Application of generalized averages to uniqueness of solutions, and to a non-local problem

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Abstract

We use an integral relation from P. Korman and Y. Li [1] to prove uniqueness of positive solutions of two-point problem, when the maximum value of solution is lying in some interval (u_0, u_1), $0 < u_0 < u_1$. If additionally one knows that the problem has exactly k solutions with values in the $(0, u_0)$ range, one may then conclude existence of exactly $k+1$ solutions with values in the $(0, u_1)$ range. In another direction, the same integral relation allows us to give a complete analysis of positive and sign-changing solutions of a non-local problem.

Key words: Uniqueness of solutions, a non-local problem.

AMS subject classification: 34B15.

1 Introduction

In our first result we give conditions for uniqueness of positive solutions of two-point problem

$$u'' + f(u) = 0, \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{1.1}$$

Our tool is the integral relation from P. Korman and Y. Li [1], which is satisfied by any positive solution of (1.1). This relation was derived by a change of the independent variable, that depended on the solution itself, and which resulted in a linear equation (on an interval whose length depended on the solution). This approach is very similar to the derivation of the time map in R. Schaaf [4]. In fact both approaches led to the same condition for the uniqueness of solutions. Both approaches allowed one to prove existence of at most two solutions, although the conditions were now different (see condition (1-4-1) in [4], and (2.14) in [1]). Since the condition (1-4-1) in [4] is hard to verify, R. Schaaf went on to develop her well-known A-B and C conditions, which involved the third derivatives of $f(u)$. Similarly, we were able to derive third order conditions, based on P. Korman and Y. Li [1]. We do not include them here, since these conditions and, to some extent, the A-B and C conditions are superseded by the following theorem, which was implicit in P. Korman, Y. Li and T. Ouyang [2], see also T. Ouyang and J. Shi [3].

Theorem 1.1 *([2])*
(i) Assume that $f(0) \geq 0$, $f''(u) < 0$ for $0 < u < u_0$, $f''(u) > 0$ for $u > u_0$. Then only turns to the left are possible on the solution curve.
(ii) Assume that $f(0) \leq 0$, $f''(u) > 0$ for $0 < u < u_0$, $f''(u) < 0$ for $u > u_0$. Then only turns to the right are possible on the solution curve.

(Of course, in both cases we conclude existence of at most two positive solutions, with the maximum values lying in the first positive hump of $f(u)$.)

In the present paper we notice that using the generalized averages, or time maps, allows for some extra flexibility, compared to the bifurcation approach, when one considers uniqueness of solution whose maximum value lies between two numbers $0 < u_0 < u_1 \leq \infty$. Indeed, our conditions restrict $f(u)$ only on the interval (u_0, u_1), which is somewhat surprising, since the range of the solution of the Dirichlet problem begins at zero. As an application of our result, we give a new exact multiplicity result, based on the cubic-like equations of P. Korman, Y. Li and T. Ouyang [2].

In another direction we use the same integral relation (2.5) to give a complete description of solutions with any number of nodes to a non-local problem

$$\begin{aligned} &u'' + u^3 = 0, \quad 0 < x < L, \\ &u(0) = 0, \quad \int_0^L u(s)\, ds = \alpha, \end{aligned}$$

where we prescribe α, which is equivalent to prescribing the average value of the solution.

Finally, we show that the conditions for the uniqueness of solutions from [4] and [1] actually imply that the solution is in addition non-degenerate.

2 A general uniqueness result

We consider positive solutions of the following two point boundary value problem ($u = u(x) > 0$)

$$u'' + f(u) = 0, \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{2.1}$$

It is well-known that positive solutions are symmetric with respect to the midpoint of the interval, $x = 1/2$, and $u'(x) > 0$ for $x \in (0, 1/2)$, so that $u(1/2)$ is the maximum value of the solution.

