \frac{4\pi\rho_0^2 R^5}{45\epsilon_0}. \end{aligned}$$

Problem 3. A neutral metal spherical shell of inner radius a and outer radius b has a point charge Q placed at its center (so in the empty central cavity). Where, how large, and in what direction are the electrostatic pressures on the metal?

Solution: The charge induces a total surface charge $-Q$ on the inner surface and, by neutrality of the metal, a total surface charge $+Q$ on the outer surface. By spherical symmetry the surfaces charges are uniform, so the inner surface charge density is $\sigma_a = -Q/(4\pi a^2)$ and the outer surface charge density is $\sigma_b = +Q/(4\pi b^2)$. The unit out-of-the-metal normal at the inner surface is $-\hat{r}$ and the unit outward normal at the outer surface is $+\hat{r}$. So the pressure at every point on the inner surface is $\vec{f}_a = -\hat{r}\sigma_a^2/(2\epsilon_0) = -\hat{r}Q^2/(32\pi^2 a^4\epsilon_0)$ and at every point on the outer surface is $\vec{f}_b = +\hat{r}\sigma_b^2/(2\epsilon_0) = +\hat{r}Q^2/(32\pi^2 b^4\epsilon_0)$.

The next four questions are about a flat insulating circular ring of inner radius R and outer radius $R+L$, lying in the $z=0$ plane and centered at the origin. It has surface charge density $\sigma(s)$ in cylindrical coordinates. Since $\sigma(s)$ is not given explicitly, the answers to questions 4–6 will be given as some integral expressions involving $\sigma(s)$. In problems 6 and 7 we will simplify things by taking the leading contribution in the limit where $R \gg L$, as described below problem 5.

Problem 4. What is the total charge, Q, of the ring?

Solution: $\rho = \sigma(s)\,\delta(z)$, so

$$Q = \int d\tau\,\rho = \int_0^\infty ds\, s \int_{-\infty}^\infty dz \int_0^{2\pi} d\phi\, \sigma(s)\,\delta(z) = 2\pi \int_R^{R+L} ds\, s\, \sigma(s).$$

Problem 5. Show that the electric potential on the ring is given by the integral expression

$$V(s) = \frac{1}{4\pi\epsilon_0}\int_R^{R+L} ds'\, s'\, \sigma(s') \int_0^{2\pi} \frac{d\phi'}{\sqrt{s^2 + (s')^2 - 2ss'\cos\phi'}}, \tag{1}$$

if you take the potential to vanish at infinity.

Solution: Use the integral formula for $V(\vec{r})$ with $\vec{r} = s\hat{x}$,

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\epsilon_0}\int_0^\infty ds'\, s' \int_{-\infty}^\infty dz' \int_0^{2\pi} d\phi'\, \frac{\sigma(s')\delta(z')}{\sqrt{s^2+(s')^2-2ss'\cos\phi'}} \\ &= \frac{1}{4\pi\epsilon_0}\int_R^{R+L} ds'\, s'\, \sigma(s') \int_0^{2\pi} \frac{d\phi'}{\sqrt{s^2+(s')^2-2ss'\cos\phi'}} \end{aligned}$$

where in the second step I used that $|\vec{r}-\vec{r}'| = \sqrt{s^2+(s')^2-2\vec{r}\cdot\vec{r}'}$ and $\vec{r}\cdot\vec{r}' = (s\hat{x})\cdot(s'\hat{s}') = ss'(\hat{x}\cdot\hat{s}') = ss'\cos\phi'$.

If you did the angular integral in eqn. (1) exactly you would find an answer in terms the special function, K, called the complete elliptic integral of the first kind. To avoid complications, in the next two problems, use the following approximation to the angular integral

$$\int_0^{2\pi} \frac{d\phi'}{\sqrt{s^2+(s')^2-2ss'\cos\phi'}} \approx \frac{1}{R}\ln\frac{8\sqrt{2}\,R}{|s-s'|}, \tag{2}$$

which is the leading term in an expansion in $L/R \ll 1$.

Problem 6. What is the total electrostatic energy of this charged ring?

