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EXISTENCE OF SOLUTIONS FOR A CLASS OF NONLINEAR NON-COERCIVE PROBLEMS

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1. Introduction.

We study the existence of periodic solutions for a nonlinear non-coercive boundary value problem:

$$
(1.1)\qquad
\begin{aligned}
u_y &= \rho(u_{xx}, u_{xz}, u_{zz}) && y = 1\\
\Delta u &= \epsilon f(x, y, z, u, u_i, u_{ij}) && 0 < y < 1\\
u &= 0 && y = 0 .
\end{aligned}
$$

Here f is 2π periodic in x, z ; ϵ is a small parameter and we are looking for a 2π periodic in x, z solution $u(x, y, z)$.

The case $\rho(u_{xx}, u_{xz}, u_{zz}) = F\, u_{xx}$, $F = \text{const} < 0$ comes from water wave theory (no surface tension). This problem is commonly believed to be ill-posed, see for example [9]. The difficulty here is that small denominators appear in Fourier series solution. However if we assume $F > 0$ we can prove

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solvability of (1.1) using simple Picard iteration even though the problem is not coercive (see [6] for details).

For the general $\rho(u_{xx}, u_{xz}, u_{zz})$ solvability of (1.1) is determined by the matrix

$$A = \begin{pmatrix} \rho_{u_{xx}} & \frac{1}{2}\rho_{u_{xz}} \\ \frac{1}{2}\rho_{u_{xz}} & \rho_{u_{zz}} \end{pmatrix} .$$

If A is positive definite then the problem (1.1) is coercive (see §5) and hence solvable by simple Picard iteration. If we only have $A \geq 0$ then the problem (1.1) is not coercive in general and then the Picard iteration fails because of the loss of derivatives at each step. But using Nash-Moser type iteration technique it is still possible to prove solvability of (1.1). Namely we prove the following:

<u>Main Theorem</u>. Assume that for the problem (1.1) with function f 2π periodic in x and z we have:

(i) $A \geq 0$

(ii) $\rho(0, 0, 0) = 0$; $f(x, y, z, 0, \dots, 0) \not\equiv 0$

(iii) $\rho, f \in C^{55}$.

Then for ϵ sufficiently small the problem (1.1) has a C^2 solution, 2π periodic in x and z .

In particular, this theorem covers the case $\rho = \rho(u_{xx})$, $\rho(0) = 0$, $\dot{\rho}(t) \geq 0$ (for $|t|$ small), while simple Picard iteration fails here.

We can also prove some general uniqueness results for (1.1), (see [6]).

2. Notations and technical lemmata.

We consider functions of three variables x, y, z which are 2π periodic in x, z and $0 \leq y \leq 1$. By V we denote the domain $0 \leq x, z \leq 2\pi$, $0 \leq y \leq 1$; its boundary we denote by ∂V and the top part of the boundary by V_t , ($V_t = \partial V \cap (y = 1)$) .

We shall also denote

$$\int f = \int_0^{2\pi}\int_0^1\int_0^{2\pi} f(x, y, z)\, dx\, dy\, dz ,$$

$$\int_t f = \int_0^{2\pi}\int_0^{2\pi} f(x, 1, z)\, dx\, dz .$$

(In other words, $\int_t$ denotes integration over V_t) .

We shall write $\|\cdot\|_m$ for m-th Sobolev norm for functions in V and $\overline{\|\cdot\|}_m$ for functions on V_t . Corresponding Sobolev spaces we denote by H^m and $\overline{H}^m$ respectively. We shall also need the norms $\boxed{\|\cdot\|}_m = \|\cdot\|_m + \overline{\|\cdot\|}_m$ and

$$|f|_N = \sum_{|\alpha| \leq N} |D^\alpha f|_{L_\infty} , \ N \geq 0 , \text{ integer.}$$

By $\overline{|f|}_N$ we understand the same norm for functions defined on V_t .

