LECTURE 4

Ch 2 Electrostatics : §2.1 $\vec{E}$-field

- Force on charge Q at point $\vec{r}$ due to other (static) electric charges is

$$\vec{F} \doteq Q\,\vec{E}(\vec{r})$$

defn — electric field

units $[Q] = C$ (Coulomb)

$[\vec{E}] = \frac{N}{C}$ (Newton)

$N \doteq kg \cdot m \cdot s^{-2}$

- Coulomb's law: electric field due to a point charge q_1 at $\vec{r}_1'$ is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 \qquad \vec{\mathscr{r}}_1 \doteq \vec{r} - \vec{r}_1'$$

- Units $[\epsilon_0] = \frac{C^2}{Nm^2}$ "permittivity of free space"

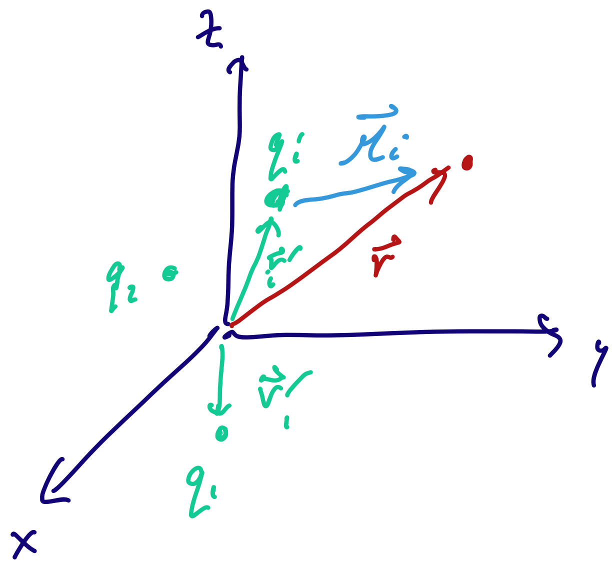
$$\epsilon_0 \cong 8.85 \times 10^{-12}\,\frac{C^2}{Nm^2}$$

(effectively defn "C" !)

- $\vec{E}$ field due to many charges is sum of $\vec{E}$ fields due to each:

number of charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}\hat{\mathscr{r}}_i \qquad \vec{\mathscr{r}}_i \doteq \vec{r} - \vec{r}_i{}'.$$

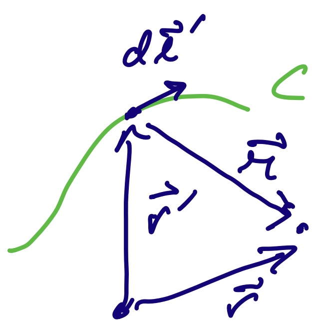


- Continuous charge distributions:

			densities	
0-dim'l:	point	charge	q	C
1-dim'l:	line	charge	λ	C/m
2-dim'l:	surface	charge	σ	C/m^2
3-dim'l:	volume	charge	ρ	C/m^3

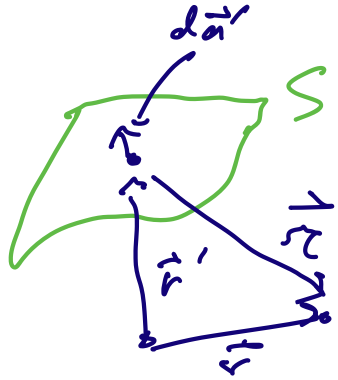
Sum over point charges becomes an integral:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_C dl'\,\lambda(\vec{r}\,')\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$



$$dl' \doteq |d\vec{l}\,'|$$

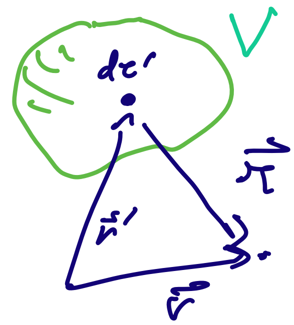
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_S da'\, \sigma(\vec{r}')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$



$$da' \doteq |d\vec{a}'|$$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V} d\tau'\, \rho(\vec{r}')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

$V \leftarrow$ optional



2.2 $\vec{\nabla}\cdot\vec{E}$ & $\vec{\nabla}\times\vec{E}$

- $$\underline{\vec{\nabla}\cdot\vec{E}(\vec{r})} = \vec{\nabla}\cdot\left(\frac{1}{4\pi\epsilon_0}\int d\tau'\, \rho(\vec{r}')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int d\tau'\, \rho(\vec{r}')\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int d\tau'\, \rho(\vec{r}')\, 4\pi\, \delta^3(\vec{\mathscr{r}})$$

$$\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{\epsilon_0}\rho(\vec{r})}$$

