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A global solution curve for a class of periodic problems, including the pendulum equation

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Abstract. Using continuation methods and bifurcation theory, we study the exact multiplicity of periodic solutions, and the global solution structure, for a class of periodically forced pendulum-like equations. Our results apply also to the first order equations. We also show that by choosing a forcing term, one can produce periodic solutions with any number of Fourier coefficients arbitrarily prescribed.

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1. Introduction

We are interested in the exact number of periodic solutions, as well as in the precise structure of the solution set, when parameters are varied, for forced pendulum-like equations

$$u''(t) + \lambda u'(t) + kg(u(t)) = \mu + e(t), \quad u(t+T) = u(t), \tag{1.1}$$

where $g(u)$ is a bounded function, whose derivative is also bounded, λ, k and μ are real parameters; $e(t)$ is a T-periodic function with $\int_0^T e(t)\, dt = 0$. The prominent example is $g(u) = \sin u$, the forced pendulum equation.

Our principal motivation comes from the paper of G. Tarantello [12], who considered the pendulum equation. Since $\sin u$ is periodic, given any periodic solution $u(t)$, $u(t) + 2n\pi$ is also a solution, so G. Tarantello has restricted to solutions whose average ξ satisfies $0 \le \xi < 2\pi$. Her result says roughly that for λ large there exist two constants $d < D$, so that the problem (1.1) has exactly two T-periodic solutions if $\mu \in (d, D)$, exactly one T-periodic solution if either $\mu = d$ or $\mu = D$, and none if $\mu \notin (d, D)$. Her method is based on the Lyapunov-Schmidt reduction. This line of research has been apparently initiated by A. Castro [1], who had established existence of T-periodic solutions, and continued by G. Fournier and J. Mawhin [6], and J. Mawhin and M. Willem [13], who proved multiplicity results. Then G. Tarantello [12] gave the exact multiplicity result, mentioned above. Perhaps one can characterize this line of research as using PDE-like methods to study periodic solutions of (1.1). More recently J. Cepicka et al [2] have extended G. Tarantello's work by using similar methods.

We apply methods of bifurcation theory to understand the exact solution structure of the problem (2.1). In the case of pendulum equation we recover the results of [12], and we illuminate these results by providing a detailed picture of the solution set. Moreover, we do not require $g(u)$ to be analytic. We describe our approach next.

When $k = 0$ and $\mu = 0$ the problem is linear, and it has a unique T-periodic solution of any average. Let us look at the solution of zero average first. We show that it continues in k, provided that $k \leq \omega\sqrt{\omega^2 + \lambda^2}$, where $\omega = \frac{2\pi}{T}$, the frequency. Namely, by applying the implicit function theorem, we show that for every k, with $k \leq \omega\sqrt{\omega^2 + \lambda^2}$, we can find $\mu = \mu(k)$ and a T-periodic solution $u = u(k)$ of (1.1) of zero average. Similarly, we produce curves of solutions of any other fixed average. Hence, for every $k \leq \omega\sqrt{\omega^2 + \lambda^2}$ we have a T-periodic solution of any average ξ. We now fix k, and show that (under another restriction on k) all these solutions are connected by a unique smooth curve of solutions, parameterized as $\mu = \mu(\xi)$. To continue the solutions in μ, we use the implicit function theorem, and a bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [4]. We recall next the Crandall-Rabinowitz bifurcation theorem [4], which is one of our main tools.

Theorem 1.1. *[4] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) =$ span $\{x_0\}$ be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

We develop similar results for the first order periodic problem

$$u'(t) + kg(u(t)) = \mu + e(t), \quad u(t+T) = u(t). \tag{1.2}$$

Our results fit in nicely with a counter example of R. Ortega [10].

It is known that the pendulum equation (i.e. the problem (1.1) with $g(u) = \sin u$) can have many geometrically different T-periodic solutions (solutions $u(t)$ and $u(t) + 2\pi n$ are considered to be geometrically equal). In case $\lambda = 0$, F. Donati [5] proved that it is possible to find a forcing term with four geometrically different T-periodic solutions, and then R. Ortega [11] showed that one can replace 4 by an arbitrary number. A. Ureña [14] showed that arbitrary many solutions can be produced in case $\lambda \neq 0$ too. We show (for any λ) that for any fixed n, one can find a forcing term ($f(t) = \mu + e(t)$), and a T-periodic solution with the first n Fourier coefficients arbitrarily prescribed. (We assume k to satisfy a certain bound, which actually gets less restrictive for larger n.) Again, we do not require $g(u)$ to be analytic, as was the case in [14].

2. Preliminary results

We consider T-periodic functions, and use $\omega = \frac{2\pi}{T}$ to denote the frequency.

Lemma 2.1. *Consider the linear problem*

$$y''(t) + \lambda y'(t) = e(t), \tag{2.1}$$

with $e(t)$ given continuous function of period T, of zero average, i.e. $\int_0^T e(s)\,ds = 0$. Then the problem (2.1) has a unique T-periodic solution of any average.

Proof. We represent $e(t)$ by its complex Fourier series $e(t) = \Sigma_{-\infty}^{\infty} e_n e^{i\omega nt}$, with $e_0 = 0$, since it is of zero average, and $\bar{e}_n = e_{-n}$, because $e(t)$ is real valued. We then compute the T-periodic solution of zero average

$$y(t) = \Sigma_{n \neq 0} c_n e^{i\omega nt}, \qquad c_n = \frac{e_n\left(-\omega^2 n^2 - i\lambda\omega n\right)}{\omega^4 n^4 + \lambda^2\omega^2 n^2}.$$

Since $\bar{c}_n = c_{-n}$, it is real valued. Adding a constant to $y(t)$, we will obtain a solution of any average. □

The following lemma is known as Wirtinger's inequality. Its proof follows easily by using the complex Fourier series, and the orthogonality of the functions $\{e^{i\omega nt}\}$ on the interval $(0, T)$.

