LECTURE 5

2.3 Electric potential

- Last lecture:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \rho(\vec{r}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad (0)$$

$$\Leftrightarrow \begin{cases} \vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{\epsilon_0}\rho(\vec{r}) & (1) \\ \vec{\nabla}\times\vec{E}(\vec{r}) = 0 & (2) \end{cases}$$

(2) $\Leftrightarrow$ exists $V(\vec{r})$ "electric pot'l" such that

$$\boxed{\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})} \qquad (3)$$

Integrate (3) $\int_C d\vec{l} \Rightarrow$

$$\boxed{V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}} \qquad (3')$$

(independent of path from $\vec{r}_0$ to $\vec{r}$).

- Can add a constant to V: if

$$V'(\vec{r}) = V(\vec{r}) + c$$

then $\vec{E} = -\vec{\nabla}V' = -\vec{\nabla}V$. This constant is arbitrary & unobservable.

- E.g. can choose it so that
$$V(\vec{r}_0) = 0$$
at any point $\vec{r}_0$.

- If charge density $\rho(\vec{r})$ does not go off to spatial infinity, then can choose

$$\boxed{V(\infty) = 0.}$$

common convention!

- Units $[V(\vec{r})] = V$ "Volts"

$$\frac{N}{C} = [\vec{E}] = [\vec{\nabla} V] = \frac{V}{m}$$

$$\Rightarrow V \doteq \frac{m \cdot N}{C} = \frac{J}{C}$$ "Joule/Coulomb"

- Rewrite ① in terms of V:

$$\frac{1}{\epsilon_0}\rho(\vec{r}) = \vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot(\vec{\nabla} V) = -\nabla^2 V \Rightarrow$$

$$\boxed{\nabla^2 V(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})} \qquad (1)'$$

"Poisson's eqn" ("Laplace eqn" if r.h.s. $=0$)

∇^2 = "laplacian"

This is a key equation of physics.

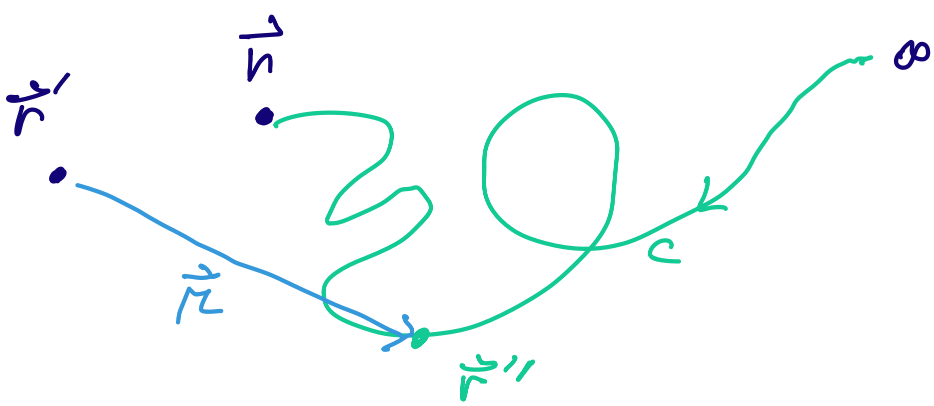
- Rewrite (0) in terms of V:

$$V(\vec{r}) = -\int_{\infty,\,C}^{\vec{r}} d\vec{l}'' \cdot \vec{E}(\vec{r}'') \qquad \text{by } (3)' \text{ w/ } \vec{r}_0 = \infty$$

$$= -\frac{1}{4\pi\epsilon_0}\int_{\infty,\,C}^{\vec{r}} d\vec{l}'' \cdot \left(\int d\tau' \rho(\vec{r}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) \qquad \text{by } (0)$$

where $\vec{\mathscr{r}} \doteq \vec{r}'' - \vec{r}'$

$$= -\frac{1}{4\pi\epsilon_0}\int d\tau' \rho(\vec{r}')\int_{\infty,\,C}^{\vec{r}} \frac{d\vec{l}''\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}$$