We shall also use the notation:

$$\|D^r f\|_m = \sum_{|\alpha| = r} \|D^\alpha f\|_m ,$$

$$|D^r f|_m = \sum_{|\alpha|=r} |D^\alpha f|_m \ ; \ r, m \ \text{are intergers} \geq 0 \, .$$

All positive constants independent of unknown functions we denote by c .

We will need the following relations between our norms.

Lemma 2.1. For any integer $n \geq 0$ and any $\epsilon > 0$ one has

(i) $\overline{\|v\|}_n \leq \|v\|_{n+1}$

(ii) $\|v\|_n \leq \epsilon\|v\|_{n+1} + c(\epsilon)\|v\|_0$

(iii) $\overline{\|v\|}_n \leq \epsilon\|v\|_{n+1} + c(\epsilon)\|v\|_0$.

Proof. Part (ii) is standard. Parts (i) and (iii) are easily derived using Schwarz inequality (see [6]).

Lemma 2.2. Suppose $f_1, f_2 \in C^r(V)$, $r \geq 0$ is an integer. Then

(i) $\|f_1 f_2\|_r \leq c[|f_1|_0 \|f_2\|_r + |f_2|_0 \|f_1\|_r]$

(ii) $|f_1 f_2|_r \leq c[|f_1|_0 |f_2|_r + |f_2|_0 |f_1|_r]$.

This lemma is standard now (see [5] for proof). Obviously, similar inequalities are true for functions on V_t .

Lemma 2.3. Suppose $w_1, \dots, w_s \in C^r(V)$. Suppose that $\varphi = \varphi(w_1, \dots, w_s)$ possesses continuous derivatives up to order $r \geq 1$ bounded by B on $\max_i |w_i| < 1$. Then

(i) $\|D^r \varphi(w_1, \dots, w_s)\|_{L^2} \leq cB\|D^r w\|_{L^2}$ for $\max_i |w_i|_{L^\infty} < 1$

(ii) $|D^r \varphi(w_1, \dots, w_s)|_{L^\infty} \leq cB|D^r w|_{L^\infty}$ for $\max_i |w_i|_{L^\infty} < 1$.

For proof of this lemma and references see [5].

Corollary 1. Assuming the conditions of the lemma 2.3 we have (for $\max_i |w_i|_0 < 1$)

(i) $\|\varphi(w_1, \dots, w_s)\|_r \leq c(\|w\|_r + 1)$

(ii) $|\varphi(w_1, \dots, w_s)|_r \leq c(|w|_r + 1)$.

Corollary 2. If in addition we assume

$$\varphi(0, \dots, 0) = 0$$

$$r \geq [n/2] + 1 \quad (n = 3 \text{ , the number of spacial dimensions}).$$

Then

$$\|\varphi(w_1, \dots, w_s)\|_r = \delta(\|w\|_r) \, ,$$

where $\delta(t) \to 0$ as $t \to 0$. (We denote $\|w\|_r = \max_i \|w_i\|_r$) .

Proof. We assume $s = 1$ and the proof is similar for the general case. By lemma 2.3

$$\|\varphi(w)\|_r^2 = \sum_{1 \leq \ell \leq r} \|D^\ell \varphi(w)\|_0^2 + \|\varphi(w)\|_0^2$$

$$\leq cB\|w\|_r^2 + \|\varphi(w)\|_0^2 \, .$$

Using Sobolev's lemma the second term is estimated as follows:

$$\|\varphi(w)\|_0 \leq c|\varphi(w)|_0 \leq c \sup_{|t| \leq |w|_0} |\varphi(t)|$$

$$\leq c \sup_{|t| \leq c\|w\|_r} |\varphi(t)| = \delta(\|w\|_r) \quad .$$

Remark. Obviously lemma 2.3 and its corollaries can also be stated for the norms $\overline{\|\cdot\|}_m$, $\overline{|\cdot|}_m$ and $\overline{\overline{\|\cdot\|}}_m$.

