LECTURE 6

3.1 Laplace's equation

Potential satisfies Poisson's equation

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

If you know $\rho(\vec{r})$, a solution is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}\,')}{\mathscr{r}} .$$

with $V(\infty) = 0$.

We want to show that this is the only solution (with $V(\infty)=0$).

Key is to consider first the equation without charges:

$$\nabla^2 V = 0 \qquad \text{Laplace's equation}$$

Then the above solution is simply $V=0$.

We want to show that this the <u>only</u> solution with $V(\infty)=0$.

- We will prove first that $V(\vec{r})$ is the average of its values over a (any) sphere of radius R centered on $\vec{r}$:

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{S_R(\vec{r})} V\, da .$$

$S_R(\vec{r}) \leftarrow$ 2-sphere rad R @ $\vec{r}$

<u>Proof</u>: Choose $\vec{r}$ to be origin of spherical coordinate system (r', θ', φ'). Then

$$0 = \int_{B_R} d\tau' \, \nabla^2 V \qquad \text{(Laplace eqn)}$$

$B_R \leftarrow$ spherical ball radius R @ 0

$$= \oint_{S_R} d\vec{a}' \cdot \vec{\nabla}' V \qquad \text{(Divergence thm)}$$

$S_R \leftarrow \partial B_R$

$$= \oint_{S_R} da' \, \hat{r}' \cdot \vec{\nabla}' V$$

$$= \iint R^2 \sin\theta' \, d\theta' d\varphi' \, \frac{\partial V}{\partial r'}(\vec{r}') \Big|_{r'=R}$$

$$= R^2 \frac{\partial}{\partial R}\left[\iint \sin\theta' d\theta' d\varphi' \, V(R\hat{r}')\right]$$

$$= 4\pi R^2 \frac{\partial}{\partial R}\left[\frac{R^2 \iint \sin\theta' d\theta' d\varphi' \, V(R\hat{r}')}{4\pi R^2}\right]$$

$$= 4\pi R^2 \frac{\partial}{\partial R}\left[\frac{\oint_{S_R} da' \, V}{4\pi R^2}\right]$$

$$\therefore \quad \frac{1}{4\pi R^2} \oint_{S_R} da' \, V = \text{independent of } R.$$

In the limit $R \to 0$, by Taylor expansion

$$V(R\hat{r}') = V(0) + \mathcal{O}(R)$$

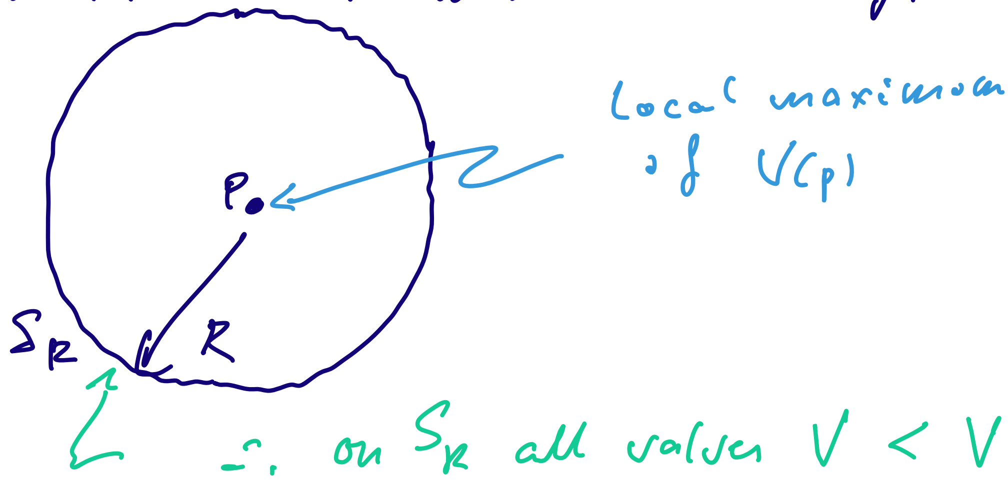
so

$$\lim_{R\to 0} \frac{\oint_{S_R} da' \, V}{4\pi R^2} = \lim_{R \to 0} \frac{4\pi R^2 (V(0) + \mathcal{O}(R))}{4\pi R^2} = V(0).$$

$$\boxed{\therefore \quad \frac{1}{4\pi R^2} \oint_{S_R} da' \, V = V(0) \qquad \text{q.e.d.}}$$

(Different from Griffith's argument: his is incorrect! Can you spot his mistake?)