Lemma 2.2. *Assume that $f(t)$ is a continuously differentiable function of period T, and of zero average, i.e. $\int_0^T f(s)\,ds = 0$. Then*

$$\int_0^T f'^2(t)\,dt \geq \omega^2 \int_0^T f^2(t)\,dt.$$

Next we consider a linear periodic problem in the class of functions of zero average

$$w''(t) + \lambda w'(t) + kh(t)w(t) = \mu, \;\; w(t+T) = w(t), \;\; \int_0^T w(s)\,ds = 0, \tag{2.2}$$

where $h(t)$ is a given continuous function of period T, k and μ are parameters. The following lemma was implicit in G. Tarantello [12], and explicitly stated and proved in J. Cepicka et al [2]. We include its proof for completeness.

Lemma 2.3. *Assume that $|h(t)| \leq M$, where $M > 0$ is a constant, and*

$$k < \frac{1}{M}\omega\sqrt{\omega^2 + \lambda^2}. \tag{2.3}$$

Then the only solution of (2.2) is $\mu = 0$ and $w(t) \equiv 0$.

Proof. Multiply the equation (2.2) by $w''(t)+\lambda w'(t)$ and integrate

$$\int_0^T \left(w''+\lambda w'\right)^2 \, dt = -k \int_0^T h(t) w \left(w''+\lambda w'\right) \, dt$$
$$\le kM \left(\int_0^T \left(w''+\lambda w'\right)^2 \, dt\right)^{1/2} \left(\int_0^T w^2 \, dt\right)^{1/2},$$

i.e. making use of the Wirtinger's inequality

$$\int_0^T \left(w''+\lambda w'\right)^2 \, dt \le k^2 M^2 \int_0^T w^2 \, dt \le \frac{k^2 M^2}{\omega^2} \int_0^T {w'}^2 \, dt. \tag{2.4}$$

On the other hand, again using the Wirtinger's inequality

$$\int_0^T \left(w''+\lambda w'\right)^2 \, dt = \int_0^T {w''}^2 \, dt + \lambda^2 \int_0^T {w'}^2 \, dt \ge (\omega^2+\lambda^2) \int_0^T {w'}^2 \, dt. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain a contradiction with our condition (2.3), unless $\int_0^T {w'}^2 \, dt = 0$, and the lemma follows. $\square$

Next we consider another linear problem (notice that here $w(t)$ is not assumed to be of zero average)

$$w''(t)+\lambda w'(t)+kh(t)w(t)=0, \quad w(t+T)=w(t). \tag{2.6}$$

The following lemma was proved in [12], and in [2]. We present a different proof, which explains the significance of the condition (2.7). Here again $h(t)$ is a given continuous function of period T.

Lemma 2.4. *Assume that* $|h(t)| \le 1$ *and*

$$k < \frac{\lambda^2}{4} + \omega^2. \tag{2.7}$$

Then any non-trivial solution of (2.6) is of one sign, i.e. we may assume that $w(t) > 0$ *for all* t.

Proof. Assume on the contrary that we have a sign changing solution $w(t)$. Since $w(t)$ is T-periodic, there exist $t_1 < t_2$, such that $t_2 - t_1 = T$, and $w(t_1) = w(t_2) = 0$. Since we also have $w'(t_1) = w'(t_2)$, it follows that $w(t)$ has at least one more root on (t_1, t_2). If we now consider an eigenvalue problem on (t_1, t_2),

$$w''(t)+\lambda w'(t)+kh(t)w(t)=0, \ \text{ for } t_1 < t < t_2, \quad w(t_1)=w(t_2)=0, \tag{2.8}$$

it follows that $w(t)$ is the second or higher eigenfunction, with k then being the second or higher eigenvalue. On the other hand, we consider the following Dirichlet eigenvalue problem on (t_1, t_2)

$$z''(t)+\lambda z'(t)+kz(t)=0, \quad z(t_1)=z(t_2)=0. \tag{2.9}$$

Its eigenvalues are $k_n = \frac{\lambda^2}{4} + \frac{n^2\omega^2}{4}$, and in particular $k_2 = \frac{\lambda^2}{4} + \omega^2$. Since all eigenvalues of (2.8) are greater than the corresponding eigenvalues of (2.9), we conclude that the second eigenvalue of (2.8) (i.e. k) must be greater than k_2, contradicting the assumption of the lemma. □

We shall also consider the adjoint linear problem

$$\nu''(t) - \lambda\nu'(t) + kh(t)\nu(t) = 0, \quad \nu(t+T) = \nu(t) = 0. \tag{2.10}$$

Lemma 2.5. *Assume the condition (2.7) holds. If the problem (2.6) has a non-trivial solution, the same is true for the adjoint problem (2.10). Moreover, we then have $\nu(t) > 0$ for all t.*

Proof. Assume that the problem (2.6) has non-trivial solution, but the problem (2.10) does not. The differential operator given by the left hand side of (2.10) is Fredholm, of index zero. Since its kernel is empty, the same is true for its co-kernel, i.e. we can find a solution $z(t)$ of

$$z''(t) - \lambda z'(t) + kh(t)z(t) = w(t), \quad z(t+T) = z(t) = 0. \tag{2.11}$$

Multiplying the equation (2.11) by $w(t)$, the equation (2.6) by $z(t)$, subtracting and integrating, we have

$$\int_0^T w^2(t)\,dt = 0,$$

a contradiction.

Positivity of $\nu(t)$ follows by the previous Lemma 2.4 (in which no assumption on the sign of λ was made). □

The following lemma is easily proved by integration.