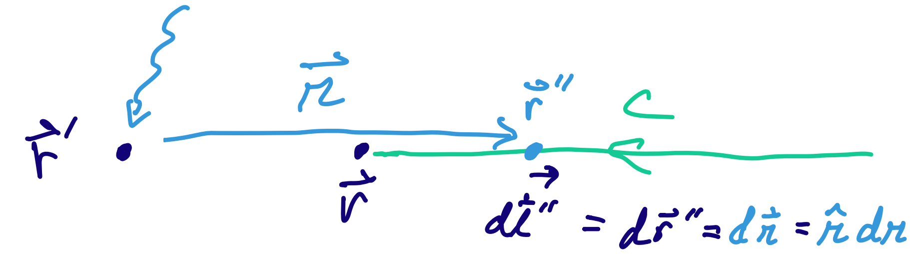


To do integral over path C, can choose any path & any coordinate system :

$\vec{\mathscr{r}} = \vec{r}\,'' - \vec{r}\,'$

$\mathscr{r} = |\vec{r}\,'' - \vec{r}\,'|$

Temporary origin of spherical coordinates (for given $\vec{r}\,'$)



$$\int_{\infty}^{\vec{r}} \frac{d\vec{l}\,''\cdot\hat{\mathscr{r}}}{\mathscr{r}^2} = \int_{\mathscr{r}=\infty}^{\mathscr{r}=|\vec{r}-\vec{r}\,'|} \frac{d\mathscr{r}\;\hat{\mathscr{r}}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2} = \int_{\infty}^{|\vec{r}-\vec{r}\,'|} \frac{d\mathscr{r}}{\mathscr{r}^2}$$

$$= \left(-\frac{1}{\mathscr{r}}\right)\Bigg|_{\infty}^{|\vec{r}-\vec{r}\,'|} = \frac{-1}{|\vec{r}-\vec{r}\,'|}.$$

$$\therefore \quad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{d\tau'\,\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau'\,\frac{\rho(\vec{r}\,')}{\mathscr{r}}} \qquad \text{☺} \qquad (\vec{\mathscr{r}} = \vec{r} - \vec{r}\,')$$

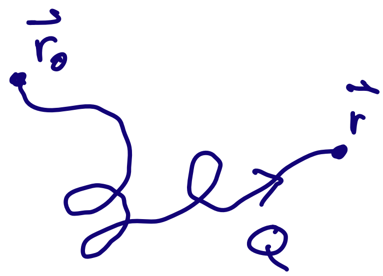
- Note: since we chose $V(\infty) = 0$ in deriving it, this formula does not apply if $\rho(\vec{r})$ extends to infinity (it diverges).

- Summary (for compact charge distributions):

$\vec{E}$	$= -\vec{\nabla} V$
$\vec{E} = \frac{1}{4\pi\epsilon_0}\int d\tau' \, \rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$	$V = \frac{1}{4\pi\epsilon_0}\int d\tau' \, \frac{\rho(\vec{r}\,')}{\mathscr{r}}$ $(V(\infty)=0)$
$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$ $\vec{\nabla}\times\vec{E} = 0$	$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$

2.4 Work & energy in electrostatics

- Work to move a charge Q in a fixed $\vec{E}$-field



$$W = \int_{\vec{r}_0}^{\vec{r}} d\vec{l}\cdot\vec{F}$$

force exerted to move Q

$$= \int_{\vec{r}_0}^{\vec{r}} d\vec{l}\cdot(-Q\vec{E})$$

"just enough to overcome force on Q from $\vec{E}$ field

$$= Q\left(V(\vec{r}) - V(\vec{r}_0)\right)$$

independent of path.

= energy change by moving Q from $\vec{r}_0$ to $\vec{r}$.

If $\vec{r}_0 = \infty$ & set $V(\infty) \doteq 0$, have

$$W = Q\,V(\vec{r}).$$ "Potential energy of Q in given $\vec{E}$ field"

- Energy of point charge distribution:

Say have charges q_i at $\vec{r}_i$ $\quad i = 1 \cdots n$.
Can assemble them by bringing them together from infinity one at a time.

$W_1 = 0$ work to bring in q_1, since $\vec{E} = 0$.

$$W_2 = q_2\,V_1(\vec{r}_2)$$ potential due to q_1 at the point $\vec{r}_2$

$$= q_2 \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{12}} \qquad r_{12} \doteq |\vec{r}_1 - \vec{r}_2|$$

$$W_3 = q_3\left(V_1(\vec{r}_3) + V_2(\vec{r}_3)\right)$$

$$= \frac{1}{4\pi\epsilon_0}\, q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

$\vdots$ (induction)

$$W = \sum_{i=1}^{n} W_i$$

$$= \frac{1}{4\pi\epsilon_0}\left\{0 + \left(\frac{q_2 q_1}{r_{21}}\right) + \left(\frac{q_3 q_1}{r_{31}} + \frac{q_3 q_2}{r_{32}}\right) + \dots\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{q_i q_j}{r_{ij}}$$

$$\therefore \quad \boxed{W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j=1 \\ i\neq j}}^{n}\frac{q_i q_j}{r_{ij}}}$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right)}_{\text{potential at } \vec{r}_i \text{ due to all other charges } j \neq i.}$$

$$(*) \quad \boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i \, V(\vec{r}_i)}$$

potential at $\vec{r}_i$ due to all <u>other</u> charges $j \neq i$.