Lemma 2.4. Let ℓ , k , m be non-negative integers, $k \leq m$. Then

(i) $\|u\|_{k+\ell} \leq c\|u\|_{m+\ell}^{k/m} \|u\|_{\ell}^{1-k/m}$

(ii) $\overline{\|u\|}_{k+\ell} \leq c\overline{\|u\|}_{m+\ell}^{k/m} \overline{\|u\|}_{\ell}^{1-k/m}$

(iii) $\overline{\overline{\|u\|}}_{k+\ell} \leq c\overline{\overline{\|u\|}}_{m+\ell}^{k/m} \overline{\overline{\|u\|}}_{\ell}^{1-k/m}$.

Proof. Inequalities (i), (ii) are standard. Inequality (iii) easily follows from them.

3. Outline of the proof.

We will prove solvability of our problem (1.1) by applying an abstract Implicit Function Theorem (IFT). We start by introducing a (standard) concept of a scale of Banach spaces.

Definition 1. Suppose we have a family of Banach spaces B^m indexed by the parameter $m \geq 0$. We say that this family

forms a Banach scale if $B^m \subset B^{m_1}$ for $m > m_1$, and $\|u\|_{m_1} \leq \|u\|_m$ for $u \in B^m$.

Definition 2. We call B^m a Banach scale with smoothing if there exists a family of smoothing operators $S(t)$, $t \geq 0$ with the properties $(0 \leq r \leq \rho)$:

(S_1) $\|S(t)u\|_\rho \leq c\, t^{\rho - r} \|u\|_r$, $u \in B^r$

(S_2) $\|(I - S(t))u\|_r \leq c\, t^{r-\rho} \|u\|_\rho$, $u \in B^\rho$.

The following theorem is a slight modification of the one in [8].

Theorem (IFT). Let B_1^m , B_2^m be two Banach scales; the first one with smoothing. Let $F(u) : B_1^m \to B_2^{m-\alpha}$ $(0 \leq 2\alpha \leq m)$ be a (non-linear) operator with the domain $D(F) = \{u \in B_1^m , \|u\|_m < \delta , \delta > 0\}$. Suppose that

(i) $F(u)$ has two continuous Frechet derivatives both bounded by c .

(ii) For any n , $m - \alpha \leq n \leq m$ there exists a map $L(u)$ with domain $D(L) = D(F)$ and range in the space $\mathcal{B}(B_2^n , B_1^{n-\alpha})$ of bounded linear operators on B_2^n to $B_1^{n-\alpha}$, such that:

(iia) $F'(u)\, L(u)h = h$, $h \in B_2^n$, $u \in D(F)$

(iib) $\|L(u)h\|_{n-\alpha} \leq c\|h\|_n$, $h \in B_2^n$, $u \in D(F)$

(iic) $\|L(u)\, F(u)\|_{m+9\alpha} \leq c(1 + \|u\|_{m+10\alpha})$, $u \in B_1^{m+10\alpha} \cap D(F)$.

Then if $F(0)$ is small enough $F(D(F))$ contains the origin.

To apply this theorem we define an operator $F(u) : B_1^{m} \to B_2^{m-\alpha}$ as follows:

$$(3.1)\quad F(u) = \begin{pmatrix} u_y - \rho(u_{xx}, u_{xz}, u_{zz}) \\ \Delta u - \epsilon f(x, y, z, u, u_i, u_{ij}) \end{pmatrix},$$

where

$$(3.2)\quad \begin{aligned} B_1^{m} &= \{u \in H^m(V) \ , \ \|u\|_m < \delta \ \text{ and } \ u(x, o, z) = 0\} , \\ B_2^{m-\alpha} &= H^{m-\alpha}(V) \times H^{m-\alpha}(V_t) . \end{aligned}$$

The constants $m \geq 2\alpha$, $\alpha \geq 5$ and δ small will be specified later. It is well known that both B_1^{m} and B_2^{m} form Banach scales with smoothing.

Now we can solve the original problem (1.1) by proving the solvability of

$$(3.3)\qquad F(u) = 0 .$$

This will be done by checking the conditions of the IFT for the operator $F(u)$.