Gauss's law (differential form)

$$\int_V d\tau \, \vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{\epsilon_0}\int_V d\tau \, \rho(\vec{r}) = \frac{1}{\epsilon_0} Q_{enc}$$

$\Vert$ ← divergence thm

total (net) charge enclosed inside S

$$\oint_{S=\partial V} d\vec{a}\cdot\vec{E}(\vec{r})$$

"E-flux through S"

$$\boxed{\oint_S d\vec{a}\cdot\vec{E}(\vec{r}) = \frac{1}{\epsilon_0} Q_{enc}}$$

Gauss's law (integral form)

- $\underline{\vec{\nabla}\times\vec{E}}$ $= \vec{\nabla}\times\left(\frac{1}{4\pi\epsilon_0}\int d\tau' \rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$

$$= \frac{1}{4\pi\epsilon_0}\int d\tau' \rho(\vec{r}\,')\, \vec{\nabla}\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0.$$

$$\boxed{\vec{\nabla}\times\vec{E}(\vec{r}) = 0}$$

$$\Rightarrow \boxed{\oint_C d\vec{l}\cdot\vec{E}(\vec{r}) = 0 \quad \text{any closed } C}$$

$$\Rightarrow \boxed{\exists\, V(\vec{r}) \text{ s.t. } \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})}$$

$V \doteq$ "electric potential" ...

· **Summary:**

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \, \frac{\rho(\vec{r}\,')\,\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \text{(Coulomb)}}$$

is equivalent* to

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E}(\vec{r}) &= \frac{1}{\epsilon_0}\rho(\vec{r}) \\ \vec{\nabla}\times\vec{E}(\vec{r}) &= 0 \end{aligned} \qquad \left(\begin{array}{l}\text{electro-}\\\text{statics}\\\text{equations}\end{array}\right)}$$

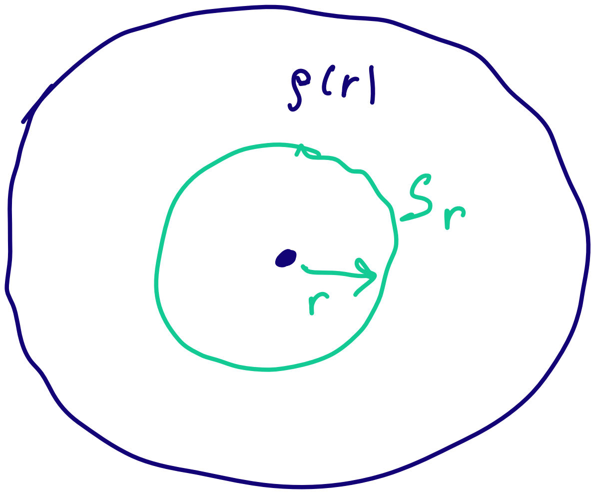
* So far, we've only shown Coulomb $\Rightarrow$ electrostatic eqns & not vice versa...

• Symmetry & Gauss's law

If problem has enough symmetry, can use integral form of Gauss's law to quickly derive $\vec{E}(\vec{r})$.

• **Example:** Spherical symmetry

$\rho(\vec{r}) = \rho(r)$, i.e. only depends on distance from origin



- Choose "Gaussian surface" S_r = sphere of radius r centered on origin.

- Spherical symmetry implies

$$\boxed{\vec{E}(\vec{r}) = E(r)\,\hat{r} \qquad \text{spherical symmetry}}$$

Why? Answer can't* depend on how rotate coord. system around origin, so indep't of θ, φ & $\hat{\theta}, \hat{\varphi}$.

* This is actually a subtle point: just because an <u>equation</u> has a symmetry, it does <u>not</u> necessarily follow that the <u>solution</u> has that symmetry! Famous simple example: what is

(the shortest possible network of roads connecting 4 towns at the corners of a square?

Challenge: find the answer.
Hint: there are 2 answers.)