- This means that $V(\vec{r})$ can have no local maxima or minima: e.g.



$\therefore$ on S_R all values $V < V(p)$

$$\Rightarrow \frac{1}{4\pi R^2} \oint_{S_R} da\, V < V(p) \qquad \#.$$

- Therefore the extreme values of $V(\vec{r})$ must occur on the boundaries.

- <u>First uniqueness theorem</u>: The solution to Laplace's eqn in some volume $\mathcal{R}$ is uniquely determined if V is specified on the boundary surface $S \doteq \partial \mathcal{R}$.

<u>Proof</u>: Assume there are 2 solutions V_1 & V_2 with the same boundary values.

Then $\nabla^2 V_1 = \nabla^2 V_2 = 0$, so

$$\nabla^2 (V_1 - V_2) = 0$$

So $V_1 - V_2$ satisfies Laplace's equ, so can have no local maxima or minima except on $\mathcal{S}$. But $(V_1 - V_2)|_{\mathcal{S}} = 0$, so we must have

$$V_1 = V_2. \qquad \text{q.e.d.}$$

In particular, is $\mathcal{R}$ = all space, $\mathcal{S}$ = "sphere at infinity", so if $V(\infty) = 0$, then $V = 0$ everywhere.

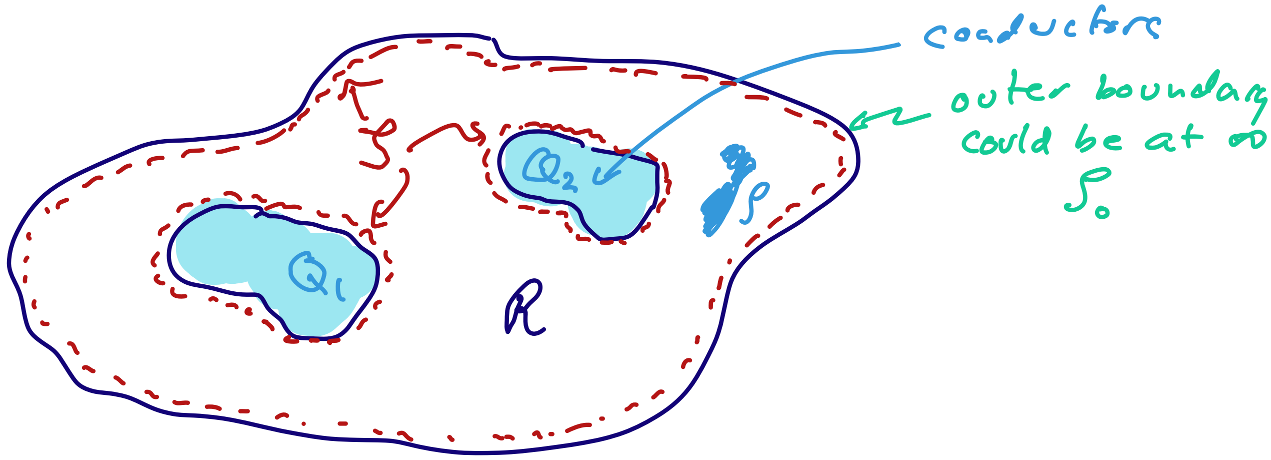
- Now consider putting in charges $\rho(\vec{r})$, so

$$\nabla^2 V(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r}).$$

Fix $\rho(\vec{r})$ and boundary values $V|_{\mathcal{S}}$. Then V is unique.

<u>Proof</u>: Assume V_1 & V_2 are 2 solutions. Then $\nabla^2(V_1 - V_2) = 0$, so use previous result to find $V_1 = V_2$. q.e.d.

- <u>2^{nd} uniqueness theorem</u> In a region $\mathcal{R}$ surrounded by conductors with specified ~~total~~ <u>charges</u>, Q_i, on each conductor, and with a specified additional fixed charge density $\rho(\vec{r})$, the electric field is uniquely determined.