Lemma 2.6. *Assume $|g(u)| \le M$, for all $u \in R$. If $u(t)$ is a T-periodic solution of (1.1), then*

$$|\mu| \le kM.$$

3. Continuation of solutions of any fixed average

We denote by C^2_T the subspace of $C^2(R)$, consisting of T-periodic functions. By $\bar{C}^2_T$ we denote the subspace of C^2_T, consisting of functions of zero average on $(0,T)$, i.e. $\int_0^T u(t)\,dt = 0$.

Theorem 3.1. *Consider the problem*

$$u''(t) + \lambda u'(t) + kg(u(t)) = \mu + e(t), \quad u(t+T) = u(t), \tag{3.1}$$

$$\frac{1}{T}\int_0^T u(t)\,dt = \xi. \tag{3.2}$$

where k, μ, λ and ξ are parameters, and the continuous function $e(t)$ is T-periodic and of zero average, i.e. $\int_0^T e(t)\,dt = 0$. Assume that the function $g(u) \in C^1(R)$

is bounded and it has a bounded derivative, i.e.

$$|g(u)|,\ |g'(u)| \le M, \ \ \textit{for all } u \in R. \tag{3.3}$$

Assume finally that

$$k \le \frac{1}{M}\omega\sqrt{\omega^2+\lambda^2}. \tag{3.4}$$

Then for any $\xi \in R$ one can find a unique μ for which the problem (3.1), (3.2) has a unique solution. (I.e. for each ξ there is a unique solution pair $(\mu, u(t))$.)

Proof. We begin by assuming that $\xi = 0$. We wish to prove that there is a unique $\mu = \mu(k)$ for which the problem (3.1) has a solution of zero average, and that solution is unique. We recast the equation (3.1) in the operator form

$$F(u, \mu, k) = e(t), \tag{3.5}$$

where $F\ : \bar{C}^2_T \times R \times R \to C_T$ is defined by

$$F(u, \mu, k) = u''(t) + \lambda u'(t) + kg(u(t)) - \mu.$$

When $k = 0$ and $\mu = 0$, the problem (3.5) has a unique T-periodic solution of zero average, according to the Lemma 2.1. We now continue this solution for increasing k, i.e. we solve (3.5) for the pair (u, μ) as a function of k. Compute the Frechet derivative

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = w''(t) + \lambda w'(t) + kg'(u(t))w(t) - \mu^*.$$

According to Lemma 2.3 the only solution of the linearized equation

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = 0, \quad w(t+T) = w(t)$$

is $(w, \mu^*) = (0, 0)$. I.e. locally we have a curve of solutions $u = u(k)$ and $\mu = \mu(k)$. To show that this curve continues for all k satisfying our condition (3.4), we only need to show that this curve cannot go to infinity at some k, i.e. we need an a priori estimate. Multiplying the equation (3.1) by u'' and integrating, we easily obtain (using boundness of $g(u)$) that $\int_0^T u''^2(t)\,dt$ is bounded, and then by the Wirtinger's inequality the same is true for $\int_0^T u'^2(t)\,dt$, and then for $\int_0^T u^2(t)\,dt$. We conclude the uniform boundness of the solution by the Sobolev embedding theorem, and then we bootstrap to the boundness in C^2_T, since μ is bounded by Lemma 2.6.

The solution (u, μ), which we found at the parameter value of k, is unique since otherwise we could continue it for decreasing k, obtaining at $k = 0$ a zero average T-periodic solution of

$$u'' + \lambda u' = \mu_0 + e(t),$$

with some constant μ_0. Clearly, $\mu_0 = 0$, and then we obtain another solution of the problem (2.1) (since curves of solutions of zero average do not intersect, in view of the implicit function theorem), contradicting Lemma 2.1.

Turning to the solutions of any average ξ, we again solve (3.5), but redefine $F : \bar{C}_T^2 \times R \times R \to C_T$ by

$$F(u, \mu, k) = u''(t) + \lambda u'(t) + kg(u(t) + \xi) - \mu.$$

As before, we obtain a solution (u, μ) at k, with u of zero average, which implies that $(u + \xi, \mu)$ solves our problem (3.1), (3.2). □

Remark. A similar result can be proved using the Lyapunov-Schmidt reduction, as was done in G. Tarantello [12], see also [2]. Our proof appears to be simpler, and we do not need $g(u)$ to be analytic, as in those papers.

4. Continuation in μ for fixed k and λ

In this section we keep k and λ fixed, and study the solutions of (3.1) as a function of μ. The following simple lemma will be important for the understanding of the solution curves.

Lemma 4.1. *Assume that a fixed k satisfies the inequality (3.4). The average value of the solution uniquely determines the corresponding μ, and the solution. I.e., for any $\xi \in R$ we can find a unique μ at which the problem (3.1) has a unique T-periodic solution $u(t)$. (In short, the value of ξ uniquely determines the solution pair $(\mu, u(t))$.)*

Proof. We begin with the unique T-periodic solution of average ξ of the linear equation, obtained from (3.1) by setting $k = 0$ and $\mu = 0$. We now continue this solution for increasing k, using the Theorem 3.1, until we reach the fixed k in our problem with the corresponding $\mu = \mu_0$, and the T-periodic solution $u_0(t)$. If there were another pair $(\mu_1, u_1(t))$, satisfying (3.1) at the same k (and the average of $u_1(t)$ is also ξ), we would continue it for decreasing k on another curve of solutions, obtaining another solution of average ξ at $k = 0$, which is a contradiction. □

Remarks

1. It follows that as we continue the solution of (3.1) in μ, the average of solution ξ must change monotonously.
2. When one considers positive solutions of the two-point problem

$$u'' + \lambda f(u) = 0, \quad x \in (-1, 1), \quad u(-1) = u(1) = 0$$

one proves that the value of the maximum of solution, i.e. $u(0)$, uniquely identifies both the value of the parameter λ and the solution $u(x)$, see e.g. [9]. The above result gives an exact analogy for the periodic solutions.

