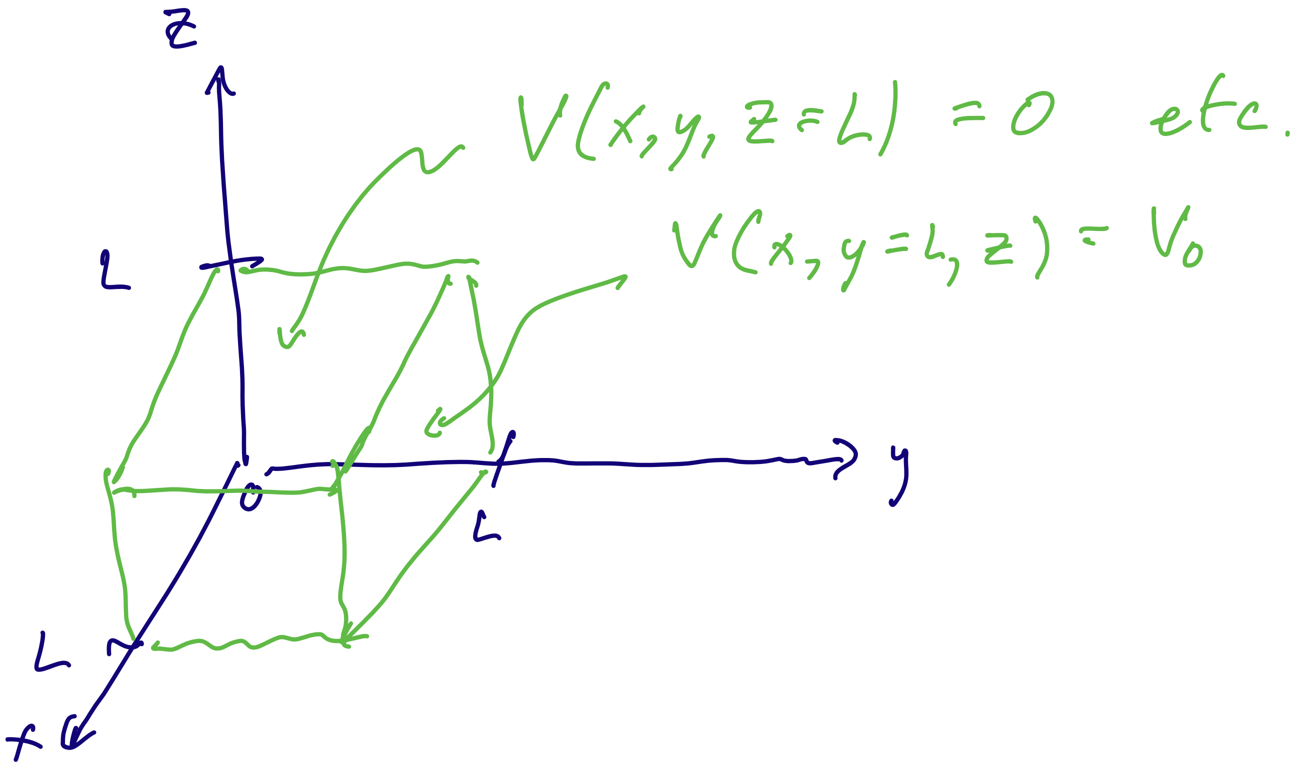
Lecture 7a Worked problem

- A cube of side L with one face at potential $V = V_0$ and the rest at potential $V = 0$. What is $V(\vec{r})$ inside the cube?

- Solution by separation of variables

(1) Pick coordinate system:

Since the boundary conditions are on orthogonal planes, use Cartesian coordinates.



(2) <u>Write Laplace eqn</u>

$$0 = \nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V(x,y,z)$$

(3) <u>Make sep'o'variables ansatz</u>

$$V(x,y,z) = A(x)\, B(y)\, C(z)$$

$$\therefore 0 = A''BC + AB''C + ABC''$$

where $'$ = derivative with respect to the argument of the function. Thus, e.g., $A' \doteq \frac{dA}{dx}$, $B' \doteq \frac{dB}{dy}$, etc.

(4) <u>Manipulate</u> so eqn is a sum of terms depending on a single variable each: $\div ABC \Rightarrow$

$$0 = \frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} + \frac{C''(z)}{C(z)}.$$

This can only be satisfied

if each term is a constant.
Therefore:

$$\frac{A''(x)}{A(x)} = -k_A^2$$

$$\frac{B''(y)}{B(y)} = -k_B^2$$

$$\frac{C''(z)}{C(z)} = -k_C^2$$

with $\boxed{k_A^2 + k_B^2 + k_C^2 = 0}$ (*)

Here $-k_i^2$ are real constants, so k_i can be <u>real</u> or <u>imaginary</u>.

(5) <u>Solve O.D.E.s</u>

$$A = a \sin(k_A x) + \tilde{a} \cos(k_A x)$$
$$B = b \sin(k_B y) + \tilde{b} \cos(k_B x)$$
$$C = c \sin(k_C z) + \tilde{c} \cos(k_C z)$$

when $a, \tilde{a}, \ldots$ are undetermined constants. For k_i real, $a, \tilde{a}$ etc are real. For k_i imaginary, a, b, c are imaginary & $\tilde{a}, \tilde{b}, \tilde{c}$ are real.

(6) Write general solution for V

$$V(x,y,z) = \sum_{k_A k_B k_C} \left(a_{k_A} \sin(k_A x) + \tilde{a}_{k_A} \cos(k_A x)\right)$$
$$\cdot \left(b_{k_B} \sin(k_B y) + \tilde{b}_{k_B} \cos(k_B y)\right)$$
$$\cdot \left(c_{k_C} \sin(k_C z) + \tilde{c}_{k_C} \cos(k_C z)\right)$$

Unknowns: set of allowed (real or imaginary) $\{k_A, k_B, k_C\}$, and values of a_{k_A} --- coefficients.