<u>Proof</u>: Suppose there a 2 solutions $\vec{E}_1$, $\vec{E}_2$ in $\mathcal{R}$:

$$\vec{\nabla}\cdot\vec{E}_1 = \frac{1}{\epsilon_0}\rho \quad \& \quad \vec{\nabla}\cdot\vec{E}_2 = \frac{1}{\epsilon_0}\rho.$$

Gauss $\Rightarrow$

$$\oint_{\mathcal{S}_i} \vec{E}_1\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_i \qquad \oint_{\mathcal{S}_i} \vec{E}_2\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_i$$

$$\oint_{\mathcal{S}_0} \vec{E}_1\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{tot} \qquad \oint_{\mathcal{S}_0} \vec{E}_2\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{tot}$$

Look at $\quad \vec{E} \doteq \vec{E}_1 - \vec{E}_2$. Then

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \& \quad \oint_{\mathcal{S}_i} \vec{E}\cdot d\vec{a} = 0 \qquad \forall\, \mathcal{S}_i$$

On $\mathcal{S}_i$ V_1 & V_2 are constants, $\therefore$

$$V\Big|_{\mathcal{S}_i} = (V_1 - V_2)\Big|_{\mathcal{S}_i} = V^{(i)} \text{ constant.} \Rightarrow$$

$$0 = \sum_i V^{(i)} \oint_{\mathcal{S}_i} \vec{E}\cdot d\vec{a} = \oint_{\mathcal{S}} V\vec{E}\cdot d\vec{a} = \int_{\mathcal{R}} \vec{\nabla}\cdot(V\vec{E})\, d\tau$$

$$= \int_{\mathcal{R}} \Big(\underbrace{\vec{\nabla}V}_{-\vec{E}}\cdot\vec{E} + V\underbrace{\vec{\nabla}\cdot\vec{E}}_{0} \Big) d\tau = -\int_{\mathcal{R}} E^2\, d\tau$$

$$\boxed{\text{But } E^2 \geq 0 \text{ everywhere, so } \int_{\mathcal{R}} E^2 d\tau = 0 \text{ implies } \vec{E} = 0 \text{ in } \mathcal{R}. \qquad \text{q.e.d.}}$$

- <u>Earnshaw's theorem</u>: a charged particle cannot be held in stable equilibrium by electrostatic forces alone.

(<u>Proof</u>: see problem set.)

§3.2 Method of images

The above uniqueness theorems tell us that if we can just find one solution, then we are done. They do not tell us how to find a solution.

If we are given a fixed charge distribution, $\rho(\vec{r})$, then the solution (with $V(\infty)=0$) is given by

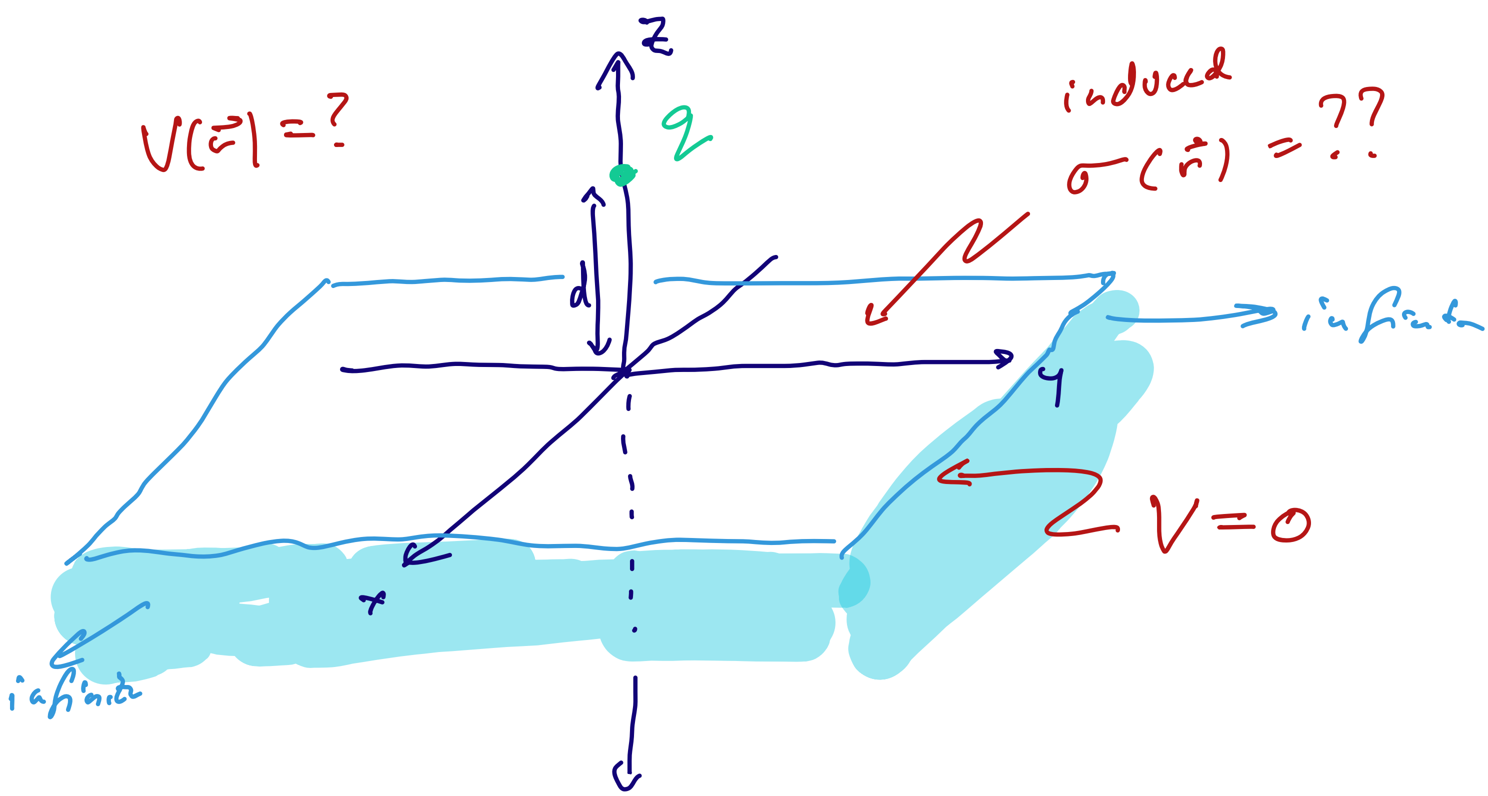
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}')}{\mathscr{r}} .$$

