Problem Set 2

Problems are worth one point each.

Find the charge density (charge per unit volume) $\rho(\vec{r})$ for the following charge distributions. In each case pick a convenient coordinate system and describe it clearly. (*Hint:* most $\rho(\vec{r})$'s will involve delta functions.)

Problem 1. Eight point charges each of charge q at the vertices of a cube of side L.

Solution: Set up the cube in Cartesian coordinates with one vertex at the origin and three edges along the x-, y-, and z-axes. Then the vertices of the cube are at $\vec{r}_1 = 0$, $\vec{r}_2 = L\widehat{x}$, $\vec{r}_3 = L\widehat{y}$, $\vec{r}_4 = L\widehat{z}$, $\vec{r}_5 = L\widehat{x} + L\widehat{y}$, $\vec{r}_6 = L\widehat{y} + L\widehat{z}$, $\vec{r}_7 = L\widehat{x} + L\widehat{z}$, $\vec{r}_8 = L\widehat{x} + L\widehat{y} + L\widehat{z}$. With these definitions, the charge density is

$$\rho(\vec{r}) = q \sum_{i=1}^{8} \delta^3(\vec{r} - \vec{r}_i).$$

Problem 3. A charge q uniformly distributed over a circular arc of radius R and length $\pi R/3$.

Solution: Use cylindrical coordinates with the center of the circular arc at the origin, the arc in the x-y plane --- i.e., at $z = 0$ --- and one end of the arc on the positive x-axis --- i.e., at $\phi = 0$. Since the arc has radius R and length $R\pi/3$, it subtends an angle $\Delta\phi = \pi/3$. Then the arc is described by the equations $z = 0$, $s = R$, and $0 \le \phi \le \pi/3$. The linear charge density (charge per unit length) along the arc is $\lambda = q/(\pi R/3)$ since the arc has length $\pi R/3$. Since the arc is localized at specific values of z and s, the charge density will be proportional to delta functions with support at those values. Thus the charge density is

$$\begin{aligned}
\rho(\vec{r}) &= \begin{cases} \lambda\,\delta(z-0)\,\delta(s-R) & \texttt{for } 0 \le \phi \le \frac{\pi}{3} \\ 0 & \texttt{otherwise} \end{cases} \\
&= \frac{3q}{\pi R}\delta(z)\,\delta(s-R) \cdot \begin{cases} 1 & \texttt{for } 0 \le \phi \le \frac{\pi}{3} \\ 0 & \texttt{otherwise} \end{cases} .
\end{aligned}$$

Problem 5. A charge q uniformly distributed over the surface of a sphere of radius R.

Solution: Use spherical coordinates and center the sphere on the origin. Then the sphere is described by the equation $r = R$ (for all θ, ϕ). The area of the sphere is $4\pi R^2$, so the surface charge density (charge per unit area) is $\sigma = q/(4\pi R^2)$. Since the sphere is localized at a specific value of r, the charge density will be proportional to a delta function with support at that value. Thus the charge density is

$$\rho(\vec{r}) = \sigma\,\delta(r-R) = \frac{q}{4\pi R^2}\delta(r-R).$$

Problem 7. A charge q uniformly distributed throughout a (three-dimensional) circular cylinder of length L and radius R.

Solution: Use cylindrical coordinates with the cylinder axis along the z-axis and the bottom of the cylinder at $z=0$. Then the interior of the cylinder is given by the equations $0 \leq z \leq L$ and $s \leq R$ for all ϕ. The volume of the cylinder is $\pi R^2 L$ so the charge density in the interior is $\rho_0 = q/(\pi R^2 L)$. Thus the charge density is

$$\begin{aligned}\rho(\vec{r}) &= \begin{cases} \rho_0 & \text{for } 0 \leq z \leq L \text{ and } s \leq R \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{q}{\pi R^2 L} \cdot \begin{cases} 1 & \text{for } 0 \leq z \leq L \text{ and } s \leq R \\ 0 & \text{otherwise} \end{cases}.\end{aligned}$$

Problem 9. A charge uniformly distributed over a (planar) logarithmic spiral from the origin out to a radius R with polar slope 1 (i.e., which intersects the circle of radius R at an angle of $\pi/4$). The linear charge density (charge per unit length) is λ.

