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Families of Solution Curves for Some Non-autonomous Problems

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Abstract The paper studies families of positive solution curves for non-autonomous two-point problems

$$u'' + \lambda f(u) - \mu g(x) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0,$$

depending on two positive parameters λ and μ. We regard λ as a primary parameter, giving us the solution curves, while the secondary parameter μ allows for evolution of these curves. We give conditions under which the solution curves do not intersect, and the maximum value of solutions provides a global parameter. Our primary application is to constant yield harvesting for diffusive logistic equation. We implement numerical computations of the solution curves, using continuation in a global parameter, a technique that we developed in (Korman in Nonlinear Anal. 93:226, 2013).

Keywords Families of solution curves · Fishing models · Numerical computations

Mathematics Subject Classification 34B15 · 92D25

1 Introduction

We study positive solutions of non-autonomous two-point problems

$$u'' + \lambda f(u) - \mu g(x) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{1.1}$$

depending on two positive parameters λ and μ. We assume that $f(u) \in C^2(\bar{R}_+)$, and $g(x) \in C^1(-1,1) \cap C[-1,1]$ satisfies

$$g(-x) = g(x), \quad \text{for } x \in (0,1), \tag{1.2}$$

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$$g(0) > 0, \quad \text{and} \quad xg'(x) \geq 0, \quad \text{for } x \in (-1, 1). \tag{1.3}$$

In case $g(x)$ is a constant, one can use the time map method, see K.C. Hung and S.H. Wang [7, 8] who have studied similar multiparameter problems, or the book by S.P. Hastings and J.B. McLeod [6]. We show that under the conditions (1.2) and (1.3) one can still get detailed results on the solution curves $u = u(x, \lambda)$, where we regard λ as a primary parameter, and on the evolution of these curves when the secondary parameter μ changes. We say that the solution curves $u = u(x, \lambda)$ are the λ-*curves*. We also consider the μ-curves, by regarding λ as the secondary parameter.

By B. Gidas, W.-M. Ni and L. Nirenberg [4], any positive solution of (1.1) is an even function, and moreover $u'(x) < 0$ for $x \in (0, 1)$. It follows that $u(0)$ is the maximum value of the solution $u(x)$. Our first result says that $u(0)$ is a *global parameter*, i.e., its value uniquely determines the solution pair $(\lambda, u(x))$ (μ is assumed to be fixed). It follows that a planar curve $(\lambda, u(0))$ gives a faithful representation of the solution set of (1.1), so that $(\lambda, u(0))$ describes the *global solution curve*. Then we show positivity of any non-trivial solution of the linearized problem for (1.1). This allows us to compute the direction of turn for convex and concave $f(u)$.

Turning to the secondary parameter μ, we show that solution curves at different μ's do not intersect, which allows us to discuss the evolution of solution curves in μ.

We apply our results to a logistic model with fishing. S. Oruganti, J. Shi, and R. Shivaji [16] considered a class of general elliptic equations on an arbitrary domain, which includes

$$u'' + \lambda u(1-u) - \mu g(x) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0.$$

They proved that for λ sufficiently close to the principal eigenvalue λ_1, the μ-curves are as in Fig. 2 below. We show that such curves are rather special, with the solution curves as in Fig. 3 below being more common. Our approach is to study the λ-curves first, leading to the understanding of the μ-curves. We obtain an exhaustive result in case $g(x)$ is a constant. The parameter $\mu > 0$ quantifies the amount of fishing in the logistic model. We also consider the case $\mu < 0$, corresponding to "stocking" of fish. Previous work on the logistic equation with harvesting also includes D.G. Costa et al. [3], and P. Girão and H. Tehrani [5].

Using the fact that $u(0)$ is a global parameter, we implement numerical computations of the solution curves, illustrating our results. We use continuation in a global parameter, a technique that we developed in [11].

2 Families of Solution Curves

The following result is included in B. Gidas, W.-M. Ni and L. Nirenberg [4], see also P. Korman [9] for an elementary proof.

Lemma 2.1 *Under the conditions* (1.2) *and* (1.3), *any positive solution of* (1.1) *is an even function*, *with* $u'(x) < 0$ *for all* $x \in (0, 1]$, *so that* $x = 0$ *is a point of global maximum.*

We begin by considering the secondary parameter μ to be fixed. To stress that, we call $h(x) = \mu g(x)$, and consider positive solutions of

$$u'' + \lambda f(u) - h(x) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{2.1}$$

Lemma 2.2 *Assume that $f(u) \in C(\bar{R}_+)$ satisfies $f(u) > 0$ for $u > 0$, and $h(x)$ satisfies the conditions (1.2) and (1.3). Then $u(0)$, the maximum value of any positive solution, uniquely identifies the solution pair $(\lambda, u(x))$.*