– If you included potential due to q_i @ $\vec{r}_i$ would get $\frac{1}{4\pi\epsilon_0}\frac{q_i}{r_{ii}} = \frac{1}{4\pi\epsilon_0}\frac{q_i}{0} = \infty$.

This is the infinite energy required to assemble a single point charge. We assume that point charges are given to us "pre-assembled" by nature, so we do not need to do this work.

- Energy of a continuous charge distribution

Take continuous limit of (*) to get

$$(*)' \quad \boxed{W = \frac{1}{2} \int d\tau\, \rho(\vec{r})\, V(\vec{r})}$$

Here $V(\vec{r})$ = potential due to all charges $\rho(\vec{r}')$ for $\vec{r}' \neq \vec{r}$, but:

$$= \frac{1}{4\pi\epsilon_0} \int_{\vec{r}' \neq \vec{r}} d\tau' \frac{\rho(\vec{r}')}{\imath}$$

$$= \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\vec{r}')}{\imath} \quad \leftarrow \text{including point } \vec{r}\text{ !}$$

They are the same since:

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\epsilon_0} \int_{B_\varepsilon(\vec{r})} d\tau' \frac{\rho(\vec{r}')}{\imath}$$

$B_\varepsilon(\vec{r}) \leftarrow$ ball centered on $\vec{r}$ of radius ε

$$= \frac{\rho(\vec{r})}{4\pi\epsilon_0} \lim_{\varepsilon \to 0} \int_0^\varepsilon d\imath\, \imath^2 \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, \frac{1}{\imath}$$

$$= \frac{\rho(\vec{r})}{\epsilon_0} \lim_{\varepsilon \to 0} \int_0^{\varepsilon} dr \cdot r$$

$$= \frac{\rho(\vec{r})}{\epsilon_0} \lim_{\varepsilon \to 0} \left(\frac{\varepsilon^2}{2}\right) = 0 .$$

So for continuous charge distributions we make no mistake by including the integral over $\vec{r}\,' = \vec{r}$.

- Rewrite $(*)'$ using $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} \Rightarrow$

$$W = \frac{\epsilon_0}{2} \int d\tau \, (\vec{\nabla} \cdot \vec{E}) \, V$$

$$= \frac{\epsilon_0}{2} \int d\tau \left(\vec{\nabla} \cdot (\vec{E} V) - \vec{E} \cdot \underbrace{\vec{\nabla} V}_{= -\vec{E}} \right)$$

$$= \frac{\epsilon_0}{2} \oint_S d\vec{a} \cdot (\vec{E} V) + \frac{\epsilon_0}{2} \int d\tau \, E^2$$

S = boundary of all space = sphere at $r \to \infty$.

But $V(\infty) = 0$ (assumed ρ compact support)

$$\therefore \quad \boxed{W = \frac{\epsilon_0}{2} \int d\tau \, E^2}$$

"Electric field carries energy density $\frac{\epsilon_0}{2} E^2$"
This interpretation becomes more convincing in electrodynamics when radiation is considered.

- Note: $\vec{E} = \vec{E}_1 + \vec{E}_2 \not\Rightarrow W = W_1 + W_2$!

$$W = \frac{\epsilon_0}{2} \int d\tau \, E^2 = \frac{\epsilon_0}{2} \int d\tau (\vec{E}_1 + \vec{E}_2)^2$$

$$= \frac{\epsilon_0}{2} \int d\tau \left(E_1^2 + 2\vec{E}_1 \cdot \vec{E}_2 + E_2^2 \right)$$

$$= W_1 + W_2 + \epsilon_0 \int d\tau \, \vec{E}_1 \cdot \vec{E}_2 .$$

2.5 Conductors

- All* <u>normal</u> <u>matter</u>** in nature is made up of positively & negatively charged particles. The positively charged particles are <u>nuclei</u> and the negatively charged particles are <u>electrons</u>.