Solution: Use cylindrical coordinates with the spiral x-y plane --- i.e., at $z=0$ --- and the center of the spiral at the origin. Also, put the end of the spiral (by rotating it in the x-y plane, if necessary) at $s=R$ on the positive x-axis --- i.e., at $\phi=0$. According to Wikipedia, a logarithmic spiral in the x-y plane has equation $s = ae^{k\phi}$ where (s,ϕ) are the polar coordinates in the plane and a and k are some real constants. Since at $\phi=0$ we want $s=R$, we have $a=R$. The spiral starts at $\phi=0$ and spirals in to the origin as $\phi \to \infty$. So points on the spiral are given by

$$\begin{aligned}\vec{r} &= s(\phi)\hat{s} + z(\phi)\hat{z} = Re^{k\phi}\hat{s} \qquad \text{for} \quad -\infty < \phi \leq 0 \\ &= Re^{k\phi}(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}),\end{aligned}$$

where in the second line I've rewritten $\hat{s}$ in terms of the Cartesian unit vectors $\hat{x}$ and $\hat{y}$.
We still have to determine the k parameter in the spiral. It is determined by the requirement in the problem that the spiral intersects the $s=R$ circle at an angle $\alpha=\pi/4$. We can determine this angle in terms of k by computing the dot product of the tangent vector to the spiral with the tangent vector to the circle where they intersect (at $\phi=0$). The tangent vector to the spiral as a function of ϕ is

$$\vec{t}(\phi) \doteq \frac{d\vec{r}(\phi)}{d\phi} = Re^{k\phi}\left[(k\cos\phi - \sin\phi)\,\hat{x} + (k\sin\phi + \cos\phi)\,\hat{y}\right].$$

To compute this derivative we needed to convert from the cylindrical $\hat{s}$ unit vector to the Cartesian $\hat{x}$ and $\hat{y}$ unit vectors since the $\hat{s}$ vector depends on ϕ. So at $\phi=0$ where the spiral intersects the $s=R$ circle, the tangent vector is $\vec{t}(0) = R(k\,\hat{x}+\hat{y})$. The unit tangent to the circle at this point is $\hat{y}$, so the angle α between them is given by $\cos\alpha = \vec{t}(0)\cdot\hat{y}/(|\vec{t}(0)|\,|\hat{y}|) = R/(R\sqrt{k^2+1})$. Setting $\alpha=\pi/4$ gives $1 = 1/\sqrt{k^2+1}$, or $k=1$. Thus, the equations for the spiral are $z=0$ and $s=Re^{\phi}$ for $-\infty < \phi \leq 0$. ϕ going to negative infinity reflects the fact that the spiral winds an infinite number of times as it spirals into the origin.
Since the linear charge density is given as the constant λ, the charge density is therefore

$$\rho(\vec{r}) \propto \lambda\,\delta(z)\,\delta(s - Re^{\phi}) \qquad \text{for } -\infty < \phi \leq 0, \text{ and 0 otherwise.}$$

I put ''$\propto$'' (''proportional to'') because, unlike the other examples where the delta functions set the coordinates to constant values, in this case it imposes a non-linear relation between s and ϕ, and it is less clear what the correct overall normalization of the density must be.

I claim the correct normalization factor for the charge density is

$$\rho(\vec{r}) = \sqrt{2}\,\lambda\,\delta(z)\,\delta(s - Re^{\phi}) \qquad \text{for } -\infty < \phi \le 0\text{, and 0 otherwise.}$$

We will check this in problem 10.

The remaining problems ask you to integrate over all space the charge densities, $\rho(\vec{r})$, found in the previous problems to get the total charge

$$q_{\rm tot} \doteq \int d\tau\, \rho(\vec{r}).$$

(The total charges in all but problem 9 are given in the statement of the problems: in those cases the exercise is to set up and do the integral in your chosen coordinate system — showing your work, of course! — to check that you recover the correct answer. In fact, you may want to do each of these problems at the same time as its corresponding problem above to determine the correct normalization of $\rho(\vec{r})$.)

Problem 2. Compute $q_{\rm tot}$ for problem 1.

Solution:

$$q_{\rm tot} \doteq \int d\tau\, \rho(\vec{r}) = \int d\tau q \sum_{i=1}^{8} \delta^3(\vec{r} - \vec{r}_i) = q \sum_{i=1}^{8} \int d\tau \delta^3(\vec{r} - \vec{r}_i) = q \sum_{i=1}^{8} 1 = 8q.$$

Problem 4. Compute $q_{\rm tot}$ for problem 3.

Solution:

$$q_{\rm tot} \doteq \int d\tau\, \rho(\vec{r}) = \int_0^{\infty} s ds \int_{-\infty}^{\infty} dz \int_0^{\pi/3} d\phi \frac{3q}{\pi R} \delta(z)\,\delta(s - R) = \frac{3q}{\pi R} \cdot R \cdot 1 \cdot \frac{\pi}{3} = q.$$

In the 2nd step I used the volume element in cylindrical coordinates with a limit on the ϕ integration coming from the charge distribution given in problem 3, and in the 3rd step I used the delta-function integration rule.

Problem 6. Compute $q_{\rm tot}$ for problem 5.

Solution:

$$q_{\rm tot} \doteq \int d\tau\, \rho(\vec{r}) = \int_0^{\infty} r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \frac{q}{4\pi R^2} \delta(r - R) = \frac{q}{4\pi R^2} \cdot R^2 \cdot 2 \cdot 2\pi = q.$$

Problem 8. Compute $q_{\rm tot}$ for problem 7.