Proof Observe from (2.1) that $f(u(0)) > 0$ for any positive solution $u(x)$. Let $(\lambda_1, v(x))$ be another solution of (2.1), with $v(0) = u(0)$, $v'(0) = u'(0) = 0$, and $\lambda_1 > \lambda$. From Eq. (2.1), $v''(0) < u''(0)$, and hence $v(x) < u(x)$ for small $x > 0$. Let $\xi \le 1$ be their first point of intersection, i.e., $u(\xi) = v(\xi)$. Clearly

$$\left|v'(\xi)\right| \le \left|u'(\xi)\right|. \tag{2.2}$$

Multiplying Eq. (2.1) by u', and integrating over $(0, \xi)$, we get

$$\frac{1}{2}u'^2(\xi) + \lambda\left[F\left(u(\xi)\right) - F\left(u(0)\right)\right] - \int_0^\xi h(x)u'(x)\,dx = 0,$$

where $F(u) = \int_0^u f(t)\,dt$. Integrating by parts, we conclude

$$\frac{1}{2}u'^2(\xi) = \lambda\left[F\left(u(0)\right) - F\left(u(\xi)\right)\right] + h(\xi)u(\xi) - h(0)u(0) - \int_0^\xi h'(x)u(x)\,dx.$$

Similarly,

$$\frac{1}{2}v'^2(\xi) = \lambda_1\left[F\left(u(0)\right) - F\left(u(\xi)\right)\right] + h(\xi)u(\xi) - h(0)u(0) - \int_0^\xi h'(x)v(x)\,dx,$$

and then, subtracting,

$$\frac{1}{2}\left[u'^2(\xi) - v'^2(\xi)\right] = (\lambda - \lambda_1)\left[F\left(u(0)\right) - F\left(u(\xi)\right)\right] + \int_0^\xi h'(x)\left(v(x) - u(x)\right)dx.$$

Since $v(x) < u(x)$ on $(0, \xi)$, the second term on the right is non-positive, while the first term on the right is negative, since $F(u)$ is an increasing function. It follows that $|u'(\xi)| < |v'(\xi)|$, which contradicts (2.2). $\square$

Lemma 2.3 *Assume that $f(u) \in C(\bar{R}_+)$, and $h(x)$ satisfies the conditions (1.2) and (1.3). Then the curves of positive solutions of (1.1) in $(\lambda, u(0))$ plane, computed at different μ's, do not intersect.*

Proof Assume, on the contrary, that $v(x)$ is a solution of

$$v'' + \lambda f(v) - \mu_1 g(x) = 0, \quad -1 < x < 1, \qquad v(-1) = v(1) = 0, \tag{2.3}$$

with $\mu_1 > \mu$, but $u(0) = v(0)$, where $u(x)$ is a solution of (1.1). Then $u''(0) < v''(0)$, and hence $u(x) < v(x)$ for small $x > 0$. Let $\xi \le 1$ be their first point of intersection, i.e., $u(\xi) = v(\xi)$. Clearly

$$\left|u'(\xi)\right| \le \left|v'(\xi)\right|. \tag{2.4}$$

Multiplying Eq. (1.1) by u', and integrating over $(0, \xi)$, we get

$$\frac{1}{2}u'^2(\xi) + \lambda\left[F\left(u(\xi)\right) - F\left(u(0)\right)\right] = \mu\int_0^\xi g(x)u'(x)\,dx.$$

Similarly, using (2.3), we get

$$\frac{1}{2}v'^2(\xi)+\lambda\big[F\big(u(\xi)\big)-F\big(u(0)\big)\big]=\mu_1\int_0^\xi g(x)v'(x)\,dx.$$

Subtracting, we obtain

$$\begin{aligned}\frac{1}{2}\big[u'^2(\xi)-v'^2(\xi)\big]&=\mu\int_0^\xi g(x)u'(x)\,dx-\mu_1\int_0^\xi g(x)v'(x)\,dx\\&>\mu_1\left[\int_0^\xi g(x)u'(x)\,dx-\int_0^\xi g(x)v'(x)\,dx\right]\\&=\mu_1\int_0^\xi g'(x)\big(v(x)-u(x)\big)\,dx>0.\end{aligned}$$

Hence, $|u'(\xi)|>|v'(\xi)|$, which contradicts (2.4). □

Corollary 1 *Assume that* λ *is fixed in* (1.1), *and* μ *is the primary parameter. Assume that* $f(u)\in C(\bar{R}_+)$ *satisfies* $f(u)>0$ *for* $u>0$, *and* $g(x)$ *satisfies the conditions* (1.2) *and* (1.3). *Then the maximum value of solution* $u(0)$ *is a global parameter, i.e., it uniquely identifies the solution pair* $(\mu,u(x))$.

Proof If at some λ_0 we had another solution pair $(\mu_1,u_1(x))$ with $u(0)=u_1(0)$, then the λ-curves at μ and μ_1 would intersect at $(\lambda_0,u(0))$, contradicting Lemma 2.2. □

The linearized problem for (1.1) is

$$w''+\lambda f'(u)w=0,\quad -1<x<1,\qquad w(-1)=w(1)=0. \tag{2.5}$$

We call the solution of (1.1) *singular* if (2.5) has non-trivial solutions. Since the solution set of (2.5) is one-dimensional (parameterized by $w'(-1)$), it follows that $w(-x)=w(x)$, and $w'(0)=0$.

Lemma 2.4 *Assume that* $f(u)\in C^1(\bar{R}_+)$, *and* $g(x)$ *satisfies the conditions* (1.2) *and* (1.3), *and let* $u(x)$ *be a positive solution of* (1.1). *Then any non-trivial solution of* (2.5) *is of one sign, i.e., we may assume that* $w(x)>0$ *for all* $x\in(-1,1)$.