*Except: <u>neutronium</u>, found in neutron stars, and relativistic matter made up of photons (light, or EM radiation) and neutrinos.

$$\text{Universe} = \left\{ \begin{array}{l} 5\% \text{ normal matter} \\ 27\% \text{ dark matter} \\ 68\% \text{ dark energy} \end{array} \right\}$$

- Insulators are materials (= collections of normal matter held together somehow) inside of which charges do not move in response to an arbitrarily small applied electric field.

- Conductors are materials inside of which charges are free to move. The mobile charges are called the conduction charges.

  - The main example are solid metals, in which the conduction charges are electrons, though there are other examples as well.

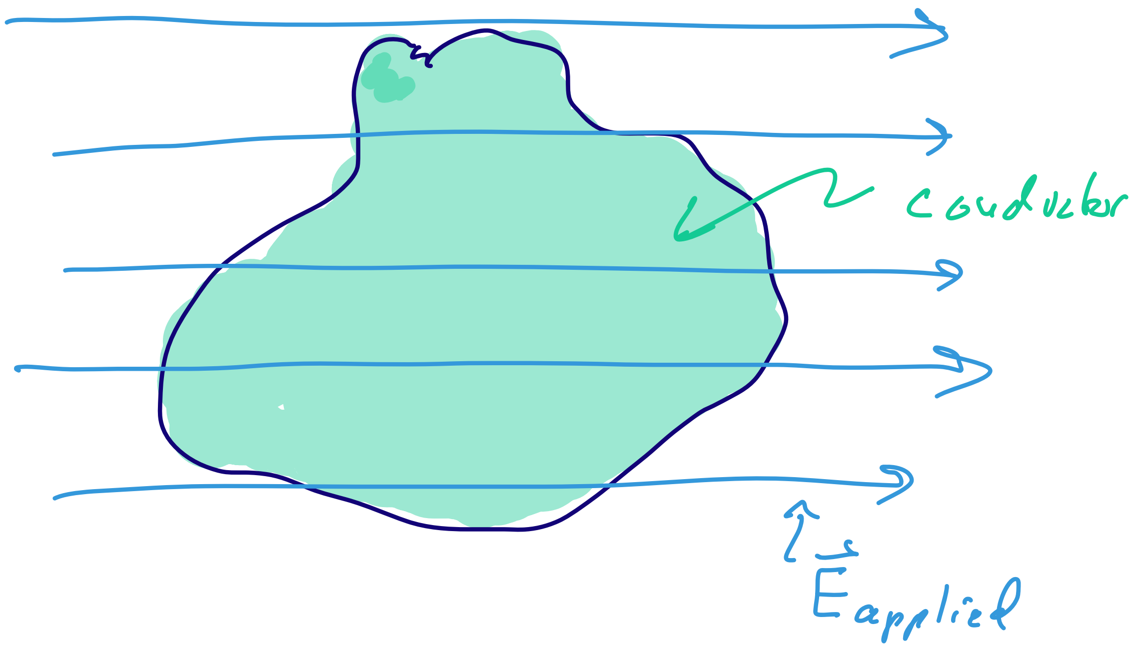
- "Free to move" means that if you apply an arbitrarily small electric field, charges will move in response in the interior of the material.
- The charges are confined to the material.
- This assumes nothing about how quickly the charges respond (measured by the <u>resistivity</u> of the material).
- In practice, if the applied $\vec{E}$ field is large enough a couple of things can happen: (1) the conductor can run out of mobile charges, in which case it becomes an insulator, and/or (2) the conduction charges can be pulled out of the material entirely.

(Likewise, if the applied $\vec{E}$ field in an insulator is large enough, some charges in its interior can be pulled free to become conduction charges.)

- In <u>static</u> (= time-independent) situations, the charges must move inside the conductor in such a way as to make the total electric field zero in the interior of the conductor:

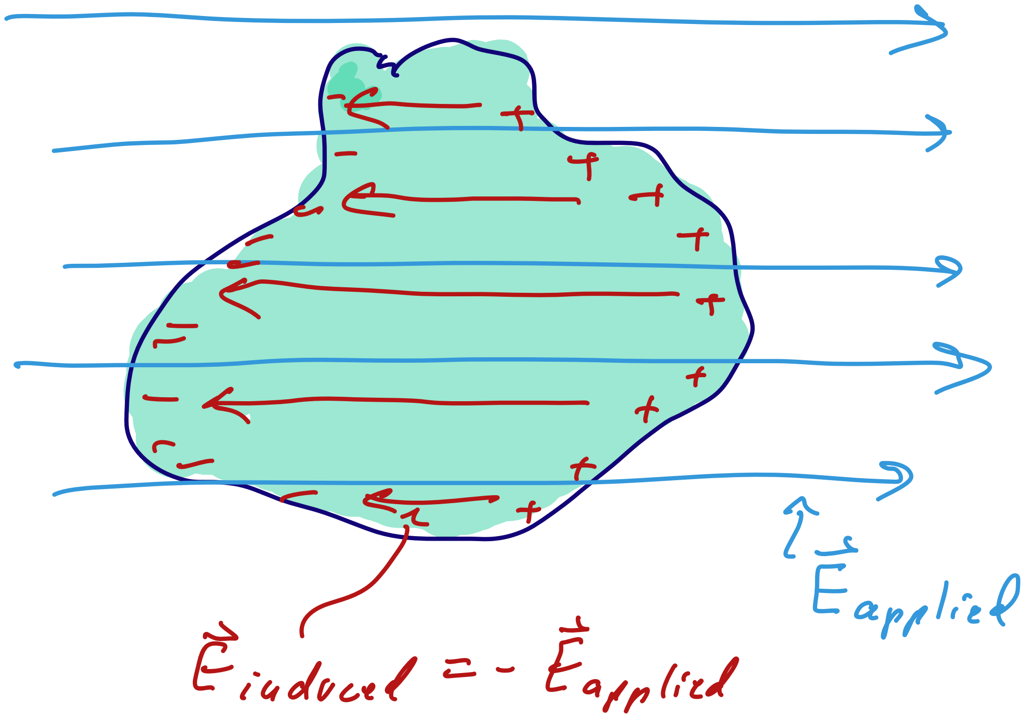
time

$t=0$



time

$t=\infty$



- If this wasn't the case, then there would be a non-zero $\vec{E}$ field inside the conductor, and charges would move, so it wouldn't be static.

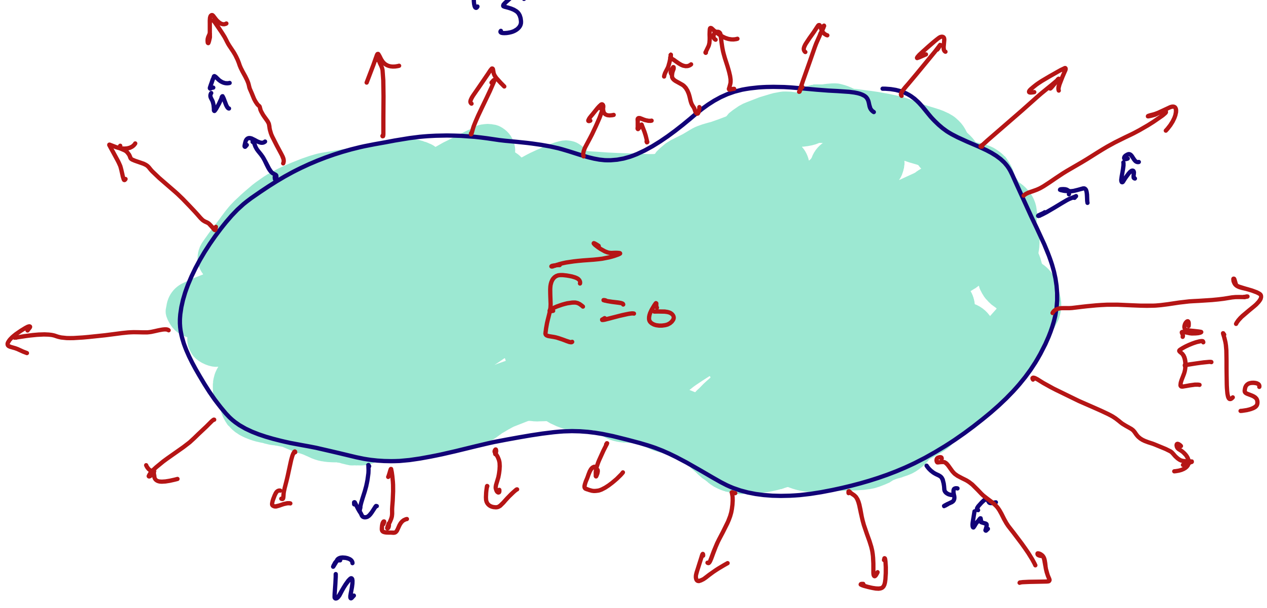
- $\Rightarrow$ $\rho = 0$ inside (static) conductor, because $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = 0$.