Theorem 2.1 *Assume that for some $0 < u_0 < u_1 \le \infty$ we have*

$$f(u) > 0, \quad \text{for } u_0 < u < u_1, \tag{2.2}$$

$$f'(u) \int_{u_0}^{u} f(t)\, dt - \frac{1}{2} f^2(u) > 0, \quad \text{for } u_0 < u < u_1. \tag{2.3}$$

(Observe that we implicitly assume that $f(u_0) = 0$.) Then the problem (2.1) has at most one positive solution, with $u_0 < u(1/2) < u_1$.

Our tool is the following integral relation from P. Korman and Y. Li [1].

Theorem 2.2 *([1]) Assume that $u(x)$ is any positive solution of*

$$u'' + f_1(u) = 0, \quad x \in (0, L), \quad u(0) = u(L) = 0, \tag{2.4}$$

where $f_1(u)$ is a continuous function, with $f_1(u) > 0$ for $u > 0$. Define $F_1(u) = \int_0^u f_1(t)\, dt$, and $g(u) = \frac{f_1(u)}{2\sqrt{F_1(u)}}$. Then for any $L > 0$,

$$\int_0^L g(u(x))\, dx = \frac{\pi}{\sqrt{2}}. \tag{2.5}$$

Remark For the particular case of

$$u'' + u^3 = 0, \quad x \in (0, L), \quad u(0) = u(L) = 0$$

the theorem implies that

$$\int_0^L u(x)\, dx = \frac{\pi}{\sqrt{2}}, \tag{2.6}$$

i.e. we obtain the average of the solution. We can then regard the formula (2.5) as giving us the generalized average of the solution.

The following lemma is well-known. For completeness we sketch an easy proof.

Lemma 2.1 *Different positive solutions of (2.1) do not intersect.*

Proof: The "energy" $E(x) = \frac{1}{2}u'^2(x) + F(u(x))$ is constant along any solution. If we have two solutions $u(x)$ and $v(x)$, and $u(x)$ is greater than $v(x)$ near $x = 0$, then $u(x)$ has higher energy than $v(x)$. At the first point of intersection we conclude that $v(x)$ must have higher energy, a contradiction. (Since both solutions are symmetric with respect to $x = 1/2$, the first point of intersection would have to lie in $(0, 1/2)$, where both $u(x)$ and $v(x)$ are increasing.)

Proof of the Theorem 2.1 Assume, on the contrary, that there are two solutions $u(x)$ and $v(x)$. By Lemma 2.1, we may assume that $u(x) > v(x)$ for all $x \in (0,1)$. Define the points ξ and η in $(0,1)$ by $u(\xi) = u(\eta) = u_0$, and the points α and β by $v(\alpha) = v(\beta) = u_0$. Clearly $(\alpha, \beta) \subset (\xi, \eta)$. Define $p(x) = u(x) - u_0$, and $q(x) = v(x) - u_0$. Then $p(x)$ satisfies

$$p'' + f_1(p) = 0, \quad x \in (\xi, \eta), \quad p(\xi) = p(\eta) = 0, \tag{2.7}$$

where $f_1(u) = f(u+u_0)$. Clearly $q(x)$ satisfies the same problem, but on the interval (α, β) instead of (ξ, η). Denote $g(u) = \frac{f_1(u)}{2\sqrt{F_1(u)}}$, as before. Observe that $g(u) > 0$ for all $u > 0$ by our condition (2.2). We then have by the Theorem 2.2 applied to (2.7)

$$\int_\alpha^\beta g(p)\,dx < \int_\xi^\eta g(p)\,dx = \frac{\pi}{\sqrt{2}},$$

$$\int_\alpha^\beta g(q)\,dx = \frac{\pi}{\sqrt{2}}.$$

Subtracting, we have

$$\int_\alpha^\beta [g(p) - g(q)]\,dx < 0. \tag{2.8}$$

Since $p(x) > q(x)$ on (α, β), this will lead to a contradiction, provided that $g(u)$ is an increasing function for all $u > 0$.