Notice that condition (iib) of IFT requires an a priori estimate for the linearized problem and (iia) its solvability. These facts are established in §4 and §5 correspondingly. Actually, we prove the estimates and the solvability for the linearized problem under rather general conditions, and in §6 we check that these are satisfied if $u \in B_1^{m}$ with δ sufficiently small. Also in §6 we verify the condition (iic), completing the proof.

4. A Priori Estimates for the Linearized Problem

Proposition 4.1. Consider the problem

$$\begin{aligned}
&(4.1a)\quad && v_y - r_1 v_{xx} - r_2 v_{xz} - r_3 v_{zz} = g , && y = 1 , \\
&(4.1b)\quad && \Delta v - \epsilon c_{ij} v_{ij} - \epsilon c_i v_i - \epsilon c_0 v = f , && 0 < y < 1 , \\
&(4.1c)\quad && v = 0 , && y = 0 .
\end{aligned}$$

Here r_i, g are functions of x, z; c_{ij}, c_i, c_0, f of x, y, z. All functions in (4.1) are assumed to be 2π-periodic in x and z. We have denoted: $v_1 = v_x$, $v_2 = v_y$, $v_3 = v_z$, $v_{1} = v_{xx}$, etc. Summation in i and j is implied.

For $k \geq 0$ integer denote:

$$(4.2)\quad \begin{aligned} c_k &= \max_{i, \ell, m} (|c_0|_k , |c_i|_k , |c_{\ell m}|_k) \\ r_k &= \max (|r_1|_k , |r_2|_k , |r_3|_k) \\ \rho_k &= c_k + r_k \end{aligned}$$

and

$$(4.3)\quad \begin{aligned} p_3 &= \rho_3 \\ p_4 &= \rho_3 p_3 + \rho_4 \\ &\cdots\cdots\cdots \\ p_n &= \rho_3 p_{n-1} + \rho_4 p_{n-2} + \rho_5 p_{n-3} + \ldots + \rho_{n-1} p_3 + \rho_n . \end{aligned}$$

The scalars c_k, r_k can be easily distinguished from the functions in (4.1), based on the context.

Assume that:

$$(4.4)\quad r_1\xi^2 + r_2\xi\eta + r_3\eta^2 \ge 0 \,,$$

for all ξ, η real and all x, z.

$$(4.5)\quad r_2 < \delta \quad (r_2 \text{ as defined by } (4.2))$$

$$c_2 < M \,,\ M > 0 \text{ a constant.}$$

Then for ϵ and δ sufficiently small we have the following a priori estimate

$$(4.6)\quad \|v\|_{n+1} \le c\left[\overline{\|F\|}_n + p_3\overline{\|F\|}_{n-1} + p_4\overline{\|F\|}_{n-2} + \dots + p_n\overline{\|F\|}_2\right]$$

($n \ge 3$ is an integer).

Proof. Let α be any derivative in x, z, $|\alpha| = n$. Differentiating (4.1b) we get:

$$\begin{cases} v^\alpha_y = rv^{\alpha+2} + r^1v^{\alpha+1} + r^2v^\alpha + \dots + r^\alpha v^2 + g^\alpha & (4.7\text{a}) \\ \Delta v^\alpha = \epsilon(c_{22}v^\alpha_{yy} + c^1_{22}v^{\alpha-1}_{yy} + c^2_{22}v^{\alpha-2}_{yy} + \dots + c^\alpha_{22}v_{yy}) & (4.7\text{b}) \\ \quad + \epsilon(c'_{ij}v^\alpha_{ij} + c'^1_{ij}v^{\alpha-1}_{ij} + c'^2_{ij}v^{\alpha-2}_{ij} + \dots + c'^\alpha_{ij}v_{ij}) & \\ \quad + \epsilon(c_iv^\alpha_i + c^1_iv^{\alpha-1}_i + c^2_iv^{\alpha-2}_i + \dots + c^\alpha_iv_i) & \\ \quad + \epsilon(c_0v^\alpha + c^1_0v^{\alpha-1} + c^2_0v^{\alpha-2} + \dots + c^\alpha_0v) + f^\alpha & \\ v^\alpha = 0 \,. & (4.7\text{c}) \end{cases}$$