- $\Rightarrow$ Only net charge can be on the surface of the conductor.

- $\Rightarrow$ $V =$ constant inside conductor, because $V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} = 0$ as long as there is a path connecting $\vec{a}$ to $\vec{b}$ inside the conductor.

- $\Rightarrow$ If the conductor is in region (volume) R with surface $S \doteq \partial R$ with unit normal $\hat{n}$, then just outside S the electric field is perpendicular to S:

$\vec{E}|_S \propto \hat{n}$



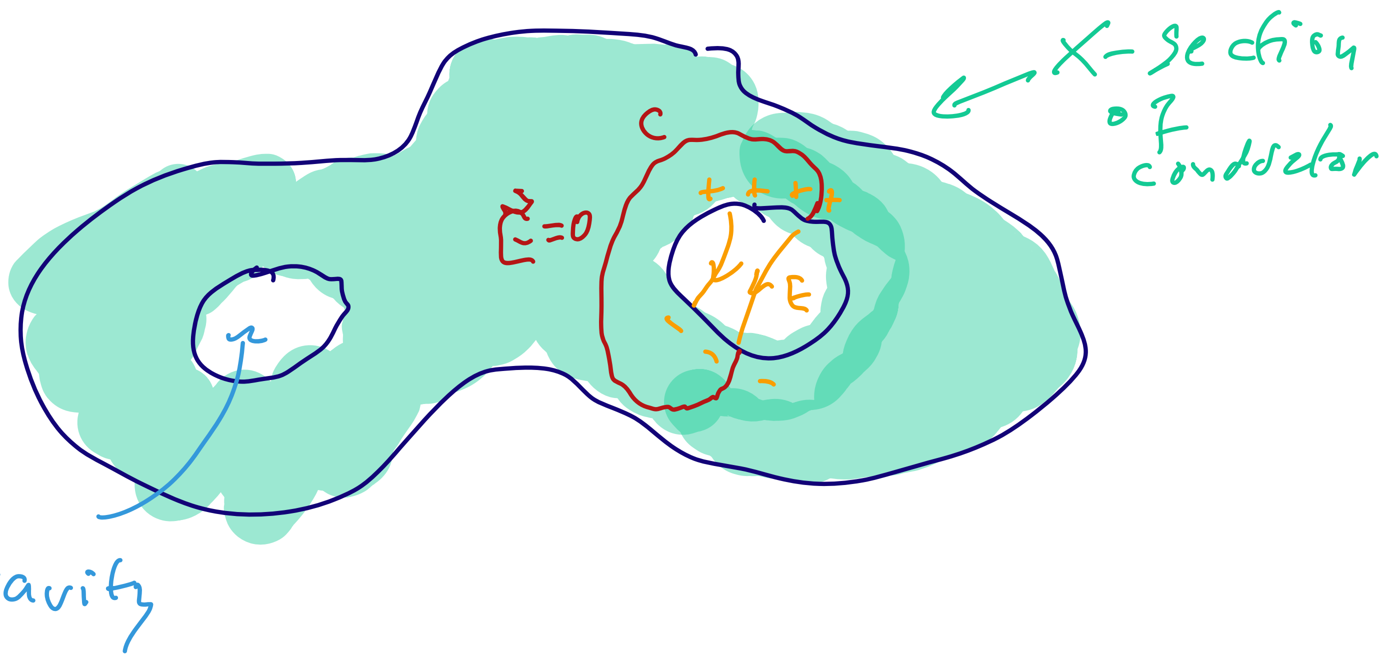
because if any component of $\vec{E}|_S$ tangent to S, then will cause conduction charges to move along S inside conductor to cancel it.

- We always assume the <u>net</u> <u>charge</u> of the conductor is zero.

- Note that since charges move to surface of conductor, there will be a separation of + & − charges. If the applied $\vec{E}$-field is non-uniform, this will mean there is a <u>net</u>

force on the conductor. We assume that something (e.g. some insulators) are holding the conductors in place.