To verify that $g(u)$ is increasing, we compute

$$2g'(u) = \frac{f_1' F_1 - \frac{1}{2}f_1^2}{F_1^{3/2}} = \frac{f'(u+u_0)\int_0^u f(t+u_0)\,dt - \frac{1}{2}f^2(u+u_0)}{F_1^{3/2}}. \tag{2.9}$$

To see that the numerator of the last fraction is positive, we set $u + u_0 = v$, where $v > u_0$. Then the numerator of (2.9) is

$$f'(v)\int_0^{v-u_0} f(t+u_0)\,dt - \frac{1}{2}f^2(v) = f'(v)\int_{u_0}^v f(\tau)\,d\tau - \frac{1}{2}f^2(v) > 0,$$

by our condition (2.3), completing the proof.

Example 1. Assume that $f(u) \in C^2(\bar{R}_+)$ satisfies

$$f(u_0) = 0, \;\; f(u) > 0 \;\text{ for } u_0 < u < u_1 \leq \infty,$$

$$f''(u) > 0 \;\text{ for } u_0 < u < u_1 \leq \infty.$$

Then the problem (2.1) has at most one positive solution, with $u_0 < u(1/2) < u_1$. In case $u_1 = \infty$, we can assert the existence of exactly one solution with $u(1/2) > u_0$. Indeed, under these assumptions the condition (2.2) holds, providing the uniqueness, while existence of solutions is well known (observe that $f(u) \to \infty$ as $u \to \infty$). This example roughly constitutes the Theorem 3.4.3 in R. Schaaf [4].

The Theorem 2.1 may be used to extend many exact multiplicity results, as the following example illustrates.

Example 2. Consider the problem

$$u'' + 0.4\,\lambda u(u-1)(u-3)(u-7) = 0, \;\; x \in (0,1), \;\; u(0) = u(1) = 0. \tag{2.10}$$

Then there exists a critical $\lambda_0 > 0$ so that for $0 < \lambda < \lambda_0$ the problem (2.10) has exactly one positive solution, it has exactly two positive solutions at $\lambda = \lambda_0$, and exactly three positive solutions for $\lambda > \lambda_0$, see the bifurcation diagram in Figure 1, where we draw $umax = u(1/2)$ versus λ. The diagram was generated by the actual computation of the solutions, using *Mathematica*. (The constant 0.4 in front of the nonlinear term makes the picture look better.) We provide a proof next.

Observe that the function $f(u) = 0.4\,u(u-1)(u-3)(u-7)$ has inflection points $u_1 \simeq 1.2$ and $u_2 \simeq 4.3$. In particular, it changes concavity exactly once on $(0,3)$. By the Theorem 2.3 of P. Korman, Y. Li and T. Ouyang [2] the solution set for $u(0) < 3$ is a parabola-like curve, i.e. there exists a critical λ_0 so the problem has zero, one or two solutions with $u(0) < 3$, depending on whether $0 < \lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Since $f(u)$ is convex for $u > 7$, by the Theorem 2.1 there is at most one positive solution with $u(0) > 7$. Since existence of solution in this range is well known, we conclude the proof.

In fact we have the following general exact multiplicity result.

Theorem 2.3 *Assume that the function $f(u) \in C^2(\bar{R}_+)$ has four roots $0 \leq a < b < c < d$.*

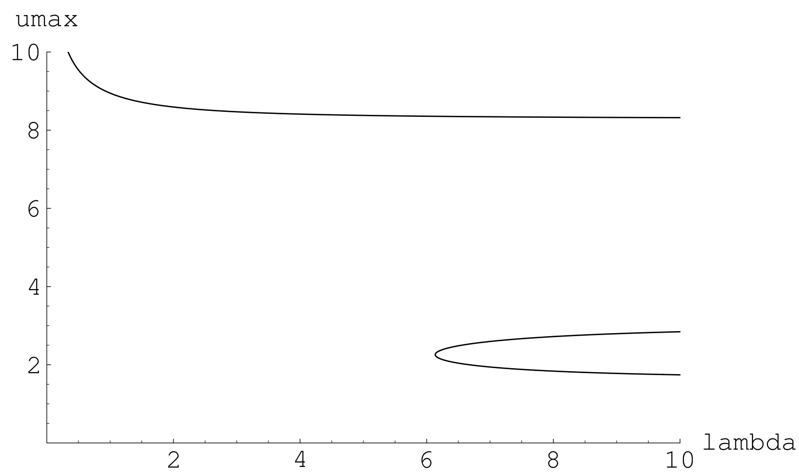

Figure 1: Bifurcation diagram for the problem (2.10).