We have introduced the notation:

$$r^\ell v^{\alpha-\ell+2} \equiv r_1^\ell v^{\alpha-\ell}_{xx} + r_2^\ell v^{\alpha-\ell}_{xz} + r_3^\ell v^{\alpha-\ell}_{zz} \quad (0 \le \ell \le n) \,,$$

where

$$r_1^\ell v^{\alpha-\ell}_{xx} \equiv \sum_{\substack{|\gamma| = \ell \\ \gamma \le \alpha}} a_\gamma r_1^\gamma v^{\alpha-\gamma}_{xx} \qquad (a_\gamma\text{-const}) \,.$$

and the same notation is employed for $r_2^\ell v^{\alpha-\ell}_{xz}$, $r_3^\ell v^{\alpha-\ell}_{zz}$, $c'^\ell_{ij}v^{\alpha-\ell}_i$, $c^\ell_i v^{\alpha-\ell}_i$, $c^\ell_0 v^{\alpha-\ell}$ (a_γ are multinomial constants). Summation in i and j is implied $(1 \le i, j \le 3)$, and c'_{ij} signifies that the $i = j = 2$ term is omitted.

Multiplying (4.7b) by v^α and integrating by parts we get:

$$(4.8)\quad -\int |\nabla v^\alpha|^2 + \int_t v^\alpha v^\alpha_y = \epsilon I_1 + \epsilon I_2 + \epsilon I_3 + \epsilon I_4 + \int v^\alpha f^\alpha \,,$$

where the integrals I_1, I_2, I_3, I_4 are defined and estimated below (integrals over the bottom and side parts of the boundary disappear by (4.1c) and periodicity of v^α).

(a) Consider

$$I_1 = \int (c_{22}v^\alpha_{yy} + c^1_{22}v^{\alpha-1}_{yy} + c^2_{22}v^{\alpha-2}_{yy} + \dots + c^\alpha_{22}v_{yy})v^\alpha$$
$$\equiv \int c_{22}v^\alpha_{yy}v^\alpha + I_{1,2} \,.$$

The first term of I_1 is integrated by parts

$$\int c_{22}v^\alpha_{yy}v^\alpha = -\int c_{22}(v^\alpha_y)^2 - \int (c_{22})_y v^\alpha_y v^\alpha + \int_t c_{22}v^\alpha v^\alpha_y$$
$$\equiv I_{1,1} + \int_t c_{22}v^\alpha_y v^\alpha$$

and by (4.5)

$$|I_{1,1}| \le cc_1\|v\|^2_{n+1} \le c\|v\|^2_{n+1} \ .$$

Also

$$(4.9)\quad |I_{1,2}| = \left|\int (c^1_{22}v^{\alpha-1}_{yy} + c^2_{22}v^{\alpha-2}_{yy} + c^3_{22}v^{\alpha-3}_{yy} + \dots + c^{\alpha}_{22}v_{yy})v^{\alpha}\right|$$
$$\le \frac{1}{2}\int (v^{\alpha})^2 + \frac{1}{2}\int (c^1_{22}v^{\alpha-1}_{yy} + c^2_{22}v^{\alpha-2}_{yy} + \dots + c^{\alpha}_{22}v_{yy})^2$$
$$\le c\|v\|^2_{n+1} + c_3^2\|v\|^2_{n-1} + c_4^2\|v\|^2_{n-2} + \dots + c_n^2\|v\|^2_2 \ .$$

So that we express

$$(4.10)\qquad I_1 = I_{1,1} + I_{1,2} + \int_t c_{22}v^{\alpha}_y v^{\alpha}$$

and

$$(4.11)\quad |I_{1,1} + I_{1,2}|$$
$$\le c\|v\|^2_{n+1} + c_3^2\|v\|^2_{n-1} + c_4^2\|v\|^2_{n-2} + \dots + c_n^2\|v\|^2_2 \ .$$

