Problem Set 3

Problems are worth one point each.

These problems give charge distributions and either ask you to use symmetry arguments to predict the behavior of the electric field at certain locations, or to find the electric field at various locations. In the former case, please state clearly the symmetries and their consequences. In the latter case, use either Coulomb's law (giving $\vec{E}$ as an integral over a charge density) or the integrated form of Gauss's law, whichever is easier, to find $\vec{E}$.

Problem 1. In spherical coordinates, a charge distribution consisting of a point charge Q at $r = 0$, a constant surface charge density σ_0 on a spherical shell $r = R$, and a constant (volume) charge density ρ_0 in the interior of a thick spherical shell given by $3R < r < 3R+L$. Find $\vec{E}$ everywhere.

Solution: This problem has spherical symmetry. The (volume) charge density is

$$\rho(r) = Q\delta^3(\vec{r}) + \sigma_0\delta(r - R) + \rho_0 \cdot \begin{cases} 1 & 3R < r < 3R + L \\ 0 & \texttt{otherwise} \end{cases}$$

We showed in lecture that for spherical symmetry

$$\vec{E}(\vec{r}) = \frac{\hat{r}}{\epsilon_0 r^2} \int_0^r dr'\,(r')^2\,\rho(r'),$$

but this formula is a bit delicate to use because of the point charge at the origin where spherical coordinates are singular. One way to deal with this is to use superposition, $\vec{E} = \vec{E}_Q + \vec{E}_{\rm rest}$, and that $\vec{E}_Q = Q\hat{r}/(4\pi\epsilon_0 r^2)$ (Coulomb's law for a point charge). Then the above formula gives for the rest:

$$\begin{aligned}
\vec{E}_{\rm rest}(\vec{r}) &= \frac{\hat{r}}{\epsilon_0 r^2}\int_0^r dr'\,(r')^2\,\cdot 0 = 0 && \texttt{if } r < R \\
&= \frac{\hat{r}}{\epsilon_0 r^2}\int_0^r dr'\,(r')^2\,\sigma_0\delta(r' - R) = \frac{\hat{r}\,\sigma_0 R^2}{\epsilon_0 r^2} && \texttt{if } R \le r < 3R \\
&= \frac{\hat{r}\,\sigma_0 R^2}{\epsilon_0 r^2} + \frac{\hat{r}}{\epsilon_0 r^2}\int_{3R}^r dr'\,(r')^2\,\rho_0 = \frac{\hat{r}\,\sigma_0 R^2}{\epsilon_0 r^2} + \rho_0\frac{\hat{r}(r^3 - (3R)^3)}{3\epsilon_0 r^2} && \texttt{if } 3R \le r < 3R + L \\
&= \frac{\hat{r}\,\sigma_0 R^2}{\epsilon_0 r^2} + \rho_0\frac{\hat{r}((3R+L)^3 - (3R)^3)}{3\epsilon_0 r^2} && \texttt{if } 3R + L \le r
\end{aligned}$$

$\vec{E}$ is then given by adding the two, so

$$\vec{E} = \frac{\hat{r}}{\epsilon_0 r^2} \cdot \begin{cases} \frac{Q}{4\pi} & r < R \\ \frac{Q}{4\pi} + \sigma_0 R^2 & R \le r < 3R \\ \frac{Q}{4\pi} + \sigma_0 R^2 + \rho_0 \frac{r^3 - (3R)^3}{3} & 3R \le r < 3R + L \\ \frac{Q}{4\pi} + \sigma_0 R^2 + \rho_0 \frac{(3R+L)^3 - (3R)^3}{3} & 3R + L \le r \end{cases}$$

Problem 2. The problem 1 charge distribution, but with the surface and volume charge densities varying as $\sigma = \sigma_0 \cos^2\theta$ and $\rho = \rho_0 \cos^2\theta$ in spherical coordinates. What can you say about $\vec{E}$ at points on the $z = 0$ plane just based on symmetry?

Solution: Since the charge distributions are no longer constant, there is no longer spherical symmetry. But since the distributions are independent of the azimuthal angle ϕ, the problem has rotational symmetry around the z-axis which I'll call ''rotational symmetry''. Since $\cos^2\theta = \cos^2(\pi-\theta)$, the problem also has a symmetry under taking $\theta \to \pi-\theta$. This corresponds in cartesian coordinates to taking $z \to -z$, leaving the x and y coordinates unchanged. I'll call this ''reflection symmetry''.