– $$\oint_{S_r} d\vec{a}\cdot\vec{E}(\vec{r}) = \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi\; r^2\,\hat{r}\cdot E(r)\,\hat{r}$$

$$= 4\pi r^2 E(r)$$

– $$Q_{enc} = \int_{V_r} d\tau'\,\rho(\vec{r}\,') = \int_{r'<r} (r')^2 dr' \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi\; \rho(r')$$

$$= 4\pi \int_0^r dr'\,(r')^2 \rho(r')$$

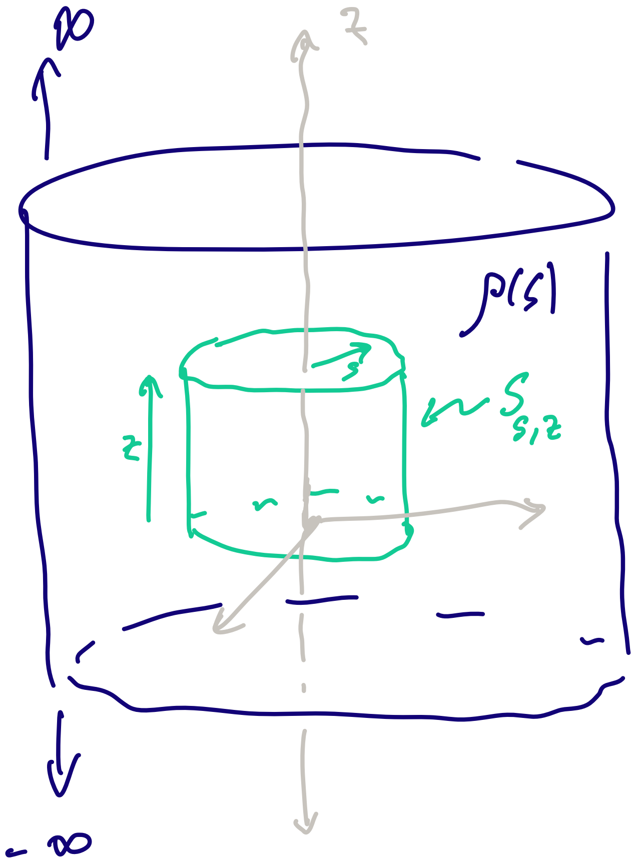
$\therefore$ Gauss's law $\Rightarrow$ $4\pi r^2 E(r) = \dfrac{4\pi}{\epsilon_0}\displaystyle\int_0^r dr'\,(r')^2\rho(r')$

$$\therefore \boxed{\vec{E}(\vec{r}) = \frac{\hat{r}}{\epsilon_0 r^2}\int_0^r dr'\,(r')^2\rho(r')\,. \quad \text{spherical symm.}}$$

- Example: Cylindrical symmetry

$$\rho(\vec{r}) = \rho(s)$$

i.e., depends only on distance from z-axis, and is translationally invariant in the z-direction.



- Choose Gaussian surface to be cylinder of height z & radius r centered on z-axis. "$S_{s,z}$"

- Cylindrical symm. implies:

$$\boxed{\vec{E}(\vec{r}) = E(s)\,\hat{s} \qquad \text{cyl. symm.}}$$

- $\displaystyle\oint_{S_{s,z}} d\vec{a}' \cdot \vec{E}(\vec{r}\,') =$

$$= \int_0^s s'ds' \int_0^{2\pi} d\phi\; \hat{z} \cdot E(s')\,\hat{s}' \qquad \text{top disk}$$

$$+ \int_0^z dz' \int_0^{2\pi} d\phi \cdot s \; \hat{s} \cdot E(s)\, \hat{s} \qquad \text{cyl. side}$$

$$+ \int_0^s s'ds' \int_0^{2\pi} d\phi \,(-\hat{z}) \cdot E(s')\, \hat{s}' \qquad \text{bottom disk}$$

$$= 0$$

$$+ \; 2\pi z s \, E(s)$$

$$+ \; 0$$

since $\hat{z} \cdot \hat{s}' = 0$

$$-Q_{enc} = \int_{V_{s,z}} d\tau' \rho(\vec{r}\,') = \int_0^z dz' \int_0^s s'ds' \int_0^{2\pi} d\phi \; \rho(s')$$