(b) Consider

$$I_2 = \int (c'_{ij}v^{\alpha}_{ij} + c'^{1}_{ij}v^{\alpha-1}_{ij} + c'^{2}_{ij}v^{\alpha-2}_{ij} + \dots + c'^{\alpha}_{ij}v_{ij})v^{\alpha}$$
$$\equiv \int c'_{ij}v^{\alpha}_{ij}v^{\alpha} + I_{2,2} \ .$$

The first term of I_2 is integrated by parts in x or z

$$\int c'_{ij}v^{\alpha}_{ij}v^{\alpha} = -\int (c'_{ij})_j v^{\alpha}_i v^{\alpha} - \int c'_{ij}v^{\alpha}_i v^{\alpha}_j \quad (j = 1 \text{ or } 3)$$

so that

$$\left|\int c'_{ij}v^{\alpha}_{ij}v^{\alpha}\right| \le c_1\|v\|^2_{n+1} \le c\|v\|^2_{n+1} \ .$$

The term $I_{2,2}$ is estimated exactly as the term $I_{1,2}$. So that

$$(4.12)\quad |I_2| \le c\|v\|^2_{n+1} + c_3^2\|v\|^2_{n-1} + c_4^2\|v\|^2_{n-2} + \dots + c_n^2\|v\|^2_2 \ .$$

(c) Estimating as before we get:

$$|I_3 + I_4| \le c\|v\|^2_n + c_1\|v\|^2_{n-1} + \dots + c_n^2\|v\|^2_0 \ .$$

If we now let $\bar{I} = I_{1,1} + I_{1,2} + I_2 + I_3 + I_4$, then

$$(4.13)\quad |\bar{I}| \le \text{RHS } (4.11) \ .$$

Now we can rewrite (4.8) as

$$(4.14)\quad -\int |\nabla v^{\alpha}|^2 + \int_t (1 - \epsilon c_{22})v^{\alpha}v^{\alpha}_y = \epsilon\bar{I} + \int v^{\alpha}f^{\alpha} \ .$$

Using (4.7a) we write (with $1 - \epsilon c_{22} \equiv \bar{c}$) .

$$(4.15)\quad \int_t (1 - \epsilon c_{22})v^{\alpha}v^{\alpha}_y$$
$$= \int_t \bar{c}(rv^{\alpha+2} + r^1v^{\alpha+1} + r^2v^{\alpha} + \dots + r^{\alpha}v^2 + g^{\alpha})v^{\alpha}$$
$$\equiv J_1 + J_2 + J_3 + \int_t v^{\alpha}g^{\alpha} \ ,$$

where the integrals J_1, J_2, J_3 are defined and estimated below.

(a) Integrating by parts we get:

$$J_1 = \int_t \bar{c} r v^{\alpha+2} v^{\alpha} = \int_t \bar{c}(r_1 v^{\alpha}_{xx} + r_2 v^{\alpha}_{xz} + r_3 v^{\alpha}_{zz}) v^{\alpha}$$

$$= -\int_t \bar{c}[r_1 (v^{\alpha}_x)^2 + r_2 v^{\alpha}_x v^{\alpha}_z + r_3 (v^{\alpha}_z)^2]$$

$$+ \frac{1}{2} \int_t [(\bar{c} r_1)_{xx} + (\bar{c} r_2)_{xz} + (\bar{c} r_3)_{zz}](v^{\alpha})^2$$

$$\equiv J_{1,1} + J_{1,2} \ ,$$

where

$$J_{1,1} = -\int_t \bar{c}[r_1 (v^{\alpha}_x)^2 + r_2 v^{\alpha}_x v^{\alpha}_z + r_3 (v^{\alpha}_z)^2] \ ,$$

