# Lecture 11 Magnetic fields in matter Ch. 6

- <u>Magnetic dipole model of matter</u> (heuristic)

  - Atoms or molecules of matter can be approximated as point dipoles "$\vec{m}$".

  - With no applied $\vec{B}$ field, some* materials' atoms have no dipole moment, but most have "permanent dipole moments".

    (* mostly in atoms with even numbers of electrons)

  - Can think of dipole moments as due to electrons orbiting nuclei so something like a small current loop, or due to electron spin. Both really need quantum mechanics to understand.

- <u>Diamagnetism</u>

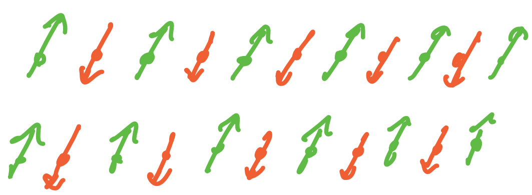
  - Material has atomic $\vec{m} = 0$ when

$\vec{B}=0$. Applied $\vec{B}\neq 0$ induces
$\vec{m} \propto -\vec{B}$ (pointing opposite to $\vec{B}$)

- Griffiths gives classical model of this effect due to change in electron orbit in presence of $\vec{B}$.

• Paramagnetism

- Material has atoms with permanent dipoles $\vec{m}\neq 0$ :



(or random)

net (average) $\vec{m}=0$ if $\vec{B}=0$

- Force on a dipole in applied $\vec{B}$:

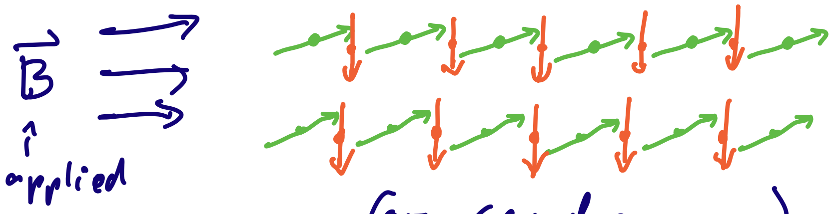
$\vec{F}=(\vec{m}\cdot\vec{\nabla})\vec{B}$ force

$\vec{N}=\vec{m}\times\vec{B}$ torque

(Recall elec. dipoles: $\vec{F}=(\vec{p}\cdot\vec{\nabla})\vec{E}$, $\vec{N}=\vec{p}\times\vec{E}$)

(See Griffiths sec 6.1.)

$\vec{N}$ tends to rotate $\vec{m}$ to be parallel to $\vec{B}$ :




} average $\vec{m}$ now $\neq 0$ $\Rightarrow$ $\langle \vec{m} \rangle$

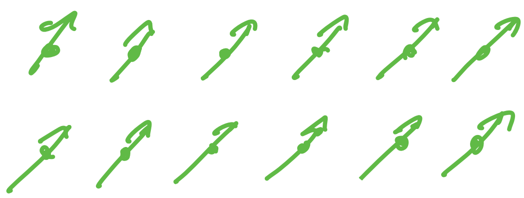
(or random ...)

So tend to develop net $\vec{m} \propto + \vec{B}$ (aligning with applied $\vec{B}$).

- Paramagnetism tends to be much stronger than diamagnetism.

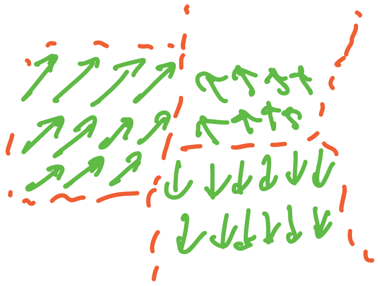
- <u>Ferromagnetism</u>

  - Material has permanent dipoles and they do not cancel on average:



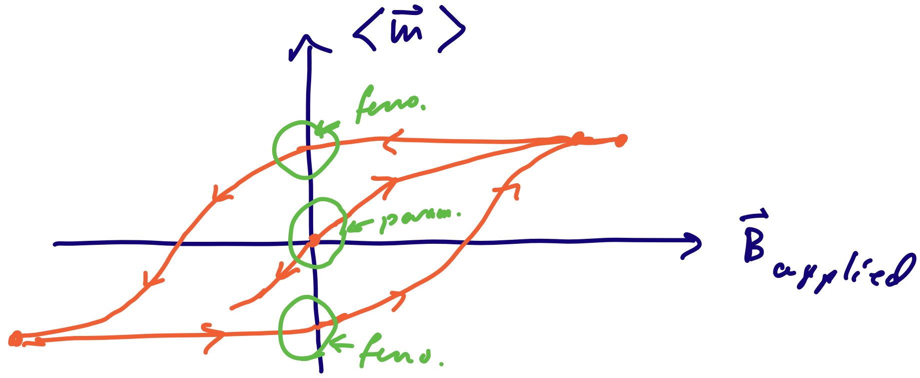
} average $\langle \vec{m} \rangle \neq 0$ even if $\vec{B} = 0$.

  - Many ferromagnetic solids look paramagnetic at large scales because ferromagnetic "domains" are randomly mis-aligned:



so average $\langle \vec{m} \rangle = 0$

Apply $\vec{B}$, has paramagnetic response as $\vec{m}$'s tend to align. For strong enough $\vec{B}$, can move/merge domains so mostly aligned. Then when turn off $\vec{B}$, remains ferromagnetic

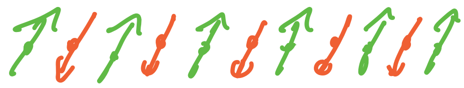



"Hysteresis" = "history dependences"

- Reason permanent magnets (ferromagnetism) occurs is that it is energetically favorable for neighboring dipoles to align:

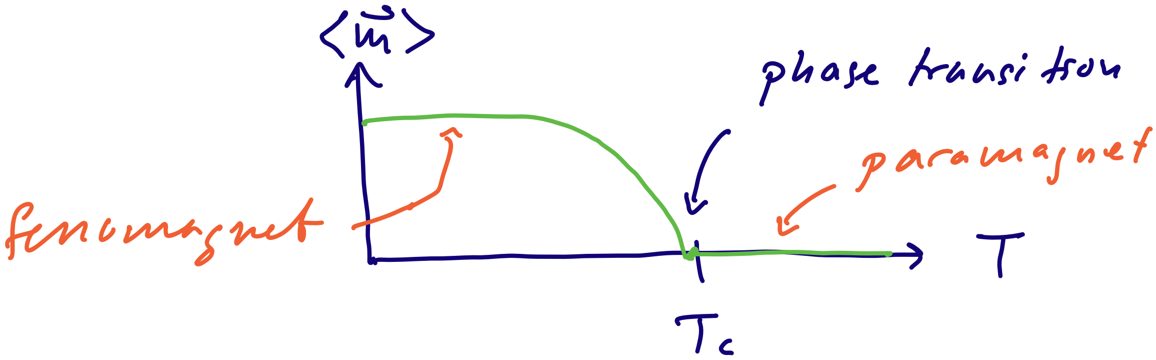
$$E(\uparrow\uparrow) < E(\uparrow\downarrow)$$

The reason is essentially quantum mechanical.

( – There are situations in which the reverse is true and dipoles tend to "anti-align": 

Then $\langle \vec{m} \rangle = 0$ if $\vec{B} = 0$ & looks paramagnetic. This is "anti-ferromagnetism." )

– At high temperatures, thermal motion tends to randomize dipoles, overcoming ferromagnetic alignment:




• <u>Linear media</u>

– Concentrate on diamagnets & paramagnets with small applied $\vec{B}$ fields.

Then expect linear response:

$$\langle \vec{m} \rangle \propto \vec{B}$$

just as with dielectrics.