(i) In case $a = 0$ we assume that that $f(u) > 0$ on $(b,c) \cup (d,\infty)$ and $f(u) < 0$ on $(0,b) \cup (c,d)$. We also assume that there is a point $\theta \in (b,c)$ so that $f(u)$ is convex on $(0,\theta)$ and concave on (θ,c). Let $\int_0^c f(u)\,du > 0$, and assume finally that

$$f'(u)\int_d^u f(t)\,dt - \frac{1}{2}f^2(u) > 0, \quad \text{for all } u > d. \tag{2.11}$$

Then there exists a critical $\lambda_0 > 0$ so that for $0 < \lambda < \lambda_0$ the problem (2.1) has exactly one positive solution, it has exactly two positive solutions at $\lambda = \lambda_0$, and exactly three positive solutions for $\lambda > \lambda_0$, and the generic bifurcation diagram is the same as in Figure 1.
(ii) In case $a > 0$ we assume that that $f(u) > 0$ on $(0,a) \cup (b,c) \cup (d,\infty)$ and $f(u) < 0$ on $(a,b) \cup (c,d)$. We also assume that there is a point $\theta \in (b,c)$ so that $f(u)$ is convex on $(0,\theta)$ and concave on (θ,c). Let $\int_a^c f(u)\,du > 0$, and assume that (2.11) holds. We define two points $\beta, r \in (b,c)$ by the relations $(\beta - a)f'(\beta) = f(\beta)$, and $\int_a^r f(u)\,du = 0$. We define $i(u) = f^2(u) - 2F(u)f'(u)$, and assume finally that either $f'(r) < 0$ or $i(\beta) > 0$ holds. Then there exists a critical $\lambda_0 > 0$ so that for $0 < \lambda < \lambda_0$ the problem (2.1) has exactly two positive solution, it has exactly three positive solutions at $\lambda = \lambda_0$, and exactly four positive solutions for $\lambda > \lambda_0$, and the generic bifurcation diagram is the same as in Figure 2.

We remark that condition (2.11) holds, provided that $f''(u) > 0$ for $u > d$.

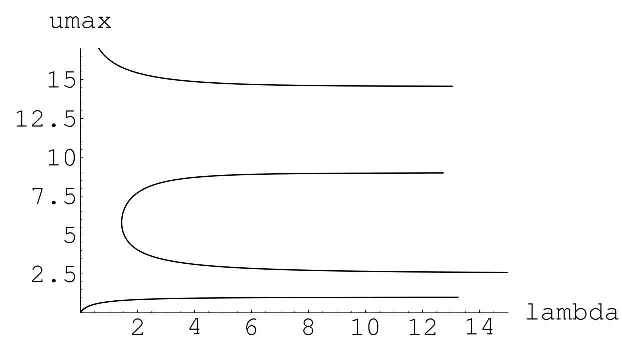

Figure 2: Bifurcation diagram for the problem (2.12).

Proof: In case $a = 0$ the proof is exactly the same as in the Example 2. In case $a > 0$ the proof is similar, this time using the Theorem 2.7 in P. Korman, Y. Li and T. Ouyang [2].

Example 3. Consider the problem

(2.12) $u'' + 0.2\,\lambda(u-1)(u-2)(u-9)(u-13) = 0, \;\; u(0) = u(1) = 0.$

For the function $f(u) = 0.2\;(u-1)(u-2)(u-9)(u-13)$ we calculate the inflection points $u_1 \simeq 3.38$ and $u_2 \simeq 9.12$. In particular, it changes concavity exactly once on $(0,9)$, and is convex for $u \geq 13$. We also calculate $\beta \simeq 4.79$ and $i(\beta) \simeq 862 > 0$. The Theorem 2.3 applies, implying existence of exactly two, exactly three, or exactly four positive solutions, depending on the value of λ. The actual bifurcation diagram, computed using the *Mathematica* software, is given in Figure 2.