The consequences of rotational symmetry are that $\vec{E}$ does not depend on ϕ and has no component in the $\widehat{\phi}$ direction, $\vec{E}(r,\theta,\phi) = \widetilde{A}(r,\theta)\widehat{r}+\widetilde{B}(r,\theta)\widehat{\theta}$ for some functions $\widetilde{A}$ and $\widetilde{B}$. The $z=0$ plane corresponds to $\theta = \pi/2$ and at this value of θ the unit vector $\widehat{\theta} = -\widehat{z}$, so $\vec{E}(r,\theta{=}\frac{\pi}{2},\phi) = A(r)\widehat{r}+B(r)\widehat{z}$ where I've defined $A(r) \doteq \widetilde{A}(r,\frac{\pi}{2})$ and $B(r) \doteq -\widetilde{B}(r,\frac{\pi}{2})$.

Since the reflection symmetry fixes (ie, leaves unchanged) the $z=0$ plane and reverses $\widehat{z} \to -\widehat{z}$, it follows that $B(r)=0$, so we have

$$\vec{E}(r,\theta{=}\frac{\pi}{2},\phi) = A(r)\widehat{r}.$$

(If you used cylindrical coordinates instead, this would read $\vec{E}(s,z{=}0,\phi) = A(s)\widehat{s}$.)

Problem 3. In spherical coordinates, a charge distribution consisting of a surface charge density $\sigma = \sigma_0\cos\theta$ on a spherical shell $r=R$. Find $\vec{E}$ at the origin.

Solution: As in the last problem, there is a rotational symmetry around the z axis which implies that at the origin $\vec{E}(0) = A\widehat{z}$ for some constant A. The fact that it points only in the $\widehat{z}$ direction follows from rotational symmetry since any component orthogonal to $\widehat{z}$ could be rotated to its negative.

To compute A we need to integrate using Coulomb's law in the form

$$\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\int d\tau'\,\frac{\rho(\vec{r}\,')\,\vec{\mathscr{r}}}{\mathscr{r}^3}$$

where, as per Griffith's notation, $\vec{\mathscr{r}} \doteq \vec{r}-\vec{r}\,' = -\vec{r}\,'$, since $\vec{r}=0$. Since by symmetry the $\widehat{x}$ and $\widehat{y}$ components of $\vec{E}$ cancel, we can restrict to $\vec{\mathscr{r}} = -z'\widehat{z}$ in the numerator of the integrand. In spherical coordinates $d\tau' = (r')^2dr'\,\sin\theta'\,d\theta'\,d\phi'$, $z' = r'\cos\theta'$, and $\mathscr{r} = |-\vec{r}\,'| = r'$. Plugging this all into Coulomb's law we get

$$\begin{aligned}\vec{E}(0) &= \frac{1}{4\pi\epsilon_0}\iiint(r')^2dr'\,\sin\theta'\,d\theta'\,d\phi'\;\;\frac{\rho(r',\theta')\,(-r'\cos\theta')\widehat{z}}{(r')^3}\\ &= -\frac{\widehat{z}\sigma_0}{2\epsilon_0}\int_0^\infty \delta(r'-R)\,dr'\int_{-1}^{1}du\,u^2 = -\frac{\widehat{z}\sigma_0}{2\epsilon_0}\cdot 1\cdot\frac{u^3}{3}\bigg|_{-1}^{1} = -\frac{\widehat{z}\sigma_0}{3\epsilon_0}.\end{aligned}$$

In the second step I did the ϕ' integral, inserted $\rho(r',\theta') = \sigma_0\cos\theta'\delta(r'-R)$, and changed variables to $u=\cos\theta'$.

Problem 4. The circle $z=0$ and $s=R$ carrying a linear charge density $\lambda = \lambda_0\cos\phi$ in cylindrical coordinates. What can you say about $\vec{E}$ at points on the x-axis just based on symmetry?