Solution:

$$q_{\rm tot} \doteq \int d\tau\, \rho(\vec{r}) = \int_0^{R} s ds \int_0^{L} dz \int_0^{2\pi} d\phi \frac{q}{\pi R^2 L} = \frac{q}{\pi R^2 L} \cdot \frac{R^2}{2} \cdot L \cdot 2\pi = q.$$

Problem 10. Compute q_{tot} for problem 9.

Solution:

$$q_{\text{tot}} \doteq \int d\tau\, \rho(\vec{r}) = \int_{-\infty}^{0} d\phi \int_{-\infty}^{\infty} dz \int_{0}^{\infty} s ds \sqrt{2}\,\lambda\,\delta(z)\,\delta(s - Re^{\phi}) = \int_{-\infty}^{0} d\phi\,\sqrt{2}\,\lambda\cdot 1\cdot Re^{\phi} = \sqrt{2}\,\lambda\,R\,e^{\phi}\Big|_{-\infty}^{0} = \sqrt{2}\,\lambda\,R.$$

To check that this is the right value, note that the total charge is given by $q_{\text{tot}} = \lambda L$ where L is the length of the spiral. We compute the length of the spiral by integrating its infinitesimal arc length from $-\infty < \phi \le 0$. The infinitesimal arc length is

$$\begin{aligned} d\ell \doteq |d\vec{l}| = |\vec{t}(\phi)|\, d\phi &= Re^{\phi}\sqrt{(\cos\phi - \sin\phi)^2 + ((\cos\phi + \sin\phi)^2}\ \ d\phi \\ &= Re^{\phi}\sqrt{2\cos^2\phi + 2\sin^2\phi}\ \ d\phi = \sqrt{2}\,Re^{\phi}\,d\phi, \end{aligned}$$

where I used the result from problem 9 for the tangent vector with $k = 1$. Then the total arc length of the spiral is

$$L \doteq \int d\ell = \int_{-\infty}^{0} d\phi\,\sqrt{2}\,Re^{\phi} = \sqrt{2}\,Re^{\phi}\Big|_{-\infty}^{0} = \sqrt{2}\,R(1-0) = \sqrt{2}\,R.$$

Thus $q_{\text{tot}} = \lambda L = \sqrt{2}\,\lambda\,R$.

[Where did the factor of $\sqrt{2}$ come from, and how could we have predicted it from first principles (and not just checked it after the fact)? The factor comes from the fact that the line charge does not lie on a curve which is given by constant coordinate values in an orthonormal coordinate system (like cartesian, spherical, or cylindrical coordinates). If a line charge is on a general curve given by some parameterized equations $\vec{r} = \vec{f}(t)$ where the components of $\vec{f}$ are some given functions of t which parameterizes the points of the curve, and the line charge density is given by some function $\lambda(t)$ of the parameter, then the (volume) charge density is

$$\rho(\vec{r}) = \int dt\, \left|\frac{d\vec{f}}{dt}\right|\, \lambda(t)\,\delta^3(\vec{r} - \vec{f}(t)).$$

This follows because the 3-dimensional delta function localizes the charge density to the points of the curve, and the measure $dt\,|d\vec{f}/dt| = |d\vec{r}| = d\ell$ is the differential arc length, so $\lambda(t)d\ell$ is the differential charge.
The above formula, though it looks very different in form from what we found in problem 9, is, in fact, the same. To see this, write the curve for the logarithmic spiral as $\vec{r}(t) \doteq (s(t), \phi(t), z(t)) = (Re^t, t, 0)$ in cylindrical coordinates. Thus we are using the cylindrical azimuthal angle coordinate as the t parameter. Then $d\vec{r}/dt = \widehat{s}(ds/dt) + \widehat{\phi}s(d\phi/dt) + \widehat{z}(dz/dt) = \widehat{s}\,Re^t + \widehat{\phi}s$, so $|d\vec{r}/dt| = \sqrt{R^2e^{2t} + s^2}$. Also, λ is constant. Plugging into the above formula and using the expression for δ^3 in cylindrical coordinates gives

$$\begin{aligned} \rho(\vec{r}) &= \int_{-\infty}^{0} dt\,\sqrt{R^2e^{2t} + s^2}\,\lambda\,\frac{1}{s}\delta(s - Re^t)\delta(\phi - t)\delta(z) \\ &= \begin{cases} \sqrt{R^2e^{2t} + (Re^t)^2}\,\lambda\,\frac{1}{Re^t}\delta(s - Re^t)\delta(z) = \sqrt{2}\,\lambda\,\delta(s - Re^t)\delta(z) & \texttt{if}\ \ -\infty < \phi < 0 \\ 0 & \texttt{otherwise} \end{cases} \end{aligned}$$

giving the result of problem 9.]