- Define $\vec{M} \doteq \dfrac{\langle \vec{m} \rangle}{\text{unit volume}}$ "Magnetization"

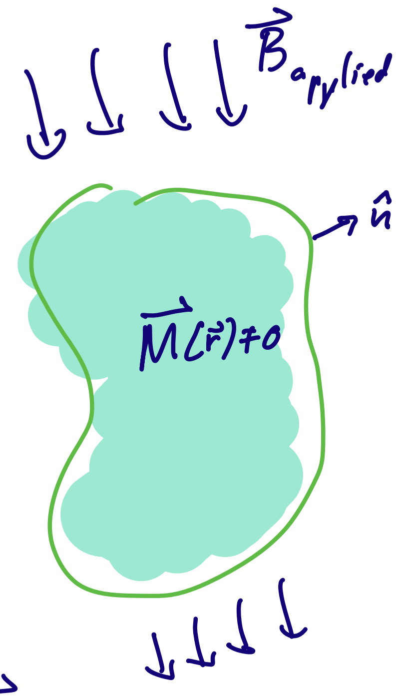
Recall $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ for dipole.

Average over many dipoles:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\tau' \, \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} .$$

$\vdots$ (just as for $V(\vec{r})$ in dielectrics)

$$\boxed{\begin{aligned} \vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \int d\tau' \, \frac{\vec{J}_b(\vec{r}\,')}{\mathscr{r}} \\ \text{w/ } \vec{J}_b &\doteq \vec{\nabla} \times \vec{M} \end{aligned}}$$




"bound current density"

($\Rightarrow \vec{\nabla} \cdot \vec{J}_b = 0$ ✓)

- At surface of material where $\vec{M}$ goes to 0 discontinuously, develop a "bound surface current"

$$\vec{K}_b = \vec{M} \times \hat{n}$$

(Parallel to dielectric polarization, $\vec{P}$
$$\rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \sigma_b = \vec{P}\cdot\hat{n}$$
)

- Say we have some magnetic material and we specify some steady "free" currents $\vec{J}_f$. Then total current is

$$\vec{J} = \vec{J}_f + \vec{J}_b \quad \leftarrow \text{"induced current"}$$

and Ampere's law $\Rightarrow$

$$\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla}\times\vec{M}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f \quad \text{w/} \quad \vec{H} \doteq \frac{1}{\mu_0}\vec{B} - \vec{M}}$$

$$\Rightarrow \oint \vec{H}\cdot d\vec{l} = I_{f\,\text{enc.}} \qquad \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

- Boundary conditions:

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}) \iff \boxed{B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}}$$

$$\boxed{\vec{H}^{\parallel}_{\text{above}} - \vec{H}^{\parallel}_{\text{below}} = \vec{K}_f \times \hat{n}} \iff \vec{B}^{\parallel}_{\text{above}} - \vec{B}^{\parallel}_{\text{below}} = \mu_0(\vec{K}\times\hat{n})$$

- Linear (isotropic, homogeneous) media:

$$\vec{M} = \chi_m \vec{H}$$

"magnetic susceptibility" (constant)

$\chi_m < 0 \Rightarrow$ Diamagnet

$\chi_m > 0 \Rightarrow$ Paramagnet

$$\Rightarrow \vec{B} = \mu \vec{H} \quad \text{w/} \quad \mu \doteq \mu_0(1+\chi_m)$$

"permeability"

- Compare to electrostatics in insulators:

| <u>insulators</u> | <u>magn. materials</u> | |
|---|---|---|
| $\vec{\nabla}\cdot\vec{D} = \rho_f$ | $\vec{\nabla}\cdot\vec{B} = 0$ | |
| $\vec{\nabla}\times\vec{E} = 0$ | $\vec{\nabla}\times\vec{H} = \vec{J}_f$ | |
| $\vec{D} \doteq \epsilon_0\vec{E} + \vec{P}$ | $\vec{H} \doteq \frac{1}{\mu_0}\vec{B} - \vec{M}$ | |
| $\Delta D^{\perp} = \sigma_f$ | $\Delta B^{\perp} = 0$ (continuous) | @ bdry |
| $\Delta \vec{E}^{\parallel} = 0$ | $\Delta \vec{H}^{\parallel} = \vec{K}_f \times \hat{n}$ | @ bdry |
| $\vec{P} = \epsilon_0\chi_e\vec{E}$ | $\vec{M} = \chi_m \vec{H}$ | linear media |
| $\vec{D} = \epsilon\vec{E}$ | $\vec{B} = \mu\vec{H}$ | linear media |
| $\epsilon = \epsilon_0(1+\chi_e)$ | $\mu \doteq \mu_0(1+\chi_m)$ | linear media |