With k and λ fixed, and k satisfying (3.4), the problem (3.1) has a T-periodic solution of any average $\xi \in R$ (with $\mu = \mu(\xi)$). This continuum of solutions "has arrived" on curves of fixed average from $k = 0$. These solutions are non-singular on the curves of fixed average (which should not be confused with unconditionally non-singular solutions). As we now vary μ, we may encounter singular solutions. However, if condition (2.7) is also satisfied, we show that the Crandall-Rabinowitz theorem applies at the singular solutions. We then show that there is a unique smooth curve of solutions, connecting all T-periodic solutions at a fixed k.

Theorem 4.1. *For the problem (3.1) assume that the conditions (2.7), (3.3) and (3.4) are satisfied. Then there exists a continuous bounded function $\phi(\xi)$, defined for all $\xi \in R$, so that for $\mu = \phi(\xi)$ the problem (3.1) has a unique T-periodic solution of average ξ.*

Proof. Starting with the solution of (3.1) of, say, zero average and the corresponding $\mu = \mu_0$, we continue this solution in μ. If the solution is non-singular we can continue using the implicit function theorem. Eventually, though, a singular solution will be reached, since this continuation in μ cannot continue indefinitely. Indeed, integrating the equation (3.1), we see that no T-periodic solutions are possible if $|\mu| > kM$.

We show next that the Crandall–Rabinowitz theorem applies at a singular solution. We recast our equation (3.1) in the operator form $F = 0$, by defining a map $F : \bar{C}^2_T \times R \to C_T$ as follows

$$F(u, \mu) \equiv u'' + \lambda u' + kg(u) - \mu - e(t) = 0.$$

The linearized equation is

$$F_u(u, \mu)w = w'' + \lambda w' + kg'(u)w = 0, \quad w(t) = w(t + T). \tag{4.1}$$

If $u(t)$ is a singular solution, i.e. the problem (4.1) has a non-trivial solution, then we may assume that $w(t) > 0$ for all t, in view of Lemma 2.4. By Lemma 2.5 the adjoint linear problem

$$\nu'' - \lambda \nu' + kg'(u)\nu = 0, \quad \nu(t) = \nu(t + T) \tag{4.2}$$

also has a T-periodic solution $\nu(t) > 0$. We claim that the null-space of F_u is one dimensional. Indeed, if we had two linearly independent solutions of (4.1) $w_1(t) > 0$ and $w_2(t) > 0$, then we could find constants c_1, c_2, so that $c_1 w_1(t) + c_2 w_2(t)$ is sign changing, a contradiction. Since the operator F is Fredholm of index zero, the codimension of F_u is also one. Finally, we check that $F_\mu \notin R(F_u)$. Indeed, assuming otherwise we could find a non-trivial solution of

$$\theta'' + \lambda \theta' + kg'(u)\theta = 1, \quad \theta(t) = \theta(t + T),$$

i.e. 1 is orthogonal to the solution of (4.2), $\int_0^T \nu(t)\, dt = 0$, a contradiction, since $\nu > 0$. Hence, the Crandall-Rabinowitz theorem applies, and we can continue the solution smoothly through the critical point. So that the solution curves can be continued indefinitely (globally).

According to the Lemma 4.1 the average of the solution ξ changes monotonously on the solution curve. Hence, we can use ξ as a parameter on the solution curve, with $(\mu, u(t))$ being a function of ξ. We claim that $(\mu, u(t))$ cannot become unbounded at a finite ξ. We already know that $|\mu| \le kM$. To get a bound on $u(t)$, we proceed as in the Theorem 3.1. Multiplying the equation (3.1) by u'' and integrating, we obtain the boundness of $\int_0^T u''^2(t)\,dt$, and then of $\int_0^T u'^2(t)\,dt$. Since the average ξ is bounded, we conclude a bound in H^1, and hence the uniform bound. It follows that we can continue the solution curve as $\xi \to \infty$, and when $\xi \to -\infty$. ☐

The above theorem shows that all solutions of (3.1) lie on a unique solution curve in $(\xi, \mu, u(t))$ "space". A good idea about the curve can be gleaned from its projection $\Gamma\,:\,(\xi, \mu)$, which is a graph of the function $\phi(\xi)$, described in the above theorem. In the present generality we cannot say much about the shape of Γ. We can get considerably more information for special type of nonlinearities $g(u)$ and the corresponding special class of solutions $u(t)$, which are motivated by G. Tarantello [12].

Definition We say that the function $g(u) \in C^2(R)$ is of class $\mathbf{T}_m$ if $g(u)$ changes sign infinitely many times, and if ρ denotes any point of local maximum (minimum) of $g(u)$ then on the interval $|u - \rho| < m$ we have

$$\begin{aligned} g(u) > 0 \quad (< 0) \\ g''(u) < 0 \quad (> 0). \end{aligned} \tag{4.3}$$

In other words, $g(u)$ has only positive maximums and negative minimums, and they are spaced out. For example, $\sin u \in \mathbf{T}_{\pi/2}$.

Definition We say that the function $u(t) \in C_T$ is of class $\mathbf{t}_m$ if, when writing $u(t) = \xi + U_\xi(t)$, with $\int_0^T U_\xi(t)\,dt = 0$, we have

$$|U_\xi(t)| < m/2 \quad \text{for all } t. \tag{4.4}$$

As was pointed out by G. Tarantello [12], see the Lemma 4.2 below, the condition (4.4) will hold if either λ is large, or $e(t)$ is small. If $u(t)$ is of class $\mathbf{t}_m$, it follows that the range of $u(t)$ belongs to an interval of length less than m. Hence, if the range of $u(t)$ includes any point of local maximum (minimum) of $g(u)$, and $g(u)$ is of class $\mathbf{T}_m$, then (4.3) holds, which implies that $g''(u)$ is of one sign. This will allow us to compute the direction of bifurcation.