Solution: The total energy is

$$\begin{aligned}
W &= \frac{1}{2}\int d\tau\, \rho(\vec{r})V(\vec{r}) = \frac{1}{2}\int_0^\infty ds\, s \int_{-\infty}^\infty dz \int_0^{2\pi} d\phi\, \sigma(s)\,\delta(z)V(\vec{r}) = \pi\int_R^{R+L} ds\, s\,\sigma(s)V(\vec{r}) \\
&= \pi\int_R^{R+L} ds\, s\,\sigma(s)\frac{1}{4\pi\epsilon_0}\int_R^{R+L} ds'\, s'\,\sigma(s')\frac{1}{R}\ln\frac{8\sqrt{2}R}{|s-s'|} = \frac{1}{4R\epsilon_0}\int_R^{R+L} ds\int_R^{R+L} ds'\, s\, s'\,\sigma(s)\,\sigma(s')\ln\frac{8\sqrt{2}R}{|s-s'|} \\
&= \frac{1}{4R\epsilon_0}\int_R^{R+L} ds\int_R^{R+L} ds'\, s\, s'\,\sigma(s)\,\sigma(s')\left(\ln(8\sqrt{2}R) - \ln|s-s'|\right) \\
&= \frac{1}{4R\epsilon_0}\left\{\frac{Q^2}{(2\pi)^2}\ln(8\sqrt{2}) - \int_R^{R+L} ds\int_R^{R+L} ds'\, s\, s'\,\sigma(s)\,\sigma(s')\ln(|s-s'|/R)\right\},
\end{aligned}$$

where in the last step I used the result from problem 4 for the total charge.

Problem 7. For fixed total charge Q, give a qualitative argument (or, for a real challenge, a quantitative argument) that the charge distribution $\sigma(s)$ which minimizes the energy has the charge spread smoothly over the ring, as opposed to the charge being concentrated on the edges of the ring.

Solution: In the expression for the energy from the last problem, the first term is a constant, so we only need to minimize the second term, which is proportional to the double integral

$$I[\sigma] := -\int_R^{R+L} ds\int_R^{R+L} ds'\, s\,\sigma(s)\, s'\,\sigma(s')\ln\frac{|s-s'|}{R},$$

subject to the constraint that $Q = 2\pi\int ds\, s\,\sigma$ is kept constant. To decide whether minimizing this term will favor the charge being spread smoothly over the ring versus being concentrated on the ring edges, let's compare the size of $I[\sigma]$ when $\sigma(s)$ is taken to be a constant, to when $\sigma(s) \propto \delta(s-R)$. In the first case $\sigma \approx Q/(2\pi L)$, so

$$\begin{aligned}
I[\sigma{=}\text{const.}] &\approx -\frac{Q^2}{4\pi^2L^2}\int_R^{R+L} ds\int_R^{R+L} ds'\, s\, s'\,\ln\frac{|s-s'|}{R} \approx -\frac{Q^2R^2}{4\pi^2L^2}\int_R^{R+L} ds\int_R^{R+L} ds'\,\ln\frac{|s-s'|}{R} \\
&= -\frac{Q^2R^2}{4\pi^2L^2}\int_R^{R+L} ds\int_R^{R+L} ds'\,\ln\frac{|s-s'|}{R} \\
&= \frac{Q^2R^2}{4\pi^2L^2}\int_R^{R+L} ds\left[L(1+\ln R) - (s-R)\ln(s-R) - (R+L-s)\ln(R+L-s)\right] \\
&= \frac{Q^2R^2}{4\pi^2}\left[\frac{3}{2} - \ln\frac{L}{R}\right],
\end{aligned}$$

where in the second step I approximated $ss' \approx R^2$ inside the integral to simplify it, which is justified in the $L \ll R$ limit. Thus we see that $I[\sigma{=}\text{const.}]$ remains finite, though it does grow logarithmically as $L/R \to 0$.
Now compare to the case where the charge is concentrated at an edge, $\sigma(s) = Q/(2\pi R)\,\delta(s-R)$,

$$\begin{aligned}
I[\sigma{=}\delta\text{-}\texttt{fnc}] &\approx -\frac{Q^2}{4\pi^2R^2}\int_R^{R+L} ds\int_R^{R+L} ds'\, s\, s'\,\delta(s-R)\,\delta(s'-R)\,\ln\frac{|s-s'|}{R} \\
&= -\frac{Q^2}{4\pi^2R^2}\int_R^{R+L} ds\, s\, R\,\delta(s-R)\,\ln\frac{|s-R|}{R} = -\frac{Q^2}{4\pi^2}\ln\frac{|s-R|}{R}\bigg|_{s=R} = +\infty.
\end{aligned}$$