(7) Apply boundary conditions

(7a) Apply V=0 b.c.'s first. So:

$V(x=0, y, z) = 0$:

$$\Rightarrow \quad 0 = \sum_{k's} \tilde{a}_{k_A} \left(b_{k_B} \sin(k_B y) + \tilde{b}_{k_B} \cos(k_B y) \right) \cdot$$

$$\cdot \left(c_{k_C} \sin(k_C z) + \tilde{c}_{k_C} \cos(k_C z) \right)$$

for all (y, z) $\Rightarrow$ $\boxed{\tilde{a}_{k_A} = 0 \quad \forall\, k_A}$

$V(x = L, y, z) = 0$:

$$\Rightarrow \quad 0 = \sum_{k's} a_{k_A} \sin(k_A L) \cdot$$

$$\cdot \left(b_{k_B} \sin(k_B y) + \tilde{b}_{k_B} \cos(k_B y) \right) \cdot$$

$$\cdot \left(c_{k_C} \sin(k_C z) + \tilde{c}_{k_C} \cos(k_C z) \right)$$

for all (y, z) $\Rightarrow$ $k_A L = n\pi \quad n \in \mathbb{Z}$

$$\Rightarrow \boxed{k_A = \frac{n\pi}{L} \qquad n \in \{1, 2, 3 \ldots\}}$$

- For $n = 0$ term vanishes, so can be dropped from sum.
- For $n < 0$, give same (up to sign) as $n > 0$, so can also be dropped as redundant.

- Unknown a_{kA} coefficients can be reabsorbed into the $b_{kA}\ldots$ coefficients.

(7b) <u>z b.c.'s</u>: Just like <u>x b.c.'s</u> $\Rightarrow$

$z=0$ b.c. $\Rightarrow$ $\boxed{\tilde{C}_{kL}=0 \quad \forall\, kL}$

$z=L$ b.c. $\Rightarrow$ $\boxed{k_C = \frac{m\pi}{2} \quad m\in\{1,2,3,\ldots\}}$

& unknown C_{kc} coefficients can be reabsorbed into b_{kB}, $\tilde{b}_{kK}$ coefficients.

(7c) "<u>Regroup</u>": With above results, our solution is

$$V(x,y,z) = \sum_{n,m=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi z}{L}\right)\cdot$$

$$\cdot\left(b_{n,m}\sin(k_B y) + \tilde{b}_{n,m}\cos(k_B y)\right)$$

But by (*) $k_B = \frac{i\pi}{L}\sqrt{n^2+m^2}$ $\leftarrow$ imaginary!

$\Rightarrow \left\{ \sin(k_B y) \propto \sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\,y\right)\right.$

$\{\cos(k_B y) \propto \cosh\left(\frac{\pi}{L}\sqrt{n^2+m^2}\, y\right)$

$\therefore$ our solution can be written as

$$V(x,y,z) = \sum_{n,m=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi z}{L}\right) \cdot$$

$$\cdot \left(B_{n,m} \sinh\left(\frac{\pi}{L}\sqrt{n^2+m^2}\, y\right) + \tilde{B}_{n,m} \cosh\left(\frac{\pi}{L}\sqrt{n^2+m^2}\, y\right)\right)$$

(7c) Impose y b.c.'s

$y=0$: $\quad 0 = V(x, y=0, z) \Rightarrow$

$$0 = \sum_{n,m} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi z}{L}\right) \tilde{B}_{nm} \qquad \forall x, z$$

$$\Rightarrow \boxed{\tilde{B}_{n,m} = 0 \qquad \forall n,m}$$

$y=L$: $\quad V_0 = V(x, y=L, z) \Rightarrow$ (*)

$$\boxed{V_0 = \sum_{n,m} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) B_{nm} \sinh\left(\pi\sqrt{n^2+m^2}\right)}$$

(8) Invert last B.C. (*) to find coeffs

- Use orthogonality of x- & z- special functions:

$$\frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right) = \delta_{n,n'}$$

& similarly for z.

- Compute $\frac{2}{L}\int_0^L dx \sin\left(\frac{n'\pi x}{L}\right)\frac{2}{L}\int_0^L dz \sin\left(\frac{m'\pi z}{L}\right)$ (A)

$\Rightarrow$ Left-hand side $= \frac{4}{L^2}\int_0^L dx\int_0^L dz \sin\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi z}{L}\right)V_0$

$$= \frac{4}{L^2}\cdot\frac{L}{n'\pi}\left(1-(-)^{n'}\right)\cdot\frac{L}{m'\pi}\left(1-(-)^{m'}\right)\cdot V_0$$

$$= \begin{cases} \frac{16V_0}{\pi^2 n'm'} & \text{if } n' \,\&\, m' \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow$ Right-hand side $= \sum_{n,m} B_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right)\delta_{nn'}\delta_{mm'}$

$$= B_{n'm'}\sinh\left(\pi\sqrt{(n')^2+(m')^2}\right).$$

$$\therefore \boxed{B_{nm} = \begin{cases} \frac{16\cdot V_0}{\pi^2 nm\sinh\left(\pi\sqrt{n^2+m^2}\right)} & \text{if } n \,\&\, m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}}$$

(9) Plug coefficients into general solution to find final answer:

$$\boxed{V(x,y,z) = \sum_{\substack{n,m=1 \\ \text{odd}}}^{\infty} \frac{16 V_0 \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \sinh\left(\sqrt{n^2+m^2}\,\frac{\pi y}{L}\right)}{\pi^2 n m \sinh\left(\pi \sqrt{n^2+m^2}\right)}}$$