The following lemma shows that a solution of class t_m stays in this class, as we continue it in μ (or in ξ). The proof is from G. Tarantello [12], p. 86, but we present it here, partly for completeness, partly to stress the uniformity in μ and ξ. Setting $u(t) = \xi + U_\xi(t)$ in (3.1), we obtain

$$U_\xi''(t) + \lambda U_\xi'(t) + kg(\xi + U_\xi(t)) = \mu + e(t), \quad U_\xi(t+T) = U_\xi(t). \tag{4.5}$$

We shall write L^2 for $L^2(0,T)$, and L^∞ for $L^\infty[0,T]$.

Lemma 4.2. *Assume that*

$$\frac{\sqrt{T}}{\sqrt{3}}\,\frac{\|e(t)\|_{L^2}}{|\lambda|} < m. \tag{4.6}$$

Then for any $\mu \in R$ and any $\xi \in R$ we have

$$\|U_\xi\|_{L^\infty} < \frac{m}{2}, \tag{4.7}$$

i.e. any T-periodic solution of (3.1) is of class $\mathbf{t}_m$.

Proof. Multiply (4.5) by U'_ξ, and integrate

$$\lambda \int_0^T {U'_\xi}^2(t)\, dt = \int_0^T U'_\xi(t) e(t)\, dt.$$

This implies

$$\|U'_\xi\|_{L^2} < \frac{\|e(t)\|_{L^2}}{|\lambda|}. \tag{4.8}$$

Since $U_\xi(t) = \Sigma_{j\neq 0} c_j e^{\frac{2\pi}{T} ijt}$, we have using (4.8)

$$\begin{aligned} \|U_\xi\|_{L^\infty} &\leq \Sigma_{j\neq 0}|c_j| \leq \left(\Sigma_{j\neq 0} \tfrac{1}{j^2}\right)^{1/2} \left(\Sigma_{j\neq 0} j^2 |c_j|^2\right)^{1/2} \\ &= \tfrac{\pi}{\sqrt{3}} \tfrac{\sqrt{T}}{2\pi} \|U'_\xi\|_{L^2} \leq \tfrac{\sqrt{T}}{2\sqrt{3}} \; \tfrac{\|e(t)\|_{L^2}}{|\lambda|} < \tfrac{m}{2}. \end{aligned} \qquad \square$$

Theorem 4.2. *Assume that conditions of the Theorem 4.1 are satisfied, and in addition $g(u) \in \mathbf{T}_m$, while $u(t) \in \mathbf{t}_m$ (the last condition will hold if (4.6) is assumed to be satisfied). Then the function $\mu = \phi(\xi)$ changes sign infinitely many times. Moreover, it is positive at any of its local maximums, and negative at its local minimums, and it has no points of inflection.*

If in addition, $g(u)$ is a periodic function, of period, say 2π, then the same is true for the function $\phi(\xi)$.

Proof. Integrating the equation (3.1),

$$k \int_0^T g(u(t))\, dt = \mu T. \tag{4.9}$$

Along the solution curve ξ changes continuously from $-\infty$ to ∞. When ξ equals to a point of local maximum of $g(u)$, then by our assumptions the integrand on the left in (4.9) is positive, and hence $\mu > 0$. Similarly $\mu < 0$, when ξ passes a point of local minimum of ξ. Since $g(u)$ changes sign infinitely many times, the same is true for $\phi(\xi)$.

Turning to the extremums of $\phi(\xi)$, we differentiate the equation (3.1) in ξ

$$u''_\xi + \lambda u'_\xi + kg'(u)u_\xi = \mu'(\xi). \tag{4.10}$$

At a critical point (ξ, μ_0) we have $\mu'(\xi_0) = 0$. We now set $\xi = \xi_0$ in (4.10). Then $w(t) \equiv u_\xi \mid_{\xi=\xi_0}$ satisfies the linearized problem (4.1), and hence by Lemma 2.4,

$w(t) > 0$. By Lemma 2.5, the adjoint linear problem (4.2) has a non-trivial solution $\nu(t) > 0$. Integrating (4.10) at $\xi = \xi_0$

$$k \int_0^T g'(u(t))w(t)\,dt = 0. \tag{4.11}$$

We differentiate (4.10) in ξ again, and set $\xi = \xi_0$

$$u''_{\xi\xi} + \lambda u'_{\xi\xi} + kg'(u)u_{\xi\xi} + kg''(u)w^2 = \mu''(\xi_0). \tag{4.12}$$

We multiply this equation by $\nu(t)$ and subtract the equation (4.2) multiplied by $u_{\xi\xi}$, then integrate

$$k \int_0^T g''(u(t))w^2(t)\nu(t)\,dt = \mu''(\xi_0) \int_0^T \nu(t)\,dt. \tag{4.13}$$

We see from (4.11) that $g'(u)$ changes sign on the range of $u(t)$, i.e. the range of $u(t)$ includes a critical point of $g(u)$. By our conditions, the range of $u(t)$ includes exactly one critical point of $g(u)$. If it is a point of maximum of $g(u)$, the first set of inequalities in (4.3) holds, and then we see from (4.9) and (4.12) that $\mu(\xi_0) > 0$ and $\mu''(\xi_0) < 0$, while the opposite inequalities hold in case of a point of minimum.

Finally, assume that $g(u)$ is 2π-periodic. If $u(t)$ is a solution of average ξ, then $u(t) + 2\pi$ is a solution of average $\xi + 2\pi$, corresponding to the same μ. Since the value of ξ uniquely identifies μ (and $u(t)$), we have $\phi(\xi + 2\pi) = \phi(\xi)$. □

Corollary 1. *Under the conditions of the theorem, the problem*

$$u'' + \lambda u' + kg(u) = e(t), \quad u(t+T) = u(t)$$

has infinitely many solutions.