3 Non-degeneracy of solutions

It was observed in R. Schaaf [4] and in P. Korman and Y. Li [1] that the problem (2.1) has at most one positive solution, provided that $I(u) \equiv f'(u)F(u) - \frac{1}{2}f^2(u)$ does not change sign, where as usual $F(u) = \int_0^u f(t)\,dt$.

Theorem 3.1 *Assume that the function $f(u) \in C^1(\bar{R}_+)$ satisfies $f(u) > 0$ for $u > 0$, and either*

$$(3.1) \qquad I(u) = f'(u)F(u) - \frac{1}{2}f^2(u) > 0 \quad \textit{for almost all } u > 0$$

or the opposite inequality holds. Then any positive solution of (2.1) is non-degenerate, i.e. the corresponding linearized problem

$$w'' + f'(u)w = 0, \quad x \in (0,1), \quad w(0) = w(1) = 0 \tag{3.2}$$

has only a trivial solution.

Proof: If $w(x)$ is a non-trivial solution of (3.2), it is well known that we may assume that $w(x) > 0$ on $(0,1)$, see e.g. [2]. We use a test function $z(x) = \sqrt{F(u(x))}$. One sees that $z(x)$ satisfies

$$z'' + f'(u)z = I\left(\frac{u'^2}{2F^{3/2}} + \frac{1}{F^{1/2}}\right), \quad x \in (0,1), \quad z(0) = z(1) = 0. \tag{3.3}$$

From the equations (3.2) and (3.3) we conclude that

$$\int_0^1 I\left(\frac{u'^2}{2F^{3/2}} + \frac{1}{F^{1/2}}\right) w\,dx = 0,$$

which is a contradiction, since the integrand has the same sign as I.

Remark The condition (3.1) may be replaced by a simpler and more general condition

$$uf'(u) - f(u) \text{ does not change sign for } u > 0. \tag{3.4}$$

(We still assume that $f(u) > 0$ for $u > 0$.)

Indeed, we begin by observing

$$\frac{d^2}{du^2}(\sqrt{F(u)}) = \frac{I}{2F^{3/2}} \equiv J(u),$$

where $J(u)$ has the same sign as $I(u)$. Integrating between some $a > 0$ and $u > 0$,

$$\frac{d}{du}(\sqrt{F(u)}) = \int_a^u J(\xi)\,d\xi + c > 0, \tag{3.5}$$

where $c = \frac{f(a)}{2\sqrt{F(a)}} > 0$. Integrating (3.5),

$$\sqrt{F(u)} = cu + c_1 + \int_a^u (u - \xi)J(\xi)\,d\xi, \tag{3.6}$$

where $c_1 = -\int_0^a \xi J(\xi)\,d\xi$. From (3.6) we find $F(u)$, and then $f(u)$ and $f'(u)$ by differentiation. We then have

$$uf'(u) - f(u) = 2\sqrt{F(u)}J(u)u + 2\int_0^u \xi J(\xi)\,d\xi \left(\int_a^u J(\xi)\,d\xi + c\right).$$

In view of (3.5) it follows that if $J(u)$ is positive (negative), so is $uf'(u) - f(u)$.

Since non-degeneracy of solutions is well known under the condition (3.4), we obtain an alternative proof of the Theorem 3.1.

4 A non-local problem

We begin with a simple observation. It is well known that for any $L > 0$ the problem (here $u = u(x)$)

$$u'' + u^3 = 0, \;\; 0 < x < L, \quad u(0) = u(L) = 0$$

has a unique positive solution, and a unique negative solution. If we now take a positive solution on the interval $(0, L/k)$, followed by the negative solution on $(L/k, 2L/k)$, and so on, then we obtain a solution with $k-1$ sign changes, for any positive integer k.