Proof Assuming the contrary, we can find a point $\xi\in(0,1)$ such that $w(\xi)=w(1)=0$, and $w(x)>0$ on $(\xi,1)$. Differentiate Eq. (1.1)

$$u'''+\lambda f'(u)u'-\mu g'(x)=0.$$

Combining this with (2.5),

$$\big(u'w'-u''w\big)'+\mu g'(x)=0. \tag{2.6}$$

Integrating over $(\xi,1)$,

$$u'(1)w'(1)-u'(\xi)w'(\xi)+\mu\int_\xi^1 g'(x)\,dx=0.$$

All three terms on the left are non-negative, and the second one is positive, which results in a contradiction. □

Lemma 2.5 *Assume that $f(u) \in C(\bar{R}_+)$, and $g(x)$ satisfies the conditions (1.2) and (1.3). Let $u(x)$ be a positive and singular solution of (1.1), with $u'(1) < 0$, and $w(x) > 0$ a solution of (2.5). Then*

$$\int_0^1 f(u)w\,dx > 0.$$

Proof By (2.6), the function $u'w' - u''w$ is non-increasing on $(0, 1)$, and hence

$$u'w' - u''w \geq u'(1)w'(1).$$

Integrating over $(0, 1)$, and expressing u'' from (1.1), gives

$$2\int_0^1 w\big(\lambda f(u) - \mu g(x)\big)\,dx \geq u'(1)w'(1) > 0,$$

which implies the lemma. □

Theorem 2.1 *Assume that $f(u) \in C^1(\bar{R}_+)$, and $g(x)$ satisfies the conditions (1.2) and (1.3). Then positive solutions of the problem (1.1) can be continued globally either in λ or in μ, on smooth solution curves, so long as $u'(1) < 0$.*

Proof At any non-singular solution of (1.1), the implicit function theorem applies (see e.g., L. Nirenberg [13], or P. Korman [10] for more details), while at the singular solutions the Crandall-Rabinowitz [2] bifurcation theorem applies, with Lemma 2.5 verifying its crucial "transversality condition", see e.g., P. Korman [10] (or [12, 15]) for more details. In either case we can always continue the solution curves. □

Theorem 2.2

(i) *Assume that $f(u) \in C^2[0, \infty)$ is concave. Then only turns to the right are possible in the $(\lambda, u(0))$ plane, when solutions are continued in λ, and only turns to the left are possible in the $(\mu, u(0))$ plane, when solutions are continued in μ.*

(ii) *Assume that $f(u) \in C^2[0, \infty)$ is convex. Then only turns to the left are possible in the $(\lambda, u(0))$ plane, when solutions are continued in λ, and only turns to the right are possible in the $(\mu, u(0))$ plane, when solutions are continued in μ.*

Proof Assume that when continuing in λ, we encounter a critical point (λ_0, u_0), i.e., the problem (2.5) has a non-trivial solution $w(x) > 0$. By Lemma 2.5, the Crandall-Rabinowitz [2] bifurcation theorem applies. This theorem implies that the solution set near (λ_0, u_0) is given by a curve $(\lambda(s), u(s))$ for $s \in (-\delta, \delta)$, with $\lambda(s) = \lambda_0 + \frac{1}{2}\lambda''(0)s^2 + o(s^2)$, and

$$\lambda''(0) = -\lambda_0 \frac{\int_{-1}^1 f''(u)w^3\,dx}{\int_{-1}^1 f(u)w\,dx},$$

see e.g., [10]. When $f(u)$ is concave (convex), $\lambda''(0)$ is positive (negative), and a turn to the right (left) occurs on the solution curve.

If a critical point (μ_0, u_0) is encountered when continuing in μ, the Crandall-Rabinowitz [2] bifurcation theorem implies that the solution set near (μ_0, u_0) is given by $(\mu(s), u(s))$ for $s \in (-\delta, \delta)$, with $\mu(s) = \mu_0 + \frac{1}{2}\mu''(0)s^2 + o(s^2)$, and

$$\mu''(0) = \lambda \frac{\int_{-1}^{1} f''(u) w^3 \, dx}{\int_{-1}^{1} g(x) w \, dx},$$

see e.g., [14]. When $f(u)$ is concave (convex), $\mu''(0)$ is negative (positive), and a turn to the left (right) occurs on the solution curve. $\square$

3 Numerical Computation of the Solution Curves

In this section we present computations of the global curves of positive solutions for the problem (1.1), which are based on our paper [11]. We assume that the conditions of Lemma 2.2 hold, so that $\alpha \equiv u(0)$ is a global parameter. We think of the parameter μ as secondary, and we begin with the problem (2.1) (i.e., we set $\mu g(x) = h(x)$). Since any positive solution $u(x)$ is an even function, we shall compute it on the half-interval (0, 1), by solving

$$u'' + \lambda f(u) - h(x) = 0 \quad \text{for } 0 < x < 1, \qquad u'(0) = u(1) = 0. \tag{3.1}$$

A standard approach to numerical computations involves continuation in λ by using the predictor-corrector methods, see e.g., E.L. Allgower and K. Georg [1]. These methods are well developed, but not easy to implement, because the solution curve $u = u(x, \lambda)$ may consist of several parts, each having multiple turns. Here λ is a local parameter, but not a global one.