The forced pendulum equation is our main example:

$$u''(t) + \lambda u'(t) + k\sin(u(t)) = \mu + e(t), \quad u(t+T) = u(t). \tag{4.14}$$

Because of periodicity of $\sin u$, without restricting the generality, we will only consider solutions, with average values satisfying $\xi \in [0, 2\pi)$. As before, $\int_0^T e(t)\,dt = 0$.

Theorem 4.3. *Assume that k satisfies the conditions $k < \frac{\lambda^2}{4} + \omega^2$ and $k \le \omega\sqrt{\omega^2 + \lambda^2}$, and the condition (4.6) is satisfied, with $m = \frac{\pi}{2}$. Then there exist two constants $d < 0 < D$, so that the problem (4.14) has exactly two T-periodic solutions if $\mu \in (d, D)$, exactly one T-periodic solution if either $\mu = d$ or $\mu = D$, and none if $\mu \notin (d, D)$. Moreover, all solutions lie on a unique smooth curve in (ξ, μ) plane, given by a function $\mu = \mu(\xi)$. This function has exactly two critical points - a point of global maximum, followed by a point of global minimum, and $\mu(2\pi-) = \mu(0)$, see Figure 1.*

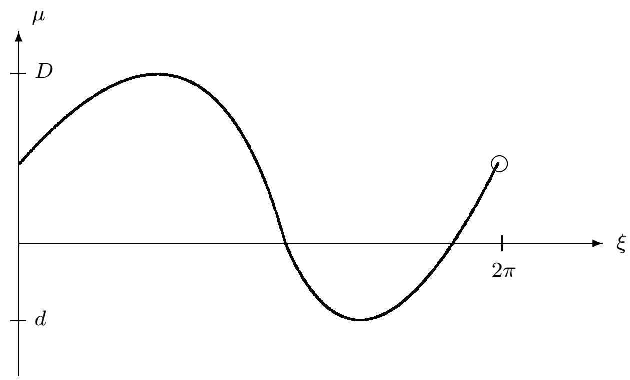

Figure 1. The solution curve for forced pendulum equation

Proof. By Lemma 4.2 any solution of our problem (4.14) is of class $\mathbf{t}_{\pi/2}$, i.e. the range of $u(t)$ belongs to an interval, whose length is less than π. Since $\sin u \in \mathbf{T}_{\pi/2}$, the previous Theorem 4.2 applies, implying the existence of solution curve $\mu = \phi(\xi)$, with $\phi(\xi)$ being 2π periodic. It remains to show that $\phi(\xi)$ has exactly two critical points on $[0, 2\pi)$ - a point of local maximum, followed by a point of local minimum. If $u(t)$ is a singular solution (which corresponds to a critical point of $\phi(\xi)$), and $w(t)$ is the solution of the corresponding linearized equation, then by (4.11) we have

$$\int_0^T w(t) \cos u(t) \; dt = 0.$$

Since by Lemma 2.4, $w(t)$ is of one sign, it follows that the range of $u(t)$ contains either $\pi/2$ or $3\pi/2$. If the range of $u(t)$ contains $\pi/2$ ($3\pi/2$), the range of $u(t)$ lies in $(0, \pi)$ (in $(\pi, 2\pi)$), where $g(u) = \sin u$ is positive (negative), and its second derivative is negative (positive). If the range of a singular solution $u(t)$ contains $\pi/2$, then by (4.13) this is a point of maximum of $\mu = \phi(\xi)$, and only one critical point is possible in this range. Similarly, when $u(t)$ is near $3\pi/2$, only one point of minimum of $\phi(\xi)$ is possible. Since $\phi(\xi)$ is 2π periodic, we have a point of local (global) maximum, followed by a point of local (global) minimum. $\square$

5. Continuation of solutions of any signature

In Theorem 3.1 we had continued in k solutions of any fixed average, by adjusting the average of the forcing term as a function of k. Average of the solution $u(t)$ is just its zeroes Fourier coefficient. We show that for any fixed n, we can continue the solution with the first n Fourier coefficients fixed, by adjusting the first n Fourier coefficients of the forcing term as a function of k.

Definition. Let $u(t)$ be a real valued function, with its complex Fourier series $u(t) = \Sigma_{j=-\infty}^{\infty} u_j e^{i\omega jt}$, $(u_{-j} = \bar{u}_j)$. We define the vector $(u_0, u_1, \ldots, u_n)$ to be the n-signature of $u(t)$ (or just signature, for short). (Observe, that we also know $(u_{-1}, \ldots, u_{-n})$, if the n-signature of $u(t)$ is given.)

We will need the following straightforward generalization of Wirtinger's inequality, Lemma 2.2.

Lemma 5.1. *Let $f(t)$ be a T-periodic function of class C^1, whose n-signature is zero (i.e. $f_0 = f_1 = \ldots = f_n = 0$ in its complex Fourier series). Then*

$$\int_0^T f'^2(t)\, dt \geq (n+1)^2 \omega^2 \int_0^T f^2(t)\, dt.$$

We now write the equation (1.1) in the form

$$\begin{aligned} u''(t) + \lambda u'(t) + kg(u(t)) = \mu + \mu_1 e^{i\omega t} + \bar{\mu}_1 e^{-i\omega t} + \ldots \\ +\mu_n e^{in\omega t} + \bar{\mu}_n e^{-in\omega t} + e(t), \qquad u(t+T) = u(t), \end{aligned} \tag{5.1}$$

where $e(t)$ has n-signature zero, i.e. $\int_0^T e(t) e^{ij\omega t}\, dt = 0$, for $j = 0, \pm 1, \ldots \pm n$.

As before, we begin by considering the linear problem

$$\begin{aligned} w''(t) + \lambda w'(t) + kh(t)w(t) = \mu_0 + \mu_1 e^{i\omega t} + \bar{\mu}_1 e^{-i\omega t} + \ldots \\ +\mu_n e^{in\omega t} + \bar{\mu}_n e^{-in\omega t}, \\ w(t+T) = w(t), \quad w_0 = w_1 = \ldots = w_n = 0. \end{aligned} \tag{5.2}$$

Here $h(t)$ is a given continuous function of period T, and we are looking for $(\mu_0, \mu_1, \ldots, \mu_n)$, and the T-periodic solution $w(t)$, whose n-signature is zero.