# Conductors

- Recall in electrostatics, in conductors:
  - $\vec{E} = \rho = 0$ in interior ($V$ = constant)
  - $\vec{E}^{\parallel}_{out} = 0 \;\big|_{\text{boundary}}$
  - $E^{\perp}_{out} = \sigma \;\big|_{\text{boundary}}$

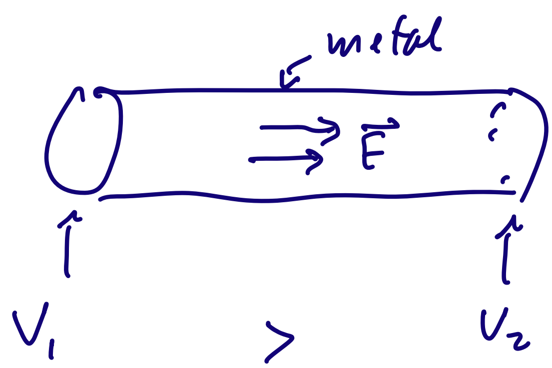
  $$\left.\begin{array}{l} \vec{E}^{\parallel}_{out} = 0 \;\big|_{\text{boundary}} \\ E^{\perp}_{out} = \sigma \;\big|_{\text{boundary}} \end{array}\right\} \Rightarrow \vec{E}_{out}\Big|_{bdry} = \frac{\sigma}{\epsilon_0}\hat{n}$$

  $\Rightarrow$ $V$ = continuous, but $\dfrac{\partial V}{\partial n} = \dfrac{\sigma}{\epsilon_0}$

  $\sigma \doteq$ induced surface charge on bdry

- But can also have steady currents in conductors, giving sources for magnetostatic fields.

- Currents are response of conduction charges (electrons) to applied $\vec{E}$ field inside conductor:



If can add/subtract conduction charges at ends of wire so as to keep $V_1 - V_2 = \text{constant}$ (e.g., use a battery) then keep steady $\vec{E} \neq 0$ inside metal! This $\vec{E}$ accelerates conduction electrons, forming current.

- In conductors there are frictional forces on the conduction electrons (e.g. scattering off impurities ...) which slows them down (dissipating their energy as heat). A given $\vec{E}$ then accelerates e's to a steady velocity $\vec{v}$ at which the frictional force counterbalances $e\vec{E}$. Then get a <u>steady</u> current $\vec{J} = \rho_e \vec{v}$.

- "Constitutive relation" $\vec{J} = \vec{J}(\vec{E})$ ?

In linear (small enough applied $\vec{E}$), homogeneous, isotropic conductor have

$$\boxed{\vec{J} = \sigma \vec{E}}$$

"Ohm's law"

$\sigma^{*} =$ constant

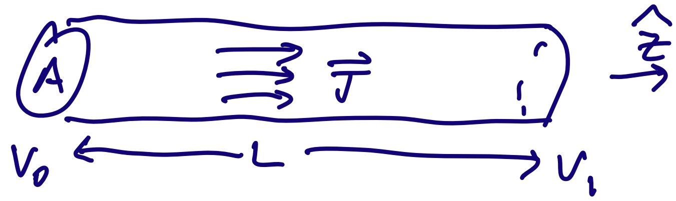
$\doteq$ "conductivity"

($1/\sigma \doteq$ "resistivity")

* NOT surface charge!