Since $\alpha = u(0)$ is a global parameter, we shall compute the solution curve $(\lambda, u(0))$ of (3.1) in the form $\lambda = \lambda(\alpha)$, with $\alpha = u(0)$. If we solve the initial value problem

$$u'' + \lambda f(u) - h(x) = 0, \quad u(0) = \alpha, \qquad u'(0) = 0, \tag{3.2}$$

then we need to find λ, so that $u(1) = 0$, in order to obtain the solution of (3.1), with $u(0) = \alpha$. Rewrite Eq. (3.2) in the integral form

$$u(x) = \alpha - \lambda \int_0^x (x - t) f\big(u(t)\big) \, dt + \int_0^x (x - t) h(t) \, dt,$$

and then the equation for λ is

$$F(\lambda) \equiv u(1) = \alpha - \lambda \int_0^1 (1 - t) f\big(u(t)\big) \, dt + \int_0^1 (1 - t) h(t) \, dt = 0. \tag{3.3}$$

We solve this equation by using Newton's method

$$\lambda_{n+1} = \lambda_n - \frac{F(\lambda_n)}{F'(\lambda_n)}.$$

We have

$$F(\lambda_n) = \alpha - \lambda_n \int_0^1 (1 - t) f\big(u(t, \lambda_n)\big) \, dt + \int_0^1 (1 - t) h(t) \, dt,$$

$$F'(\lambda_n) = -\int_0^1 (1 - t) f\big(u(t, \lambda_n)\big) \, dt - \lambda_n \int_0^1 (1 - t) f'\big(u(t, \lambda_n)\big) u_\lambda \, dt,$$

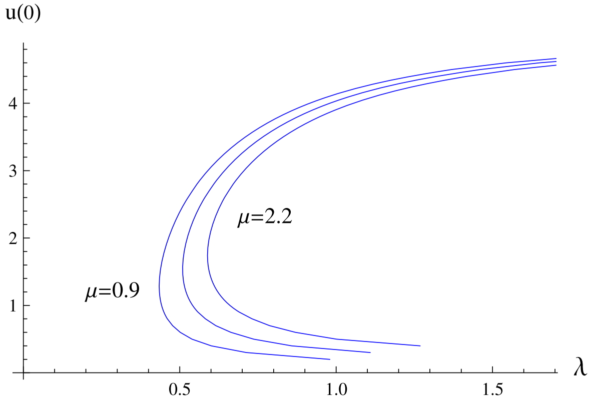

Fig. 1 The curve of positive solutions for the problem (3.6) at $\mu = 0.9$, $\mu = 1.5$ and $\mu = 2.2$

where $u(x, \lambda_n)$ and u_λ are respectively the solutions of

$$u'' + \lambda_n f(u) - h(x) = 0, \qquad u(0) = \alpha, \qquad u'(0) = 0, \tag{3.4}$$

$$u''_\lambda + \lambda_n f'\big(u(x, \lambda_n)\big)u_\lambda + f\big(u(x, \lambda_n)\big) = 0, \qquad u_\lambda(0) = 0, \qquad u'_\lambda(0) = 0. \tag{3.5}$$

(As we vary λ, we keep $u(0) = \alpha$ fixed, that is why $u_\lambda(0) = 0$.) This method is very easy to implement. It requires repeated solutions of the initial value problems (3.4) and (3.5) (using the NDSolve command in *Mathematica*).

Example Using *Mathematica* software, we have computed the solution curves in $(\lambda, u(0))$ plane for the problem

$$u'' + \lambda u(10 - 2u) - \mu\big(1 + 0.2x^2\big) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{3.6}$$

at $\mu = 0.9$, $\mu = 1.5$ and $\mu = 2.2$. Results are presented in Fig. 1. (The curve in the middle corresponds to $\mu = 1.5$.)

We now discuss numerical continuation of solutions in the secondary parameter. We consider again

$$u'' + \lambda f(u) - \mu g(x) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{3.7}$$

and assume λ to be fixed, and we continue the solutions in μ. By Corollary 1, $\alpha = u(0)$ is a global parameter, and we shall compute the solution curve $(\mu, u(0))$ of (3.1) in the form $\mu = \mu(\alpha)$. As before, to find $\mu = \mu(\alpha)$, we need to solve

$$F(\mu) \equiv u(1) = \alpha - \lambda \int_0^1 (1-t) f\big(u(t)\big)\, dt + \mu \int_0^1 (1-t) g(t)\, dt = 0.$$

We solve this equation by using Newton's method

$$\mu_{n+1} = \mu_n - \frac{F(\mu_n)}{F'(\mu_n)},$$

with

$$F(\mu_n) = \alpha - \lambda \int_0^1 (1-t) f\big(u(t, \mu_n)\big)\, dt + \mu_n \int_0^1 (1-t) g(t)\, dt,$$

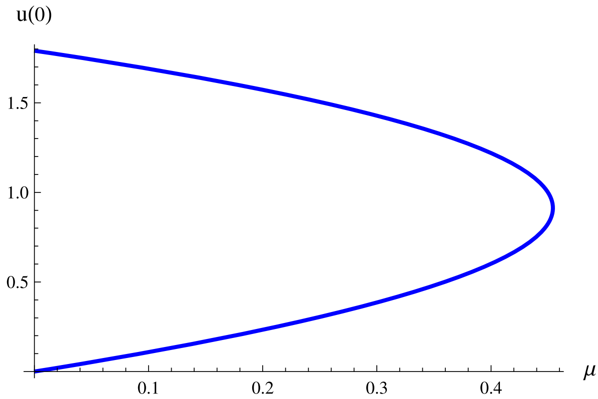

Fig. 2 The curve of positive solutions for the problem (3.8)