Lemma 5.2. *Let $|h(t)| \leq M$ for all $t \in R$, and*

$$k < \frac{1}{M}(n+1)\omega\sqrt{\omega^2(n+1)^2 + \lambda^2}. \tag{5.3}$$

Then the only solution of (5.2) is

$$\mu_0 = \mu_1 = \ldots = \mu_n = 0, \quad \textit{and} \quad w(t) \equiv 0.$$

Proof. Multiply the equation (5.2) by its solution $w(t)$ (i.e. $w(t)$ has n-signature zero). Integrate over $(0, T)$. The integral on the right is zero. Then we proceed exactly as in the proof of Lemma 2.3, using the generalized Wirtinger's inequality, Lemma 5.1, twice. (Once we prove that $w(t) \equiv 0$, it follows that $\mu = \mu_1 = \ldots = \mu_n = 0$, by uniqueness of Fourier series.) ☐

Similarly to the Theorem 3.1, we prove the following result, which is an extension of the one in A. Ureña [14].

Theorem 5.1. *Consider the problem (5.1), where $e(t)$ is a T-periodic continuous function of n-signature zero. Assume that the function $g(u) \in C^1(R)$ satisfies*

$$|g(u)|,\ |g'(u)| \leq M, \ \ \textit{for all } u \in R.$$

Assume finally that (5.3) holds. Then for any vector $(u_0, u_1, \ldots, u_n)$*, of any length* n*, one can find a unique vector* $(\mu(k), \mu_1(k), \ldots, \mu_n(k))$*, so that the problem (5.1) has a unique* T*-periodic solution of* n*-signature* $(u_0, u_1, \ldots, u_n)$*.*

Proof. We begin by assuming that $u_0 = u_1 = \ldots = u_n = 0$. By $\hat{C}_T^2$ we denote the subspace of C_T^2, consisting of functions, whose n-signature is zero. We also denote $\mu = (\mu_0, \mu_1, \ldots, \mu_n) \in R^{n+1}$. We recast the equation (5.1) in the operator form

$$F(u, \mu, k) = e(t), \tag{5.4}$$

where $F\ : \hat{C}_T^2 \times R^{n+1} \times R \to C_T$ is defined by

$$F(u, \mu, k) = u''(t) + \lambda u'(t) + kg(u(t)) - \mu_0 - \mu_1 e^{i\omega t} - \bar{\mu}_1 e^{-i\omega t} - \ldots$$
$$-\mu_n e^{i\omega n t} - \bar{\mu}_n e^{-i\omega n t}.$$

When $k = 0$ and $\mu = 0$, the problem (5.1) has a unique T-periodic solution of n-signature zero , as easily follows by Fourier series, similarly to Lemma 2.1. We now continue this solution for increasing k, i.e. we solve (3.5) for $(u(t), \mu_0, \mu_1, \ldots \mu_n)$ as a function of k. Compute the Frechet derivative

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = w''(t) + \lambda w'(t) + kg'(u(t))w(t) - \mu_0^*$$
$$-\mu_1^* e^{i\omega t} - \bar{\mu}_1^* e^{-i\omega t} - \ldots - \mu_n^* e^{i\omega n t} - \bar{\mu}_n^* e^{-i\omega n t}.$$

According to Lemma 5.2 the only solution of the linearized equation

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = 0, \quad w(t+T) = w(t), \quad w_0 = w_1 = \ldots = w_n = 0$$

is $(w, \mu^*) = (0, 0)$. I.e. locally we have a curve of solutions of zero signature $u = u(k)$ and $\mu = \mu(k)$.

To show that this curve continues for all k satisfying the condition (5.3), we need an a priori estimate. We begin by showing that $\mu = \mu(k)$ is bounded. Integrating the equation (5.1), we conclude as before that $|\mu_0| \le kM$. If we multiply the equation (5.1) by $e^{-ij\omega t}$, for any $j = 1, \ldots n$, and integrate, we have (keep in mind that $u(t)$ has n-signature zero)

$$\mu_j T = k \int_0^T g(u(t))\, e^{-ij\omega t}\, dt,$$

i.e. $|\mu_j| \le kM$. If we now denote by $f(k, t)$ the right hand side of the equation (5.1), then $f(k, t)$ is bounded in $L^2(0, T)$, uniformly in k. We now multiply the equation (5.1) by u'', integrate, and conclude the uniform boundness of solution, exactly as before.

The solution $(u(t), \mu)$ that we found at the parameter value k is unique. Indeed, any other solution we could continue for decreasing k, obtaining at $k = 0$ a solution of n-signature zero for the problem

$$u''(t) + \lambda u'(t) = \mu_0^0 + \mu_1^0 e^{i\omega t} + \bar{\mu}_1^0 e^{-i\omega t} + \ldots$$
$$+\mu_n^0 e^{i\omega n t} + \bar{\mu}_n^0 e^{-i\omega n t} + e(t), \quad u(t+T) = u(t),$$

for some vector $\mu^0 = (\mu_0^0, \mu_1^0, \dots \mu_n^0)$. Multiplying this equation by $e^{-ij\omega t}$, for any $j = 0, 1, \dots n$, and integrating, we conclude that $\mu^0 = 0$, and then we obtain another solution of n-signature zero for the linear problem

$$u''(t) + \lambda u'(t) = e(t), \quad u(t+T) = u(t),$$

which is impossible.