But in a problem with conductors, we only know the total charge Q_i on the conductors, and not the static surface charge distributions $\sigma_i(\vec{r})$ on each conductor. The 2nd uniqueness thm tells us there is a unique answer. How to find it?

In a few very special cases there is a <u>trick</u>, "the method of images", that allows us to get the solution.

2 cases where it works are for an infinite conducting plane and for a conducting sphere. I will discuss only the plane here, and will leave the sphere for the problem set.

So consider a conductor filling $z<0$:



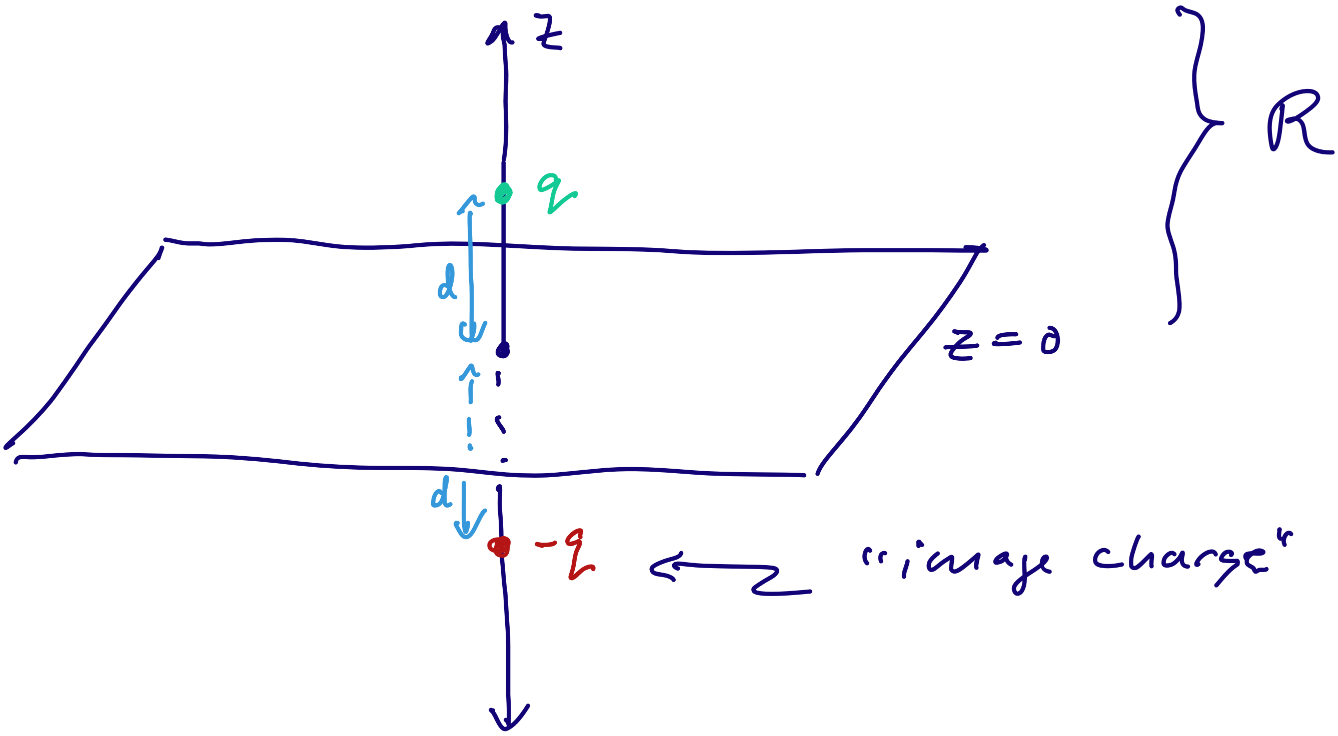
with a charge q a distance d above.