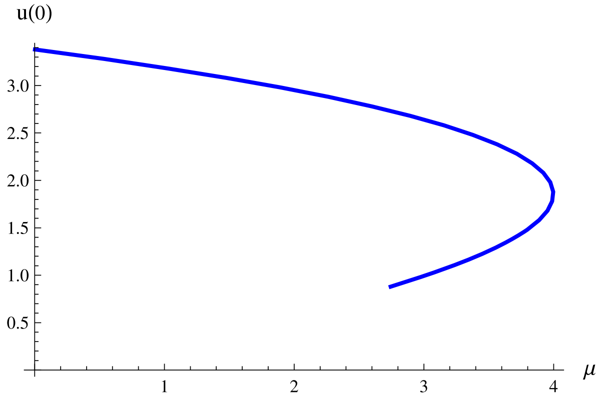

Fig. 3 The curve of positive solutions for the problem (3.9)

$$F'(\mu_n) = -\lambda \int_0^1 (1-t) f'\big(u(t,\mu_n)\big) u_\mu \, dt + \int_0^1 (1-t) g(t)\, dt,$$

where $u(x,\mu_n)$ and u_μ are respectively the solutions of

$$u'' + \lambda f(u) - \mu_n g(x) = 0, \qquad u(0) = \alpha, \qquad u'(0) = 0,$$
$$u''_\mu + \lambda f'\big(u(x,\mu_n)\big) u_\mu - g(x) = 0, \qquad u_\mu(0) = 0, \qquad u'_\mu(0) = 0.$$

(As we vary μ, we keep $u(0) = \alpha$ fixed, that is why $u_\mu(0) = 0$.)

Example We have continued in μ the positive solutions of

$$u'' + u(4-u) - \mu\big(1+x^2\big) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{3.8}$$

The curve of positive solutions is given in Fig. 2.

Example We have continued in μ the positive solutions of

$$u'' + 2.4u(4-u) - \mu\big(1+x^2\big) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{3.9}$$

The curve of positive solutions is given in Fig. 3. At $\mu \approx 2.28634$, the solutions become sign changing, negative near $x = \pm 1$.

4 Diffusive Logistic Equation with Harvesting

Recall that the eigenvalues of

$$u'' + \lambda u = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0$$

are $\lambda_n = \frac{n^2\pi^2}{4}$, and in particular $\lambda_1 = \frac{\pi^2}{4}$, $\lambda_2 = \pi^2$.

We consider positive solutions of

$$u'' + \lambda u(1-u) - \mu = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{4.1}$$

with positive parameters λ and μ. It is easy to see that no positive solutions exist if $\lambda \le \lambda_1 = \frac{\pi^2}{4}$, and by maximum principle any positive solution satisfies $0 < u(x) < 1$. The following result gives a complete description of the set of positive solutions.

Theorem 4.1 *For any fixed μ the set of positive solutions of (4.1) is a parabola-like curve opening to the right in $(\lambda, u(0))$ plane (the λ-curves). The upper branch continues for all λ after the turn, while the solutions on the lower branch become sign-changing after some $\lambda = \bar{\lambda}$ ($u_x(\pm 1, \bar{\lambda}) = 0$, see Fig. 1). For any fixed λ the set of positive solutions of (4.1) is a parabola-like curve opening to the left in $(\mu, u(0))$ plane (the μ-curves). Different λ-curves (and different μ-curves) do not intersect. The λ-curves and the μ-curves share the turning points. Namely, if at $\mu = \mu_0$, the λ-curve turns at the point (λ_0, α), then at $\lambda = \lambda_0$, the μ-curve turns at the point (μ_0, α).*

If $\lambda \in (\lambda_1, \lambda_2]$, then the μ-curve joins the point $(0, \mu_1)$, with some $\mu_1 > 0$, to $(0, 0)$, with exactly one turn to the left at some μ_0 (as in Fig. 2). If $\lambda > \lambda_2$, then the μ-curve joins the point $(0, \mu_1)$, with some $\mu_1 > 0$, to some point $(\bar{\mu} > 0, \alpha > 0)$, with exactly one turn to the left at some $\mu_0 > \bar{\mu}$ (as in Fig. 3). Solutions on the lower branch become sigh-changing for $\mu < \bar{\mu}$.

We remark that in case $\lambda \in (\lambda_1, \lambda_2]$, a more general result was given in J. Shi [16], by a more involved method.