$$= 2\pi z \int_0^s ds' \; s' \rho(s')$$

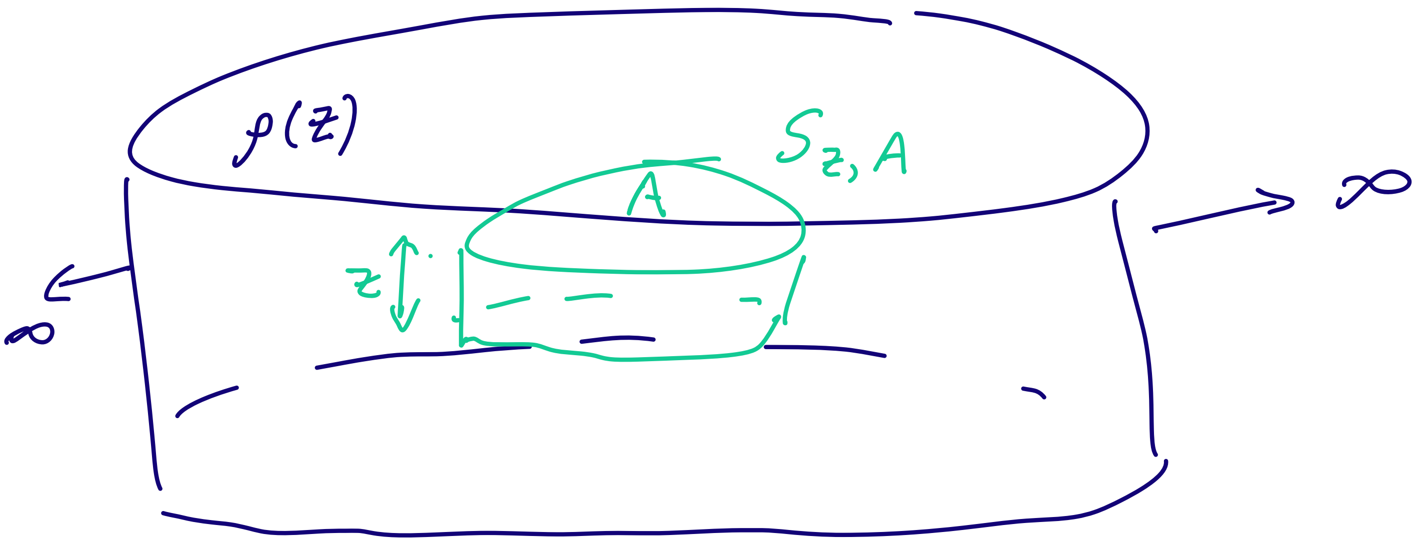
$\therefore$ G's law: $\quad 2\pi z s\, E(s) = \dfrac{1}{\epsilon_0} 2\pi z \int_0^s ds' \, s' \rho(s')$

$$\therefore \quad \boxed{\vec{E}(\vec{r}\,) = \frac{\hat{s}}{\epsilon_0 s} \int_0^s ds' \; s' \rho(s') \qquad \text{cyl. symm.}}$$

• <u>Example:</u> Planar symmetry

i.e. independent of

$\rho(\vec{r}) = \rho(z)$ translations in the x- or y- directions.



- Choose Gaussian surface to be a cylinder of arbitrary x-sectional shape "A" and height z along z-axis.

- Planar symmetry implies

$$\boxed{\vec{E}(\vec{r}) = E(z)\,\hat{z} \qquad \text{planar symmetry}}$$

- $\displaystyle\oint_{S_{z,A}} d\vec{a}' \cdot \vec{E}(\vec{r}\,') =$

$$= \int_A dx\,dy\; \hat{z} \cdot E(z)\,\hat{z} \quad \leftarrow \text{top A region}$$

$$+ \int_0^z dz' \int dl'\; \hat{n}' \cdot E(z')\,\hat{z} \quad \leftarrow \text{side cylinder region}$$

$$+ \int_A dx\,dy\,(-\hat{z}) \cdot E(0)\,\hat{z}$$ ← bottom A region

$\hat{n}'$ $\perp$ to side cylinder so point in x-y plane: $\hat{n}' = a\,\hat{x} + b\,\hat{y}$

$\therefore\ \hat{n}' \cdot \hat{z} = 0$.