- Cavities (empty volumes in the interior of the conductor – i.e., completely surrounded by conductor)



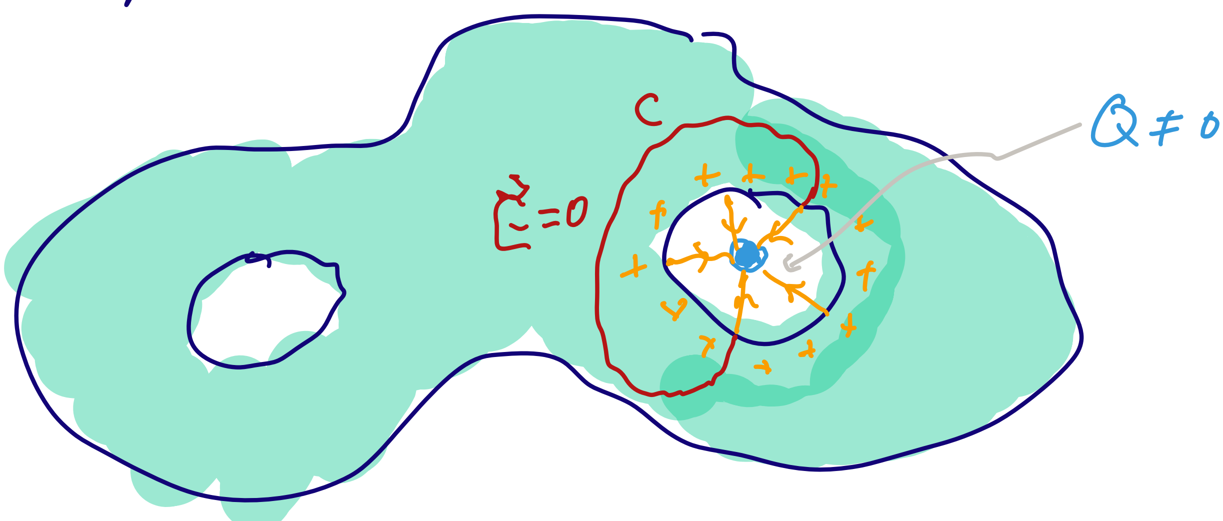
- If no charges inside cavity $\Rightarrow \vec{\nabla}\cdot\vec{E}=0$ there, so $\vec{E}$-field lines (flux along thin cylinders) must reach across cavity. But then

$$\oint_C \vec{E}\cdot d\vec{l} > 0$$

which is a contradiction.

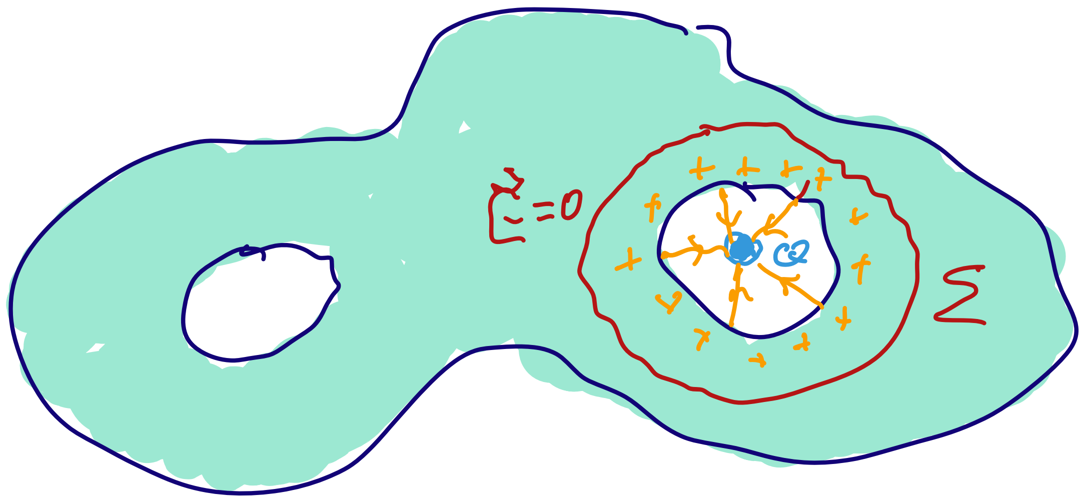
$\boxed{\begin{array}{l}\therefore \quad \vec{E} = 0 \text{ inside empty cavity} \\ \text{\& no surface charge on cavity} \\ \text{surface \& } V = \text{const. inside cavity}\end{array}}$

- If there <u>are</u> charges inside cavity $\Rightarrow \vec{\nabla}\cdot\vec{E} \neq 0$ inside & above argument fails: can get $\oint_C \vec{E}\cdot d\vec{l} = 0$.