So we see the energy diverges as the charge gets concentrated at the edges. Thus we conclude that, most likely, the minimum energy configuration favors the charge being spread out over the ring.

The next three problems explore the break-down of metal conductors in which electric fields on the surface of the conductor get so strong that they pull conduction electrons out of the metal. This is governed by the "work function" (which is, despite its name, a constant), Φ, of the metal, and a separation distance d. Φ is simply the minimum energy necessary to pull a single conduction electron a distance d off the surface of the metal. d is the minimum distance where the electron attraction to the metal can be neglected in typical situations. Both Φ and d are measured properties of a given metal.

Problem 8. What is the minimum magnitude and direction of a static surface electric field needed to pull electrons out of a metal with work function Φ. Give your answer in terms of the electron charge, $-e$, and d. (Don't put in numbers yet.)

Solution: An electric field $\vec{E} = E\widehat{n}$ normal to the surface of the metal ($\widehat{n}$ is the unit normal vector pointing out of the metal) would do work $W = (-e)\int_0^d d\vec{l} \cdot \vec{E} \approx -eEd$ while moving an electron of charge $-e$ a distance d from the surface. (This is the opposite sign compared to the work used to move a charge in a given electric field as computed in the lecture: there is was the work exerted *against* the electric force, and here it is the work done *by* the electric field.) The minimum energy needed to do this is Φ, so we need at least an electric field $\vec{E} = -\Phi\widehat{n}/(ed)$ to pull the electron out of the metal. The field thus points into the metal.

Problem 9. What is the magnitude and sign of the minimum charge, Q, which when put on a solid metal sphere of radius R and work function Φ will cause electrons to leave the surface of the metal?

Solution: A charge Q on a metal sphere will spread uniformly over its surface, by spherical symmetry. Thus the surface charge density will be $\sigma = Q/(4\pi R^2)$, and so the surface electric field will be $\vec{E} = (\sigma/\epsilon_0)\widehat{n} = Q\widehat{n}/(4\pi\epsilon_0 R^2)$. Thus, from the last problem, we need a charge $Q = -4\pi\epsilon_0\Phi R^2/(ed)$ to start pushing electrons off the sphere.

Problem 10. Now put some typical numbers into the last problem: take $R = 1\text{cm}$, $\Phi = 5\text{eV}$, $d = 5\text{nm}$, and use the measured value of the electron charge. What is Q is Coulombs? How many electrons need to be added to the sphere to achieve this break-down charge?

Solution: Using $\epsilon_0 = 9\times 10^{-12}$ C^2/Nm^2, $\Phi = 5\text{eV} = 5\times 1.6\times 10^{-19}\text{J} = 8\times 10^{-19}\text{Nm}$, $R = 1\text{cm} = 10^{-2}\text{m}$, $e = 1.6\times 10^{-19}\text{C}$, and $d = 5\text{nm} = 5\times 10^{-9}\text{m}$, we have

$$Q = -\frac{4\pi \cdot 9 \cdot 10^{-12} \cdot 8 \cdot 10^{-19} \cdot 10^{-2\cdot 2}}{1.6 \cdot 10^{-19} \cdot 5 \cdot 10^{-9}}\text{C} \approx 10^{-5}\text{C}.$$

This is $Q/e = 10^{-5}/(1.6\times 10^{-19}) \approx 6\times 10^{13}$ electrons. In a typical metal, there are about 10^{23} conduction electrons per cm^3, so in a 1cm sphere there will be approximately 4×10^{23} conduction electrons. Therefore we only need to add one extra conduction electron for every 10^{10} already in the neutral metal to create a field large enough to start pulling electrons off the metal!