We now consider a non-local problem, where instead of a second boundary condition we prescribe the average value of the solution on some interval $(0, L)$

$$\begin{gathered} u'' + u^3 = 0, \;\; 0 < x < L, \\ u(0) = 0, \\ \textstyle\int_0^L u(s)\, ds = \alpha, \end{gathered} \tag{4.1}$$

where α is a prescribed constant. We are interested in both positive, negative and sign-changing solutions, i.e. we shall talk of solutions with k sign changes, where $k \geq 0$. Without loss of generality we may assume $\alpha \geq 0$ (otherwise, consider $v = -u$). If $\alpha = 0$, it is clear that there exists exactly two solution of (4.1) with k sign changes, for any odd $k \geq 1$. Indeed, a solution of the equation in (4.1) with $u(0) = u(L) = 0$ having an odd number of roots inside $(0, L)$, and its negative, provide the desired solutions of (4.1). So that we may assume $\alpha > 0$.

Theorem 4.1 *For any $0 < \alpha < \frac{\pi}{\sqrt{2}}$ there exists exactly one solution of (4.1) with k sign changes, for any $k \geq 0$. For $\alpha = \frac{\pi}{\sqrt{2}}$ there exists exactly one solution with k sign changes, for any even $k \geq 0$, and no solutions if k is odd. For any $\alpha > \frac{\pi}{\sqrt{2}}$ the problem (4.1) has no solutions.*

Proof: The problem "scales right". Setting $x = bt$, and $u = \frac{1}{b}v$, we see that $v = v(t)$ satisfies

$$
\begin{aligned}
(4.2) \qquad & v'' + v^3 = 0, \;\; 0 < t < \tfrac{L}{b}, \\
& v(0) = 0, \\
& \textstyle\int_0^{\frac{L}{b}} v(s)\,ds = \alpha.
\end{aligned}
$$

Comparing with (4.1), we see that only the length of the interval has changed. Hence we have a one-to-one map between the solution sets on any two intervals. So consider a solution $U(x)$ of the equation $u'' + u^3 = 0$, with $u(0) = 0$, which has k sign changes, whose roots are $x = 1, 2, \ldots$, and such that $U(x) > 0$ on $(0,1)$, $U(x) < 0$ on $(1,2)$, and so on. According to the formula (2.6), the integral of $U(x)$ over any of its positive humps is equal to $\frac{\pi}{\sqrt{2}}$, while the integral of $U(x)$ over any of its negative humps is $-\frac{\pi}{\sqrt{2}}$. Imagine cutting this solution with a sliding vertical line $x = \xi$. By continuity, for any $\alpha \in (0, \frac{\pi}{\sqrt{2}}]$ we can find a unique $\xi \in (0,1]$ so that $U(x)$ is positive solution of (4.1) on the interval $(0,\xi)$. We then map this solution to the original interval $(0,L)$ by the above transformation. Similarly, for any $\alpha \in (0, \frac{\pi}{\sqrt{2}})$ we can find a unique $\xi \in (1,2)$ so that we have a solution of (4.1) on the interval $(0,\xi)$, with exactly one sign change. We then map $U(x)$ to the original interval, as before. Similarly we construct solutions with arbitrarily many sign changes.

By the Theorem 2.1, no solution is possible in case $\alpha > \frac{\pi}{\sqrt{2}}$.

References

[1] P. Korman and Y. Li, Generalized averages of solutions of two-point Dirichlet problems, *J. Math. Anal. Applic.* **239**, 478-484 (1999).

[2] P. Korman, Y. Li and T. Ouyang, Exact multiplicity results for boundary-value problems with nonlinearities generalising cubic, *Proc. Royal Soc. Edinburgh, Ser. A* **126A**, 599-616 (1996).

[3] T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, II *J. Differential Equations* **158** no. 1, 94-151 (1999).

[4] R. Schaaf, Global Solution Branches of Two Point Boundary Value Problems, Lecture Notes in Mathematics, no. **1458**, Springer-Verlag (1990).

[5] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations* **39**, 269-290 (1981).

[6] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, *J. Differential Equations* **77**, 199-202 (1989).