The proof will depend on several lemmas, which we state for a more general problem

$$u'' + \lambda f(u) - \mu = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{4.2}$$

Lemma 4.1 *Assume that $f(u) \in C^1(\bar{R}_+)$ satisfies $f(0) = 0$, and $f(u(x)) > 0$ for any positive solution of (4.2), for all $x \in (-1, 1)$. Assume that $u(x, \lambda)$ arrives at the point λ_0 where the positivity of solutions is lost (i.e., $u_x(\pm 1, \lambda_0) = 0$) with the maximum value $u(0, \lambda)$ decreasing along the solution curve. Then the positivity is lost forward in λ at λ_0. (I.e., $u(x, \lambda) > 0$ for all $x \in (-1, 1)$ if $\lambda < \lambda_0$, and $u(x, \lambda)$ is sign-changing for $\lambda > \lambda_0$.)*

Proof Since for positive solutions $u_x(x, \lambda_0) < 0$ for $x \in (0, 1)$, the only way for solutions to become sign-changing is to have $u_x(\pm 1, \lambda_0) = 0$. Assume, on the contrary, that positivity is lost backward in λ. Then by our assumption

$$u_\lambda(0, \lambda_0) \ge 0. \tag{4.3}$$

Differentiating Eq. (4.2) in λ, we have

$$u_\lambda'' + \lambda f'(u) u_\lambda = -f(u), \quad -1 < x < 1, \qquad u_\lambda(-1) = u_\lambda(1) = 0. \tag{4.4}$$

Differentiating Eq. (4.2) in x, gives

$$u_x'' + \lambda f'(u) u_x = 0. \tag{4.5}$$

Combining Eqs. (4.4) and (4.5),

$$\left(u_\lambda' u' - u_\lambda u''\right)' = -f(u) u_x > 0, \quad \text{for } x \in (0, 1).$$

It follows that the function $q(x) \equiv u_\lambda' u' - u_\lambda u''$ is increasing, with $q(0) \geq 0$ by (4.3), and $q(1) = 0$, a contradiction. □

Lemma 4.2 *Assume that $f(u) \in C^1(\bar{R}_+)$ satisfies $f(0) = 0$, and $f(u(x)) > 0$ for any positive solution of (4.2), for all $x \in (-1, 1)$. Assume that $u(x, \mu)$ arrives at the point μ_0 where the positivity is lost (i.e., $u_x(\pm 1, \mu_0) = 0$) with the maximum value $u(0, \mu)$ decreasing along the solution curve. Then the positivity is lost backward in μ at μ_0. (I.e., $u(x, \mu) > 0$ for all $x \in (-1, 1)$ if $\mu > \mu_0$, and $u(x, \mu)$ is sign-changing for $\mu < \mu_0$.)*

Proof Assume, on the contrary, that positivity is lost forward in μ. Then by our assumption

$$u_\mu(0, \mu_0) \leq 0. \tag{4.6}$$

Differentiating Eq. (4.2) in μ, we have

$$u_\mu'' + \lambda f'(u) u_\mu = 1, \quad -1 < x < 1, \qquad u_\mu(-1) = u_\mu(1) = 0. \tag{4.7}$$

Combining Eqs. (4.5) and (4.7),

$$\left(u_\mu' u' - u_\mu u''\right)' = u_x < 0, \quad \text{for } x \in (0, 1).$$

It follows that the function $r(x) \equiv u_\mu' u' - u_\mu u''$ is decreasing, with $r(0) \leq 0$ by (4.6), and $r(1) = 0$, a contradiction. □

Lemma 4.3 *Assume that $f(u) \in C^1[0, \infty)$, $f(0) = 0$, and $f'(u)$ is a decreasing function for $u > 0$. Then for any $\lambda > 0$ there is at most one solution pair $(\mu, u(x))$, with $u'(\pm 1) = 0$.*

Proof Assume, on the contrary, that there are two solution pairs $(\mu_1, u(x))$ and $(\mu_2, v(x))$, with $\mu_2 > \mu_1$, satisfying

$$u'' + \lambda f(u) - \mu_1 = 0, \quad -1 < x < 1, \qquad u(\pm 1) = u'(\pm 1) = 0, \tag{4.8}$$

$$v'' + \lambda f(v) - \mu_2 = 0, \quad -1 < x < 1, \qquad v(\pm 1) = v'(\pm 1) = 0. \tag{4.9}$$

Since $v''(1) = \mu_2 > \mu_1 = u''(1)$, we have $v(x) > u(x)$ for x close to 1. Two cases are possible.

(i) $v(x) > u(x)$ for $x \in (0, 1)$. Differentiating Eqs. (4.8) and (4.9), we get

$$u_x'' + \lambda f'(u) u_x = 0, \quad u_x < 0 \quad \text{on } (0, 1), \qquad u_x(0) = u_x(1) = 0,$$
$$v_x'' + \lambda f'(v) v_x = 0, \quad v_x < 0 \quad \text{on } (0, 1), \qquad v_x(0) = v_x(1) = 0.$$

Since $f'(u) > f'(v)$, we have a contradiction by Sturm's comparison theorem.

(ii) There is $\xi \in (0,1)$ such that $v(x) > u(x)$ for $x \in (\xi, 1)$, while $v(\xi) = u(\xi)$ and $u'(\xi) \le v'(\xi) < 0$. Multiply Eq. (4.8) by u', and integrate over $(\xi, 1)$ (with $F(u) = \int_0^u f(t)\,dt$)

$$-\frac{1}{2}u'^2(\xi) - \lambda F\big(u(\xi)\big) + \mu_1 u(\xi) = 0.$$

Similarly, from (4.9)

$$-\frac{1}{2}v'^2(\xi) - \lambda F\big(u(\xi)\big) + \mu_2 u(\xi) = 0.$$

Subtracting

$$(\mu_2 - \mu_1)u(\xi) = \frac{1}{2}\big(v'^2(\xi) - u'^2(\xi)\big).$$

The quantity on the left is positive, while the one on the right is non-positive, a contradiction. □