Also, $\iint_A dx\,dy \doteq$ area of region $A \doteq$ "A"

Therefore,

$$\oint_{S_{z,A}} d\vec{a}' \cdot \vec{E}(\vec{r}\,') = A\,E(z) - A\,E(0).$$

$$- \quad Q_{enc} = \int_{V_{z,A}} d\tau'\,\rho(\vec{r}\,') = \iint_A dx\,dy \int_0^z dz'\,\rho(z')$$

$$= A \int_0^z dz'\,\rho(z').$$

$$\therefore \text{Gauss} \Rightarrow \quad A\big(E(z) - E(0)\big) = \frac{1}{\epsilon_0} A \int_0^z dz'\,\rho(z')$$

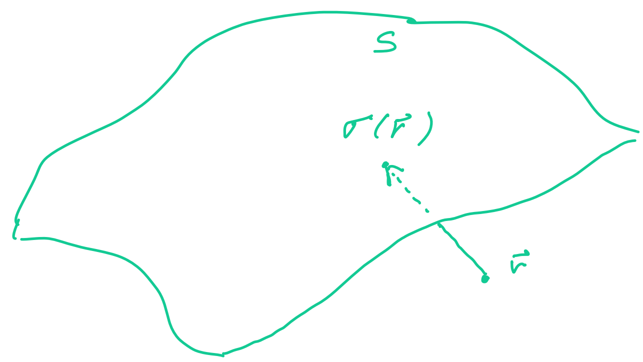
$$\therefore \quad \boxed{\vec{E}(\vec{r}) = \frac{\hat{z}}{\epsilon_0}\left(\int_0^z dz'\,\rho(z')\right) + \vec{E}(0)} \quad \text{planar symm.}$$

LECTURE 4 (Continued)

(From §2.3:) Discontinuity of $\vec{E}$ at a surface charge

- This is a useful result that follows most easily from the integrated forms of the $\vec{\nabla}\cdot\vec{E}$ & $\vec{\nabla}\times\vec{E}$ laws.

Consider a surface S with a surface charge density $\sigma(\vec{r})$:

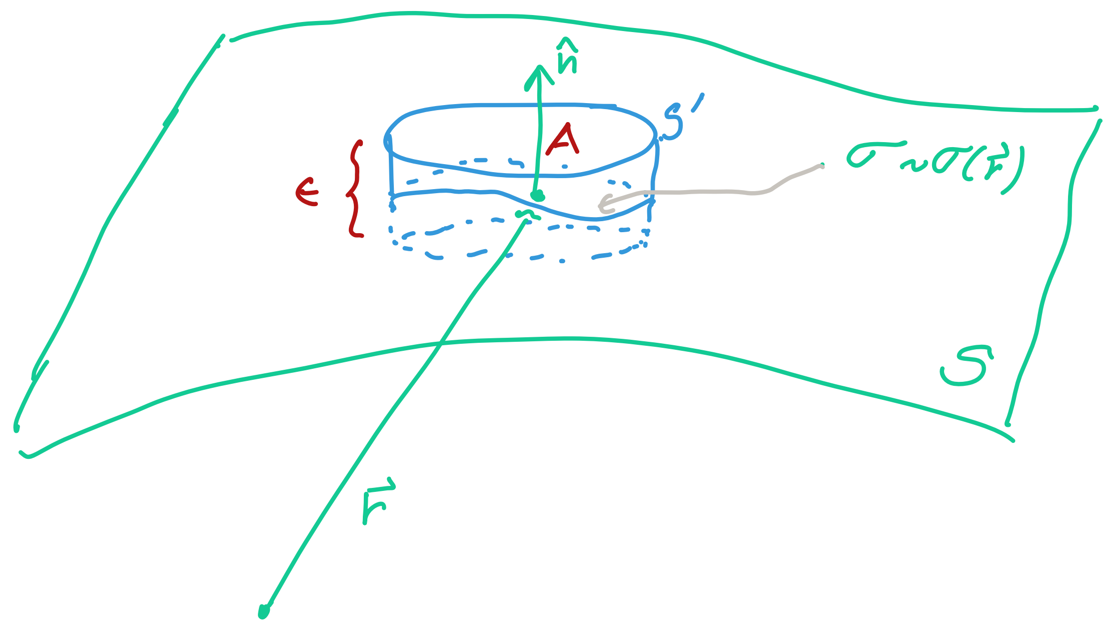


We will show that at a point $\vec{r}\in S$, the difference between the electric field just above S, $\vec{E}_+(\vec{r})$ and just below S, $\vec{E}_-(\vec{r})$, is

$$\boxed{\vec{E}_+(\vec{r}) - \vec{E}_-(\vec{r}) = \frac{1}{\epsilon_0}\sigma(\vec{r})\,\hat{n}} \qquad (*)$$

where $\hat{n}$ is the unit normal vector to S pointing to the "above" ("+") side.

