Vector Calculus Formulas

Vector Identities

$$
\begin{aligned}
\vec{A}\cdot(\vec{B}\times\vec{C}) &= \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \\
\vec{A}\times(\vec{B}\times\vec{C}) &= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} \\
(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) &= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})
\end{aligned}
$$

Differential Identities

$$
\begin{aligned}
\vec{\nabla}(fg) &= g\vec{\nabla}f + f\vec{\nabla}g \\
\vec{\nabla}\cdot(f\vec{A}) &= \vec{A}\cdot\vec{\nabla}f + f\vec{\nabla}\cdot\vec{A} \\
\vec{\nabla}\times(f\vec{A}) &= \vec{\nabla}f\times\vec{A} + f\vec{\nabla}\times\vec{A} \\
\vec{\nabla}(\vec{A}\cdot\vec{B}) &= (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) \\
\vec{\nabla}\cdot(\vec{A}\times\vec{B}) &= \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \\
\vec{\nabla}\times(\vec{A}\times\vec{B}) &= \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}\times\vec{\nabla}f &= 0 \\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) &= 0 \\
\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}
\end{aligned}
$$

Integral Identities

$$
\begin{aligned}
\int_C d\vec{\ell}\cdot\vec{\nabla}f &= f(\vec{r}_f) - f(\vec{r}_i) \qquad (\partial C = \vec{r}_f - \vec{r}_i) \\
\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) &= \oint_{\partial S} d\vec{\ell}\cdot\vec{A} \\
\int_S d\vec{a}\times\vec{\nabla}f &= \oint_{\partial S} d\vec{\ell}\, f \\
\int_V d\tau\,\vec{\nabla}\cdot\vec{A} &= \oint_{\partial V} d\vec{a}\cdot\vec{A} \\
\int_V d\tau\,\vec{\nabla}f &= \oint_{\partial V} d\vec{a}\, f \\
\int_V d\tau\,\vec{\nabla}\times\vec{A} &= \oint_{\partial V} d\vec{a}\times\vec{A}
\end{aligned}
$$

Coordinate Systems

$$(x, y, z) = (s\cos\phi, s\sin\phi, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

Cartesian $(-\infty < x, y, z < \infty)$

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad d\tau = dx\,dy\,dz \qquad \delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

$$\vec{A} = \widehat{x}A_x + \widehat{y}A_y + \widehat{z}A_z \qquad d\vec{\ell} = \widehat{x}dx + \widehat{y}dy + \widehat{z}dz \qquad \vec{r} = \widehat{x}x + \widehat{y}y + \widehat{z}z$$

$$\vec{\nabla}f = \widehat{x}\frac{\partial f}{\partial x} + \widehat{y}\frac{\partial f}{\partial y} + \widehat{z}\frac{\partial f}{\partial z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{A} = \widehat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \widehat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \widehat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical $(0 \le s < \infty\ ,\ 0 \le \phi < 2\pi\ \&\ \phi \sim \phi + 2\pi\ ,\ -\infty < z < \infty)$

$$ds^2 = ds^2 + s^2 d\phi^2 + dz^2 \qquad d\tau = s\,ds\,d\phi\,dz \qquad \delta(\vec{r}) = \frac{1}{s}\delta(s)\delta(\phi)\delta(z)$$

$$\vec{A} = \widehat{s}A_s + \widehat{\phi}A_\phi + \widehat{z}A_z \qquad d\vec{\ell} = \widehat{s}ds + \widehat{\phi}s\,d\phi + \widehat{z}dz \qquad \vec{r} = \widehat{s}s + \widehat{z}z$$

$$\vec{\nabla}f = \widehat{s}\frac{\partial f}{\partial s} + \widehat{\phi}\frac{1}{s}\frac{\partial f}{\partial \phi} + \widehat{z}\frac{\partial f}{\partial z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{A} = \widehat{s}\left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \widehat{\phi}\left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right) + \widehat{z}\frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right)$$

$$\nabla^2 f = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical $(0 \le r < \infty\ ,\ 0 \le \theta \le \pi\ ,\ 0 \le \phi < 2\pi\ \&\ \phi \sim \phi + 2\pi)$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \qquad d\tau = r^2\sin\theta\, dr\,d\theta\,d\phi \qquad \delta(\vec{r}) = \frac{1}{r^2\sin\theta}\delta(r)\delta(\theta)\delta(\phi)$$

$$\vec{A} = \widehat{r}A_r + \widehat{\theta}A_\theta + \widehat{\phi}A_\phi \qquad d\vec{\ell} = \widehat{r}dr + \widehat{\theta}r\,d\theta + \widehat{\phi}r\sin\theta\, d\phi \qquad \vec{r} = \widehat{r}r$$

$$\vec{\nabla}f = \widehat{r}\frac{\partial f}{\partial r} + \widehat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \widehat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\vec{\nabla}\times\vec{A} = \widehat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right] + \widehat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \widehat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$