Lemma 4.4 *For any $\alpha \in (0, \frac{3}{4})$ there exists a unique pair $(\bar\lambda, \bar\mu)$, with $\bar\lambda > \lambda_2$ and $\bar\mu > 0$, and a positive solution of (4.1) with $u(0) = \alpha$ and $u'(\pm 1) = 0$. Moreover, if $\bar\mu \to 0$, then $\bar\lambda \downarrow \lambda_2 = \pi^2$.*

Proof Multiplying Eq. (4.1) by u', we see that the solution with $u'(\pm 1) = 0$ satisfies

$$\frac{1}{2}u'^2 + \bar\lambda\left(\frac{1}{2}u^2 - \frac{1}{3}u^3\right) - \bar\mu u = 0. \tag{4.10}$$

Evaluating this at $x = 0$

$$\bar\lambda\left(\frac{1}{2}\alpha - \frac{1}{3}\alpha^2\right) = \bar\mu. \tag{4.11}$$

We also express from (4.10)

$$\frac{du}{dx} = -\sqrt{2\bar\mu u - \bar\lambda\left(u^2 - \frac{2}{3}u^3\right)}, \quad \text{for } x \in (0,1).$$

We express $\bar\mu$ from (4.11), separate the variables and integrate, getting

$$\int_0^\alpha \frac{du}{\sqrt{(\alpha - \frac{2}{3}\alpha^2)u - (u^2 - \frac{2}{3}u^3)}} = \sqrt{\bar\lambda}.$$

Setting here $u = \alpha v$, we express

$$\bar\lambda = \left(\int_0^1 \frac{dv}{\sqrt{(1 - \frac{2}{3}\alpha)v - (v^2 - \frac{2}{3}\alpha v^3)}}\right)^2. \tag{4.12}$$

The formulas (4.12) and (4.11) let us compute $\bar\lambda$, and then $\bar\mu$, for any $\alpha \in (0, \frac{3}{4})$. (For $\alpha \in (0, \frac{3}{4})$, the quantity inside the square root in (4.12), which is $v(1-v)[1 - \frac{2}{3}\alpha(1+v)]$, is positive for all $v \in (0,1)$.)

If $\bar{\mu} \to 0$, then from (4.11), $\alpha \to 0$ (recall that $\lambda > \lambda_1$), and then from (4.12)

$$\bar{\lambda} \downarrow \left(\int_0^1 \frac{dv}{\sqrt{v - v^2}} \right)^2 = \pi^2,$$

completing the proof. □

Proof of Theorem 4.1 It is easier to understand the λ-curves, so we assume first that μ is fixed. It is well known that for λ large enough the problem (4.1) has a positive stable ("large") solution, with $u(0, \lambda)$ increasing in λ (see e.g., [14]). Let us continue this solution for decreasing λ. This curve does not continue to λ's $\le \lambda_1$, and it cannot become sign-changing while continued to the left, by Lemma 4.1, hence a turn to the right must occur. After the turn, standard arguments imply that solutions develop zero slope at ± 1, and become sign-changing for $\lambda > \bar{\lambda}_\mu$, see e.g., [10]. By Theorem 2.2, exactly one turn occurs on each λ-curve, and by Lemma 4.4, $\inf_\mu \bar{\lambda}_\mu = \lambda_2 = \pi^2$.

Turning to the μ-curves, for any fixed $\tilde{\lambda} > \lambda_1$ we can find a positive solution on the curve $\mu = 0$ (the curve that bifurcates from the trivial solution at $\lambda = \lambda_1$). We now slide down from this point in the $(\lambda, u(0))$ plane, by varying μ. As we increase μ (keeping $\tilde{\lambda}$ fixed), we slide to different λ-curves. At some μ we reach a λ-curve which has its turn at $\lambda = \tilde{\lambda}$. After that point, μ begins to decrease on the λ-curves. If $\tilde{\lambda} \in (\lambda_1, \lambda_2]$, we slide all the way to $\mu = 0$, by Lemma 4.4. Hence, the μ-curve at $\tilde{\lambda}$ is as in Fig. 2. In case $\tilde{\lambda} > \lambda_2$, by Lemma 4.4, we do not slide all the way to $\mu = 0$, and hence the μ-curve at $\tilde{\lambda}$ is as in Fig. 3. By Lemma 4.3, this curve exhausts the set of all positive solutions of (4.1) (any other solution would lie on a solution curve with no place to go, when continued in μ). □

Remark Our results also imply that the μ-curves described in Theorem 4.1 continue without turns for all $\mu < 0$. (Observe that Lemma 2.4 holds for autonomous problems, regardless of the sign of μ, see e.g., [10].) Negative μ's correspond to "stocking" of fish, instead of "fishing". In Fig. 4 we present the solution curve of the problem

$$u'' + 6u(1-u) - \mu = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{4.13}$$

Observe that $u(0) > 1$ for $\mu < \mu_0$, for some $\mu_0 < 0$.

For the non-autonomous version of the fishing problem

$$u'' + \lambda u(1-u) - \mu g(x) = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0 \tag{4.14}$$

we were not able to extend any of the above lemmas. Still it appears easier to understand the λ-curves first. Here we cannot rule out the possibility of the λ-curves losing their positivity backward, and thus never making a turn to the right.