Turning to the solutions of an arbitrary n-signature $(u_0, u_1, \dots u_n)$, we define

$$\bar{u}(t) = u(t) + u_0 + u_1 e^{i\omega t} + \bar{u}_1 e^{-i\omega t} + \dots + u_n e^{i\omega n t} + \bar{u}_n e^{-i\omega n t},$$

and find as before the solution of n-signature zero for the problem (5.4), where we redefine $F \ : \hat{C}_T^2 \times R^{n+1} \times R \to C_T$ by

$$F(u, \mu, k) = u''(t) + \lambda u'(t) + kg(\bar{u}(t)) - \mu_0 - \mu_1 e^{i\omega t} - \bar{\mu}_1 e^{-i\omega t} - \dots$$
$$-\mu_n e^{i\omega n t} - \bar{\mu}_n e^{-i\omega n t}.$$

Then $(\bar{u}(t), \mu)$ gives solution of our problem (5.1), with the n-signature equal to $(u_0, u_1, \dots u_n)$. □

6. First order equations

One can develop similar results for the first order periodic problem

$$u'(t) + kg(u(t)) = \mu + e(t), \quad u(t+T) = u(t). \tag{6.1}$$

As before, k and μ are parameters, and $e(t)$ is a given continuous function of period T. Given the similarity with the second order case, we only sketch the proofs. The following analog of Lemma 2.1 is easily proved by integration.

Lemma 6.1. *Consider the linear problem*

$$y'(t) = e(t), \tag{6.2}$$

with $e(t)$ given continuous function of period T, of zero average, i.e. $\int_0^T e(s)\, ds = 0$. Then the problem (6.2) has a unique T-periodic solution of any average.

Again, we begin by considering a linear periodic problem in the class of functions of zero average

$$w'(t) + kh(t)w(t) = \mu, \ \ w(t+T) = w(t), \ \ \int_0^T w(s)\, ds = 0, \tag{6.3}$$

where $h(t)$ is a given continuous function of period T, k and μ are parameters.

Lemma 6.2. *Assume that $|h(t)| \le M$ and*

$$k < \frac{\omega}{M}. \tag{6.4}$$

Then the only solution of (6.3) is $\mu = 0$ and $w(t) \equiv 0$.

Proof. Multiply the equation (6.3) by w', integrate over $(0,T)$,

$$\int_0^T {w'}^2\,dt = -k\int_0^T h(t)ww'\,dt \le kM\left(\int_0^T w^2\,dt\right)^{1/2}\left(\int_0^T {w'}^2\,dt\right)^{1/2}.$$

Using the Wirtinger's inequality, we then conclude

$$\int_0^T {w'}^2\,dt \le k^2M^2\int_0^T w^2\,dt \le \frac{k^2M^2}{\omega^2}\int_0^T {w'}^2\,dt,$$

i.e. $w(t)\equiv 0$. □

The positivity for the linearized problem is now "free", as follows by simple integration.

Lemma 6.3. *Any non-trivial solution $w(t)$ of the linearized problem*

$$w'(t)+kh(t)w(t)=0,\ \ w(t+T)=w(t), \tag{6.5}$$

is of one sign, for any $k>0$, and any continuous function $h(t)$.

The following lemma is also easy to prove by integration.

Lemma 6.4. *The problem*

$$\nu'(t)-kh(t)\nu(t)=0,\ \ \nu(t+T)=\nu(t), \tag{6.6}$$

has a non-trivial solution iff the problem (6.5) does. (That happens iff $\int_0^T h(s)\,ds = 0$.)

The Theorem 3.1, as well as Lemma 4.1, now follow verbatim for the first order problem (6.1), with the condition (3.4) replaced by (6.4).

The following results follow similarly to the second order case. We shall only sketch the proof of the lemma.

Theorem 6.1. *For the problem (6.1) assume that the conditions (3.3) and (6.4) are satisfied. Then there exists a continuous bounded function $\phi(\xi)$, defined for all $\xi\in R$, so that for $\mu=\phi(\xi)$ the problem (6.1) has a unique T-periodic solution of average ξ.*

Lemma 6.5. *Assume that*

$$\frac{\sqrt{T}}{\sqrt{3}}\,\|e(t)\|_{L^2} < m. \tag{6.7}$$

Then any T-periodic solution of (6.1) is of class $\mathbf{t}_m$.

Proof. Setting as before $u(t)=\xi+U_\xi(t)$, with $\int_0^T U_\xi(t)\,dt=0$, we have

$$U_\xi'(t)+kg(\xi+U_\xi(t))=\mu+e(t),\quad U_\xi(t+T)=U_\xi(t).$$

Multiplying by $U'_\xi(t)$ and integrating

$$\int_0^T {U'_\xi}^2(t)\,dt = \int_0^T U'_\xi(t) e(t)\,dt.$$

The rest of the proof is exactly the same as in Lemma 4.2. □

Theorem 6.2. *Assume that conditions (3.3), (6.4) and (6.7) are satisfied, and in addition $g(u) \in \mathbf{T}_m$. Then the function $\mu = \phi(\xi)$ (defined in Theorem 6.1) changes sign infinitely many times. Moreover, it is positive at any of its local maximums, and negative at its local minimums, and it has no points of inflection. If in addition, $g(u)$ is a periodic function, of period, say 2π, then the same is true for the function $\phi(\xi)$. In case $g(u) = \sin u$, the solution picture is given by Figure 1.*

Example. Consider

$$u'(t) + k\sin(u(t)) = \mu + e(t), \quad u(t+T) = u(t).$$

Here $M = 1$, and hence if we assume that $k < \omega$ and (6.7) holds, then the Theorem 6.2 applies. It implies, in particular, that the problem

$$u'(t) + k\sin(u(t)) = e(t), \quad u(t+T) = u(t) \tag{6.8}$$

has exactly two solutions, whose average $\xi \in [0, 2\pi)$. On the other hand, a result of R. Ortega [10] says that for every $k \geq \omega$ one can find $e(t)$, so that the problem (6.8) has no solutions.

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