What is the induced charge $\sigma(\vec{r})$ on the surface ($z=0$) and $V(\vec{r})$ above the surface?

We know $V=$ constant on the conductor, and, since the overall additive constant in V is undetermined, <u>choose</u>: $V\big|_{z=0}=0$. Then, also, $V \to 0$ at infinity.

So we want to solve for V in the region $\mathcal{R}=\{z>0\}$ with $V=0$ on $\partial\mathcal{R}$, and with a point charge q at $\vec{r}=d\hat{z}$. The uniqueness thm implies the solution is unique, so if we can find any V satisfying these boundary conditions, then we're done.

- <u>Trick</u>: put an "image charge" $-q$ at $\vec{r}=-d\hat{z}$:



The image charge is not real: we have removed the conductor and put this fictitious charge in its place. Since these fictitious changes are not in R, we have not messed up the problem there.

From the reflection symmetry $z \to -z$ it should be clear that the potential of the image charge, $-q$, will be equal and opposite to that of q on $z=0$:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-d\hat{z}|} + \frac{-q}{|\vec{r}+d\hat{z}|}\right)}$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|x\hat{x}+y\hat{y}+(z-d)\hat{z}|} - \frac{1}{|x\hat{x}+y\hat{y}+(z+d)\hat{z}|}\right)$$

$$\therefore \; V(x,y,z=0) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+d^2}} - \frac{1}{\sqrt{x^2+y^2+d^2}}\right) = 0.$$

So we have found our solution!

- Induced surface charge:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$$

$$= \cdots = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}.$$

Total induced charge:

$$Q = \int da \cdot \sigma = -\frac{qd}{2\pi}\int \frac{dx\,dy}{(x^2+y^2+d^2)^{3/2}} = \cdots = -q. \quad \checkmark$$

- Electric field:

$$\vec{E} = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{\vec{r}-d\hat{z}}{|\vec{r}-d\hat{z}|^3} - \dfrac{\vec{r}+d\cdot\hat{z}}{|\vec{r}+d\cdot\hat{z}|^3}\right) & z>0 \\ 0 & z<0 \end{cases}$$

Note: $\vec{E} \neq \vec{E}_{\substack{\text{image}\\\text{problem}}}$ for $z<0$!

- Force on q:

$$\vec{F} = q\,\vec{E}(d\hat{z}) = q\left(\frac{-q}{4\pi\epsilon_0}\cdot\frac{\hat{z}}{(2d)^2}\right) = \frac{-q^2\,\hat{z}}{16\pi\epsilon_0 d^2}.$$

- Energy stored:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int_{z>0} E^2 d\tau$$

But, by symmetry q & $-q$ gives $|\vec{E}(x,y,-z)| = |\vec{E}(x,y,z)|$, so

$$W = \frac{\epsilon_0}{2}\left(\frac{1}{2}\int_{\forall z} E^2_{\text{imag.}}\, d\tau\right) = \frac{1}{2} W_{\substack{\text{image}\\\text{problem}}}$$

$$= \frac{1}{2}\left(\frac{-1}{4\pi\epsilon_0}\,\frac{q^2}{2d}\right) = \frac{-q^2}{16\pi\epsilon_0 d}.$$