We prove next that a turn to the right does occur for solution curves of (4.14) that are close to the curve bifurcating from zero at $\lambda = \lambda_1$. Let $\bar{\lambda}_\mu$ be the value of λ, at which positivity is lost for a given μ (so that $u_x(\pm 1, \bar{\lambda}_\mu) = 0$). We claim that $\inf_{\mu > 0} \bar{\lambda}_\mu > \lambda_1$. Indeed, assuming otherwise, we can find a sequence $\{\mu_n\} \to 0$, with $\bar{\lambda}_{\mu_n} \to \lambda_1$. By a standard argument, $\frac{u(x, \bar{\lambda}_{\mu_n})}{u(0, \bar{\lambda}_{\mu_n})} \to w(x) > 0$, where

$$w'' + \lambda_1 w = 0, \qquad w(\pm 1) = w'(\pm 1) = 0,$$

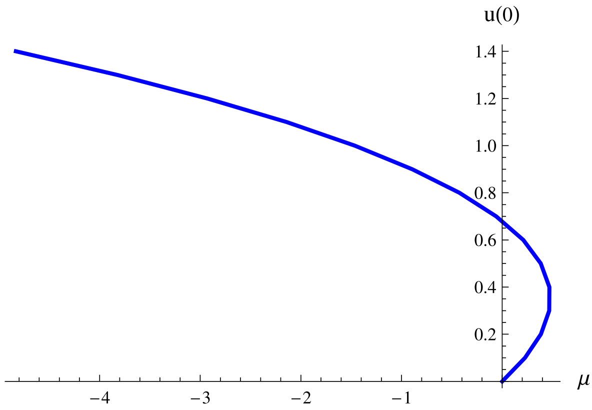

Fig. 4 The solution curve for the problem (4.13)

which is not possible. It follows that for a fixed $\lambda \in (\lambda_1, \inf_{\mu>0} \bar{\lambda}_\mu]$, the μ-curve for (4.14) is as in Fig. 2. (Notice that this also implies that the λ-curves for small μ do turn.) A similar result for general PDE's was proved in S. Oruganti, J. Shi, and R. Shivaji [14]. In case $\lambda > \inf_{\mu>0} \bar{\lambda}_\mu$, the μ-curves are different, although we cannot prove in general that they are as in Fig. 3. (For example, we cannot rule out the possibility that the μ-curves consist of several pieces.)

References

1. Allgower, E.L., Georg, K.: Numerical Continuation Methods. An Introduction. Springer Series in Computational Mathematics, vol. 13. Springer, Berlin (1990)
2. Crandall, M.G., Rabinowitz, P.H.: Bifurcation, perturbation of simple eigenvalues and linearized stability. Arch. Ration. Mech. Anal. **52**, 161–180 (1973)
3. Costa, D.G., Drábek, P., Tehrani, H.: Positive solutions to semilinear elliptic equations with logistic type nonlinearities and constant yield harvesting in R^n. Commun. Partial Differ. Equ. **33**, 1597–1610 (2008)
4. Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Commun. Math. Phys. **68**, 209–243 (1979)
5. Girão, P., Tehrani, H.: Positive solutions to logistic type equations with harvesting. J. Differ. Equ. **247**(2), 574–595 (2009)
6. Hastings, S.P., McLeod, J.B.: Classical Methods in Ordinary Differential Equations. With Applications to Boundary Value Problems. Graduate Studies in Mathematics, vol. 129. Am. Math. Soc., Providence (2012)
7. Hung, K.C., Wang, S.H.: Classification and evolution of bifurcation curves for a multiparameter p-Laplacian Dirichlet problem. Nonlinear Anal. **74**(11), 3589–3598 (2011)
8. Hung, K.C., Wang, S.H.: Bifurcation diagrams of a p-Laplacian Dirichlet problem with Allee effect and an application to a diffusive logistic equation with predation. J. Math. Anal. Appl. **375**(1), 294–309 (2011)
9. Korman, P.: Symmetry of positive solutions for elliptic problems in one dimension. Appl. Anal. **58**(3–4), 351–365 (1995)
10. Korman, P.: Global Solution Curves for Semilinear Elliptic Equations. World Scientific, Hackensack (2012)
11. Korman, P.: Exact multiplicity and numerical computation of solutions for two classes of non-autonomous problems with concave-convex nonlinearities. Nonlinear Anal. **93**, 226–235 (2013)
12. Korman, P., Li, Y., Ouyang, T.: Exact multiplicity results for boundary-value problems with nonlinearities generalising cubic. Proc. R. Soc. Edinb., Sect. A **126A**, 599–616 (1996)
13. Nirenberg, L.: Topics in Nonlinear Functional Analysis. Courant Institute Lecture Notes. Am. Math. Soc., Providence (1974)

14. Oruganti, S., Shi, J., Shivaji, R.: Diffusive logistic equation with constant yield harvesting. I. Steady states. Transl. Am. Math. Soc. **354**(9), 3601–3619 (2002)
15. Ouyang, T., Shi, J.: Exact multiplicity of positive solutions for a class of semilinear problems, II. J. Differ. Equ. **158**(1), 94–151 (1999)
16. Shi, J.: A radially symmetric anti-maximum principle and applications to fishery management models. Electron. J. Differ. Equ. **27** (2004), 13 pp. (electronic)