Solution: This problem has a reflection symmetry under $z\to -z$ (keeping x and y fixed) and a separate reflection symmetry under $y \to -y$ (keeping x and z fixed). The y reflection is because $\cos\phi = \cos(-\phi)$ and $\phi \to -\phi$ maps $y\to -y$ and $x \to x$ since $x = s\cos\phi$ and $y = s\sin\phi$.

The $\vec{E}$ field on the x-axis is $\vec{E}(x\widehat{x}) = E_x(x)\widehat{x}+E_y(x)\widehat{y}+E_z(x)\widehat{z}$, in cartesian coordinates. But since the x-axis is unchanged under the two reflections, we must have $E_y = E_z = 0$. Thus $\vec{E}(x\hat{x}) = E_x(x)\hat{x}$ for some function E_x.

(This can also be written in cylindrical coordinates, though it will look a bit more complicated since the x axis is $z = 0$ and $\phi = 0$ (for positive x) or $\phi = \pi$ (for negative x) and $s = |x|$. Also $\hat{x} = \pm\hat{s}$. Thus $\vec{E}(s, \phi{=}0, z{=}0) = A(s)\hat{s}$ and $\vec{E}(s, \phi{=}\pi, z{=}0) = B(s)\hat{s}$ is the consequence of the symmetry in these coordinates.)

Problem 5. The problem 4 charge distribution. Find $\vec{E}$ at the point $\vec{r} = 2R\hat{z}$.

Solution: By Coulomb's law for a line charge distribution

$$\vec{E}(2R\hat{z}) = \frac{1}{4\pi\epsilon_0}\int d\ell' \frac{\lambda(\vec{r}\,')\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} R d\phi' \frac{\lambda_0 \cos\phi'(2R\hat{z} - R\hat{s}')}{5^{3/2}R^3},$$

where in the second step I used that the differential arc length is $d\ell' = Rd\phi'$, $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = 2R\hat{z} - R\hat{s}'$, and so $\mathscr{r} = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = \sqrt{4R^2 + R^2} = \sqrt{5}\,R$; here I used that $\hat{z}\cdot\hat{s}' = 0$. From the y-reflection symmetry $\vec{E}(2R\hat{z})$ can have no $\hat{y}$ component. So, since $\hat{s}' = \cos\phi'\hat{x} + \sin\phi'\hat{y}$, we can drop the $\hat{y}$ component in the numerator of the integrand (ie, it will cancel by symmetry), so we have

$$\vec{E}(2R\hat{z}) = \frac{\lambda_0}{4\cdot 5^{3/2}\pi R\epsilon_0}\int_0^{2\pi} d\phi' \, \cos\phi'(2\hat{z} - \cos\phi'\hat{x}) = \frac{\lambda_0}{4\cdot 5^{3/2}\pi R\epsilon_0}\{0\cdot\hat{z} - \pi\cdot\hat{x}\} = -\frac{\lambda_0\hat{x}}{4\cdot 5^{3/2}R\epsilon_0},$$

where I used that $\int_0^{2\pi} d\phi' \, \cos\phi' = 0$ and $\int_0^{2\pi} d\phi' \, \cos^2\phi' = \pi$.

Problem 6. A line segment given by $x = y = 0$ and $-L < z < L$ carrying linear charge density $\lambda = \lambda_0\, z/L$. What can you say about $\vec{E}$ at points on the $z = 0$ plane just based on symmetry?

Solution: This problem has rotational symmetry about the z-axis. (It is not symmetric under $z \to -z$ reflections, since λ changes sign.) So, in cylindrical coordinates $\vec{E}$ at $z = 0$ (the x-y plane) is independent of ϕ and can point in the $\hat{s}$ and $\hat{z}$ directions. So $\vec{E}(s\hat{s}) = E_s(s)\hat{s} + E_z(s)\hat{z}$ for some functions $E_s(s)$ and $E_z(s)$.

Problem 7. The problem 6 charge distribution. Find $\vec{E}$ at all points on the z-axis.