[ – If conductor moves w/ steady velocity $\vec{v}$ in $\vec{B}$-field:
then $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$. ]

– Apply to wire:




$$A J \hat{z} = A \vec{J} = A \sigma \vec{E} = A \sigma \frac{V_0 - V_1}{L} \hat{z}$$

$$\Rightarrow \quad \underbrace{A J}_{\text{current} \rightarrow I} = \underbrace{\frac{A\sigma}{L}}_{\frac{1}{R}\ (\text{resistance})^{-1}} \cdot \underbrace{(V_0 - V_1)}_{V \leftarrow \text{voltage difference}} \qquad \text{(Ohm's law)}$$

– Conductivities can vary $0 \lesssim \sigma \leq \infty$

$0$ ↑ insulator; $\infty$ ↑ superconductor ...

– Conductors can be: diamagnetic (eg. Ag), paramagnetic (eg. Al), or ferromagnetic (eg. Fe).

# Superconductors

- "$\sigma \to \infty$ limit of a conductor."
  - No friction: applied $\vec{E} \neq 0$ in conductor accelerates charges without limit: $\vec{E} \propto \frac{\partial \vec{J}}{\partial t}$ so no Ohm's law
  - What about magnetic response of superconductors. They still satisfy $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (static) and can have a steady static if $\vec{E} = 0$ inside.

- London eqns: inside superconductor:

$$\boxed{\mu_0 \frac{\partial \vec{J}}{\partial t} = \frac{1}{\lambda^2} \vec{E} \quad (1) \quad \& \quad \mu_0 \vec{\nabla} \times \vec{J} = -\frac{1}{\lambda^2} \vec{B} \quad (2)}$$

$[\lambda] = $ Length $\qquad \lambda$ : "London penetration depth".

Typically $50\,nm \lesssim \lambda \lesssim 500\,nm$ (atom $\sim 0.1\,nm$)

- L.E. (1) intuitive from Lorentz force law:

$$\vec{F} = m_e \dot{\vec{v}} = -e \vec{E} \Rightarrow \dot{\vec{J}} = -n_e e \dot{\vec{v}} = \frac{n_e e^2}{m_e} \vec{E}$$

$n_e$: $e^-$ number density

- Then $\vec{\nabla} \times$ (1) & Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow$

$$0 = \mu_0 \vec{\nabla} \times \frac{\partial \vec{J}}{\partial t} - \frac{1}{\lambda^2} \vec{\nabla} \times \vec{E} = \mu_0 \vec{\nabla} \times \frac{\partial \vec{J}}{\partial t} + \frac{1}{\lambda^2} \frac{\partial \vec{B}}{\partial t} \Rightarrow$$

$$0 = \frac{\partial}{\partial t}\left(\mu_0 \vec{\nabla}\times\vec{J} + \frac{1}{\lambda^2}\vec{B}\right)$$

- Then L.E. (2) consistent.
- Only honest derivation is from quantum mechanics...

## – London penetration & Meissner effect

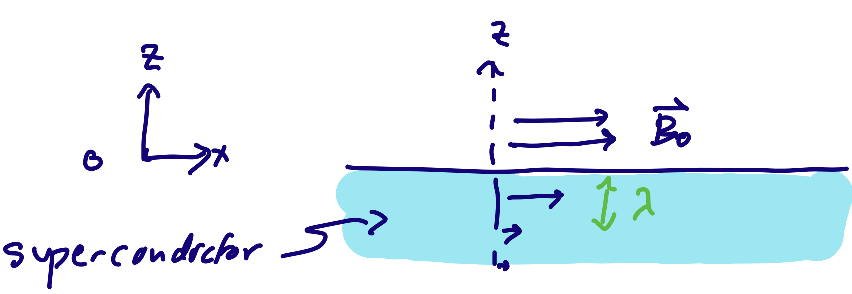
- Ampere $\Rightarrow$ $\mu_0\vec{J} = \vec{\nabla}\times\vec{B}$, plug in L.E. (2) $\Rightarrow$

$$0 = \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) + \frac{1}{\lambda^2}\vec{B}$$

$$= \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{B}}) - \nabla^2\vec{B} + \frac{1}{\lambda^2}\vec{B} \Rightarrow$$

$$\boxed{\nabla^2\vec{B} = +\frac{1}{\lambda^2}\vec{B}} \qquad (*)$$

- Solution to (*) is $\vec{B}(z) = \vec{B}_0\, e^{z/\lambda}$:




$\vec{B} = B(z)\,\hat{x}$

& $B(z=0) = B_0$

Then $\nabla^2\vec{B} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) B(z)\,\hat{x} = \frac{d^2 B(z)}{dz^2}\,\hat{x}$

& $\frac{1}{\lambda^2}\vec{B} = \frac{B(z)}{\lambda^2}\,\hat{x}$

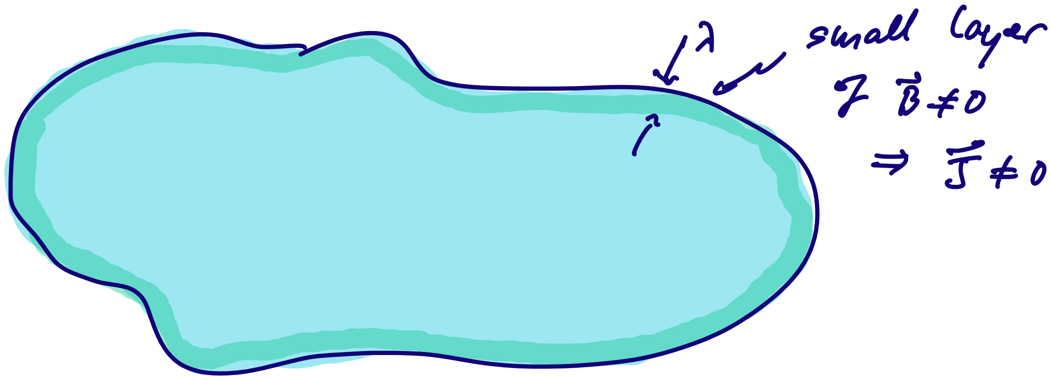
$\therefore$ (*) $\Rightarrow$ $B''(z) = +\frac{1}{\lambda^2}B(z)$

$\Rightarrow$ gen. sol'n $\quad B(z) = \alpha e^{z/\lambda} + \beta e^{-z/\lambda}$

Determine $\alpha, \beta$ by boundary conditions:

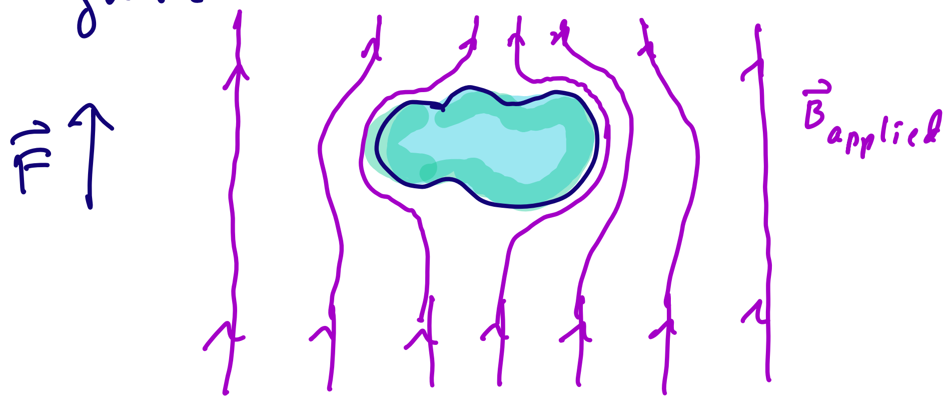
$$\left.\begin{aligned} &@\, z \to -\infty \quad B \not\to \infty, \text{ so } \beta = 0. \\ &@\, z = 0 \quad B = B_0, \text{ so } \alpha = B_0. \end{aligned}\right\} \Rightarrow B(z) = B_0 e^{z/\lambda} \quad \checkmark$$

- This implies $\vec{B}$-field $\to 0$ (exponentially fast) inside superconductor: "Meissner effect"




So current restricted to within London penetration depth of surface of superconductor.

- Superconductors expel magn fields $\Rightarrow$ "magnetic levitation"




- Steady currents w/o applied $\vec{E}$ $\Rightarrow$ electromagnets without energy loss/heating.