- Even though there is no symmetry in this problem, we can still use Gauss's law in integrated form by choosing the Gaussian surface to be an arbitrarily pill-box centered on a point $\vec{r} \in S$:



- Gauss's law: $\oint_{S'} d\vec{a}' \cdot \vec{E}(\vec{r}\,') = \frac{1}{\epsilon_0} Q_{enc}$

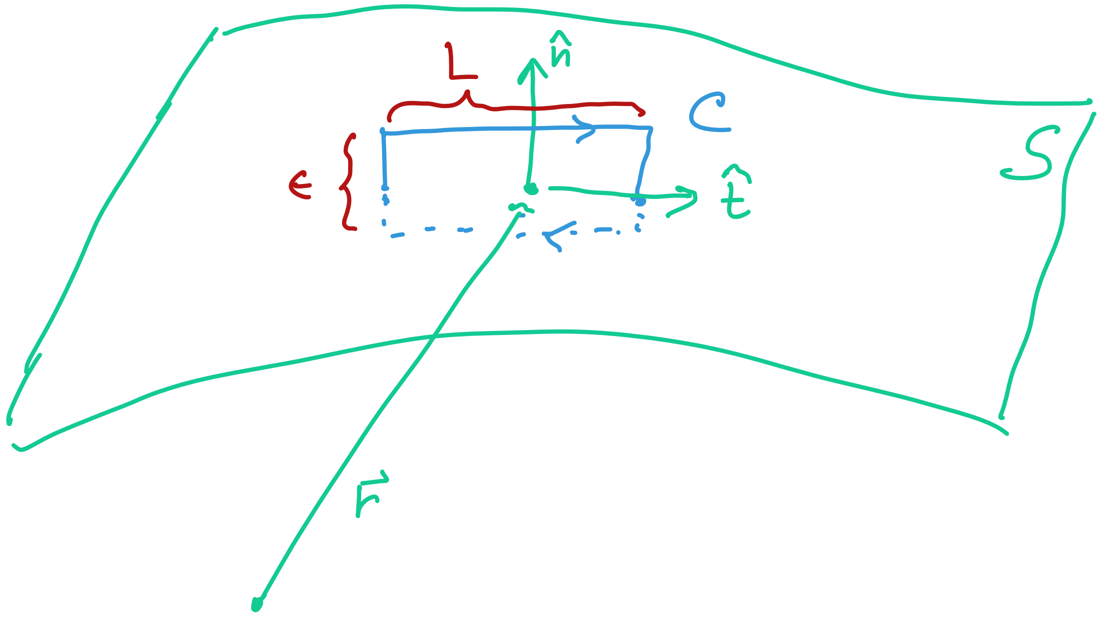
$$\begin{cases} \oint_{S'} d\vec{a}' \cdot \vec{E}(\vec{r}\,') = A\,\hat{n} \cdot \vec{E}_+(\vec{r}) - A\,\hat{n} \cdot \vec{E}_-(\vec{r}) + O(\varepsilon) \\ \frac{1}{\epsilon_0} Q_{enc} = \frac{1}{\epsilon_0} A\,\sigma(\vec{r}) + O(\varepsilon) \end{cases}$$

Take $\varepsilon \to 0$ limit $\Rightarrow$

$$\boxed{\hat{n} \cdot \left(\vec{E}_+(\vec{r}) - \vec{E}_-(\vec{r})\right) = \frac{1}{\epsilon_0} \sigma(\vec{r})} \qquad (1)$$

- Integrated form of curl eqn: $\oint_C d\vec{l}' \cdot \vec{E}(\vec{r}\,') = 0$

Choose C to be small rectangular loop along a tangential direction $\hat{t}$ to S:



Then $\oint_C d\vec{l}' \cdot \vec{E}(\vec{r}\,') = L\hat{t}\cdot\vec{E}_+(\vec{r}) - L\hat{t}\cdot\vec{E}_-(\vec{r}) + O(\varepsilon)$,

so, taking $\lim \varepsilon \to 0$, get

$$\boxed{\hat{t}\cdot\left(\vec{E}_+(\vec{r}) - \vec{E}_-(\vec{r})\right) = 0} \qquad (2)$$

for any $\hat{t}$ tangent to S.

(1) & (2) $\Rightarrow$ (*) since $\hat{n}\cdot$(*) $=$ (1), and $\hat{t}\cdot$(*) $=$ (2) (using $\hat{t}\cdot\hat{n}=0$), and since $\{\hat{n}, \hat{t}, \hat{t}'\}$ form a basis at $\vec{r}$ for any 2 linearly independent tangent vectors $\hat{t}, \hat{t}'$.