Solution: By Coulomb's law for a line charge distribution

$$\vec{E}(z\hat{z}) = \frac{1}{4\pi\epsilon_0}\int d\ell' \frac{\lambda(\vec{r}\,')\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} dz' \frac{\lambda_0(z'/L)(z\hat{z} - z'\hat{z})}{|z - z'|^3} = \frac{\lambda_0\hat{z}}{4\pi L\epsilon_0}\int_{-L}^{L} dz' \frac{z'(z - z')}{|z - z'|^3},$$

where in the second step I used that the differential path length is $d\ell' = dz'$, $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = z\hat{z} - z'\hat{z}$, and so $\mathscr{r} = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = |z - z'|$. Split it into three cases: (i) $z > L$, (ii) $L > z > -L$, and (iii) $z < -L$. In case (i) $|z - z'| = z - z'$ for all z' in the integration region, so

$$\vec{E}_{(i)}(z\hat{z}) = \frac{\lambda_0\hat{z}}{4\pi L\epsilon_0}\int_{-L}^{L} dz' \frac{z'}{(z - z')^2} = \frac{\lambda_0\hat{z}}{4\pi L\epsilon_0}\int_{-L/z}^{L/z} d\zeta \frac{\zeta}{(1 - \zeta)^2} = \frac{\lambda_0\hat{z}}{4\pi L\epsilon_0}\left\{\frac{2Lz}{z^2 - L^2} + \ln\left(\frac{z + L}{z - L}\right)\right\},$$

where in the second step I changed variables to $\zeta = z'/z$, and in the 3rd step I looked up the integral. (It is also sometimes written in terms of arctanh using that $\operatorname{arctanh}(a) = \frac{1}{2}\log\frac{1+a}{1-a}$.)

Case (iii) is very similar, but with $|z - z'| = z' - z$, so

$$\vec{E}_{(iii)}(z\hat{z}) = -\frac{\lambda_0 \hat{z}}{4\pi L\epsilon_0} \int_{-L}^{L} dz' \frac{z'}{(z - z')^2} = \frac{\lambda_0 \hat{z}}{4\pi L\epsilon_0} \int_{-L/|z|}^{L/|z|} d\zeta \frac{\zeta}{(1-\zeta)^2} = \frac{\lambda_0 \hat{z}}{4\pi L\epsilon_0} \left\{ \frac{2L|z|}{z^2 - L^2} + \ln\left(\frac{|z| + L}{|z| - L}\right) \right\}$$
$$= -\frac{\lambda_0 \hat{z}}{4\pi L\epsilon_0} \left\{ \frac{2Lz}{z^2 - L^2} + \ln\left(\frac{z + L}{z - L}\right) \right\},$$

where in the second step I changed variables to $\zeta = z'/z$ and switched the limits of the integration (that's why the overall minus sign disappears), and the in last step I used that $|z| = -z$.
Case (ii) is where the point is in the interior of the line charge, so you might expect the electric field to diverge. Indeed, since the indefinite integral $\int d\zeta\, \zeta(1-\zeta)^{-2} = (1-\zeta)^{-1} + \ln(\zeta - 1)$, it diverges if $\zeta = 1$ is inside the region of integration. Thus we can write the final answer as

$$\vec{E}(z\hat{z}) = \frac{\lambda_0 \hat{z}}{4\pi L\epsilon_0} \left\{ \frac{2Lz}{z^2 - L^2} + \ln\left(\frac{z + L}{z - L}\right) \right\} \cdot \begin{cases} +1 & \text{if } z > L \\ \infty & \text{if } L > z > -L \\ -1 & \text{if } -L > z \end{cases}.$$

Problem 8. A charge distribution consisting of a uniform surface charge density σ_0 everywhere on the $z = 0$ plane plus a volume charge density $\rho = \rho_0 z/L$ for $L \leq z \leq 2L$ for all x and y, where ρ_0 is a constant. Find $\vec{E}$ everywhere.

Solution: This problem has planar symmetry, so using the solution found from Gauss's law in the lectures,

$$\vec{E}(\vec{r}) = \frac{\hat{z}}{\epsilon_0} \int_{-\infty}^{z} dz' \rho(z') + \vec{E}(-\infty\hat{z}),$$

where I have chosen the arbitrary additive constant electric field to be its value at infinity instead of at $\vec{r} = 0$ since we are putting a surface charge density right at $z = 0$. Using that

$$\rho(z') = \sigma_0 \delta(z') + \rho_0 \frac{z'}{L} \cdot \begin{cases} 1 & \text{if } L < z < 2L, \\ 0 & \text{otherwise,} \end{cases}$$

and splitting the integral into regions, you immediately get

$$\vec{E}(\vec{r}) - \vec{E}_{-\infty} = \frac{\hat{z}}{\epsilon_0} \cdot \begin{cases} 0 & \text{if } z < 0, \\ \sigma_0 & \text{if } 0 < z < L, \\ \sigma_0 + \rho_0 \frac{z^2 - L^2}{2L} & \text{if } L < z < 2L, \\ \sigma_0 + \rho_0 \frac{3L}{2} & \text{if } 2L < z. \end{cases}$$