and choose any closed surface Σ inside the conductor & surrounding the cavity. Then

$$\oint_\Sigma \vec{E}\cdot d\vec{a} = 0 = \frac{1}{\epsilon_0} Q_{enc}$$



But $Q_{enc} = Q +$ surface charge,
so

$$\boxed{\text{Total charge induced on surface of cavity = minus the total charge inside the cavity.}}$$

- Last lecture we showed that at a surface charge σ the discontinuity of the $\vec{E}$-field across the surface is

$$\vec{E}_+(\vec{r}) - \vec{E}_-(\vec{r}) = \frac{1}{\epsilon_0}\sigma(\vec{r})\,\hat{n}$$

At the surface of a conductor,

take $\hat{n}$ to point out of the conductor. Then $\vec{E}_-(\vec{r}^s) = 0$, since $\vec{E} = 0$ inside the conductor. Thus, at the surface of a conductor the field outside is

$$\boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}.} \quad *$$

Using $\vec{E} = -\vec{\nabla} V$ this implies

$$\frac{\partial V}{\partial n} = \hat{n}\cdot\vec{\nabla} V = -\frac{\sigma}{\epsilon_0}, \text{ or}$$

$$\boxed{\sigma = -\epsilon_0 \frac{\partial V}{\partial n}} \quad *$$

These will be useful formulas for determining the surface charge σ induced on conductors.

- Recalling that $\vec{F} = q\vec{E}$, we get that the pressure $\doteq$ force per area on a surface charge is $\vec{f} = \sigma\vec{E}$. But since there is a discontinuity in

$\vec{E}$ at a surface charge you have to be a bit careful. The right answer is

$$\vec{f} = \sigma \frac{1}{2} (\vec{E}_+ + \vec{E}_-)$$

i.e. the <u>average</u> of the field above & below the surface. (See Griffiths for the argument.)

For conductors, since $\vec{E}_- = 0$, we get the pressure on a conductor is

$$\vec{f} = \frac{\sigma}{2} \vec{E} = \frac{\sigma^2}{2\epsilon_0} \hat{n} = \frac{\epsilon_0}{2} E^2 \hat{n}$$

by (*). $(P \doteq |\vec{f}|$ in Griffiths.)

This is an outward-pointing electrostatic pressure on the surface of the conductor. The <u>net force</u> on a conductor in an $\vec{E}$-field (assuming this pressure does not rip the conductor apart) is

$$\vec{F} = \oint_S da\, \vec{f}$$

$$= \frac{1}{2\epsilon_0} \oint_S d\vec{a}\, \sigma^2 = \frac{\epsilon_0}{2} \oint_S d\vec{a}\, E^2$$

To find this, need to solve for σ or E everywhere at the surface S of the conductor.

- Capacitors: Two capacitors, given equal & opposite excess charges $\pm Q$

Then each conductor will be at some constant potential, call them $V_\pm$. The capacitance is defined as

$$\boxed{C \doteq \frac{Q}{V}}$$

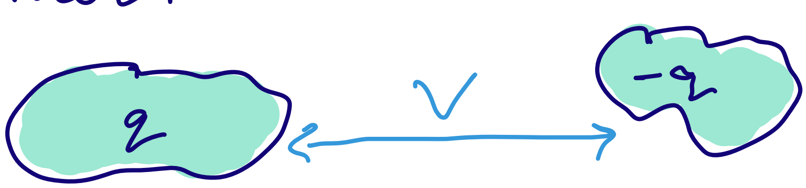
[$F \doteq \frac{C}{V}$ ← "Coulomb" ← "Volt"] units

↑ "farad"

where $V \doteq V_+ - V_-$. (Only potential differences have any meaning, anyway.)

- Capacitance is an interesting quantity because it is independent of Q — if you double Q, you double V — so it depends only on the <u>geometry</u> of the two conductors.

- The electrostatic energy stored in a capacitor holding charge Q can be computed by building up Q gradually by moving a small charge dq from one conductor to the other:



$\Updownarrow$

transfer dq

$\Downarrow$

Work done in this step is

$$dW = dq \cdot V = dq \cdot \frac{q}{C}.$$

So total work (energy) is

$$W = \int_0^Q dq \left(\frac{q}{C}\right) = \frac{1}{2}\frac{Q^2}{C}$$

using that C is constant (independent of q).