The arbitrary constant $\vec{E}_{-\infty}$ cannot be determined from the problem.

Problem 9. Two parallel lines given by $z = 0$ and $y = \pm L/2$ (for all x) carry uniform linear charge densities $\pm\lambda_0$, respectively (i.e., the signs are correlated). Find $\vec{E}$ at the origin.

Solution: The solution for the electric field $\vec{E}'$ of a uniform line charge λ_0 on the z axis is by Gauss's law (cylindrical symmetry) $\vec{E}' = \frac{\hat{s}}{\epsilon_0 s} \int_0^s ds'\, s'\, \rho(s') = \lambda_0 \hat{s}/(2\pi\epsilon_0 s)$ in cylindrical coordinates. At points along the y axis $\hat{s}/s = \hat{y}/y$, so

$$\vec{E}'(y\hat{y}) = \frac{\lambda_0 \hat{y}}{2\pi\epsilon_0 y}. \quad (1) \quad \boxed{\texttt{zaxis}}$$

Now shift this solution along the y-axis so the line lies at $y = L/2$. This is equivalent to shifting $y \to y - L/2$, so

$$\vec{E}_0(y\hat{y}) = \frac{\lambda_0 \hat{y}}{\pi\epsilon_0(2y - L)}.$$

Similarly, if we shift the line instead to $-L/2$ and change $\lambda_0 \to -\lambda_0$ we have

$$\vec{E}_{-1}(y\hat{y}) = -\frac{\lambda_0 \hat{y}}{\pi\epsilon_0(2y + L)}.$$

The electric field at $y = 0$ for the problem at hand is then the sum of these two, giving

$$\vec{E}(0) = \vec{E}_0(0) + \vec{E}_{-1}(0) = \frac{\lambda_0 \hat{y}}{\pi\epsilon_0(2 \cdot 0 - L)} - \frac{\lambda_0 \hat{y}}{\pi\epsilon_0(2 \cdot 0 + L)} = -\frac{2\lambda_0 \hat{y}}{\pi\epsilon_0 L}.$$

Problem 10. A charge distribution similar to that of problem 9, but now with an infinite number of parallel lines intersecting the y-axis at $y = (n + \frac{1}{2})L$ for all integers n carrying constant linear charge densities $(-1)^n\lambda_0$. Find $\vec{E}$ at the origin.

Solution: This is very similar to the last problem. Start from the solution for the electric field, $\vec{E}'$, of a line charge λ_0 placed along the z-axis at points on the y-axis given in (zaxis 1) above, then shift $y \to y-(n+\frac{1}{2})L$ and $\lambda_0 \to (-)^n\lambda_0$ and evaluate at $y = 0$ to get the contribution at the origin of the line at $y = (n + \frac{1}{2})L$:

$$\vec{E}_n(0) = \frac{(-)^n \lambda_0 \hat{y}}{2\pi\epsilon_0(-(n + \frac{1}{2})L)}.$$

Now sum these all up to get

$$\vec{E}(0) = \sum_{n=-\infty}^{\infty} \vec{E}_n(0) = -\sum_{n=-\infty}^{\infty} \frac{(-)^n \lambda_0 \hat{y}}{\pi\epsilon_0(2n + 1)L} = -\frac{\lambda_0 \hat{y}}{\pi\epsilon_0 L} \sum_{n=-\infty}^{\infty} \frac{(-)^n}{2n + 1} = -\frac{\lambda_0 \hat{y}}{\pi\epsilon_0 L} \cdot \frac{\pi}{2} = -\frac{\lambda_0 \hat{y}}{2\epsilon_0 L},$$

where in the fourth step I looked up the infinite sum.