(4.16)

$$J_{1,2} = \frac{1}{2} \int_t [(\bar{c} r_1)_{xx} + (\bar{c} r_2)_{xz} + (\bar{c} r_3)_{zz}](v^{\alpha})^2 \ .$$

Clearly

(4.17) $$|J_{1,2}| \le c c_2 r_2 \overline{\| v \|}^2_n \le c\delta \overline{\| v \|}^2_n \ .$$

(b) The second integral is also integrated by parts:

$$J_2 = \int_t \bar{c} r^1 v^{\alpha+1} v^{\alpha} = -\int_t (\bar{c} r^1)' v^{\alpha} v^{\alpha} - \int_t \bar{c} r^1 v^{\alpha} v^{\alpha+1} \ .$$

The first term on the right is bounded by $c\delta \overline{\| v \|}^2_n$. The second one is of the same type as the original J_2. Integrate it by

parts in the same way. Eventually we will rearrange derivatives so as to get

(4.18) $$J_2 = \pm \int_t r v^{\gamma} (v^{\gamma})' + R = \mp \frac{1}{2} \int_t r' (v^{\gamma})^2 + R \ ,$$

where R is collection of terms each bounded by $c\delta \overline{\| v \|}^2_n$, $|\gamma| = n$ and " ' " denotes a derivative in x or z. It is clear from (4.18) that

(4.19) $$|J_2| < c\delta \overline{\| v \|}^2_n \ .$$

(c) As before we estimate

(4.20) $$|J_3| = \left| \int_t \bar{c}(r^2 v^{\alpha} + r^3 v^{\alpha-1} + \dots + r^{\alpha} v^2) v^{\alpha} \right|$$

$$\le \epsilon_1 \overline{\| v \|}^2_n + c(r_2^2 \overline{\| v \|}^2_n + r_3^2 \overline{\| v \|}^2_{n-1} + \dots + r_n^2 \overline{\| v \|}^2_2) \ .$$

If we now denote

$$\bar{J} = J_{1,2} + J_2 + J_3$$

then (4.15) takes the form

(4.21) $$\int_t (1 - \epsilon c_{22}) v^{\alpha} v^{\alpha}_y = J_{1,1} + \int_t (1 - \epsilon c_{22}) v^{\alpha} g^{\alpha} + \bar{J} \ ,$$

and

(4.22) $$|\bar{J}| \le C[(\delta + \epsilon_1) \overline{\| v \|}^2_n + r_3^2 \overline{\| v \|}^2_{n-1} + \dots + r_n^2 \overline{\| v \|}^2_2] \ .$$

Now we can rewrite (4.14) as

$$(4.23)\quad -\int|\nabla v^{\alpha}|^2 - \int_t (1-\epsilon c_{22})[r_1(v_x^{\alpha})^2 + r_2 v_x^{\alpha} v_z^{\alpha} + r_3(v_z^{\alpha})^2]$$

$$- \epsilon\bar{I} + \bar{J} + \int_t (1-\epsilon c_{22}) v^{\alpha} g^{\alpha} = \int f^{\alpha} v^{\alpha} \quad .$$

Notice that for ϵ sufficiently small the second term on the left is non-positive by (4.4). Then using (4.13), (4.22) and Lemma 2.1 we estimate the left-hand side of (4.23) as follows:

$$(4.24)\quad |\mathrm{LHS}(4.23)| \geq \int|\nabla v^{\alpha}|^2 - \epsilon c(\|v\|_{n+1}^2 + c_3^2\|v\|_{n-1}^2 + \ldots + c_n^2\|v\|_2^2$$

$$- c[(\delta+\epsilon_1)\overline{\|v\|}_n^2 + r_3^2\overline{\|v\|}_{n-1}^2 + \ldots + r_n^2\overline{\|v\|}_2^2]$$

$$- \frac{1}{8}\int|\nabla v^{\alpha}|^2 - c\int_t (g^{\alpha})^2$$

$$\geq \frac{7}{8}\int|\nabla v^{\alpha}|^2 - c[\epsilon\|v\|_{n+1}^2 + (\delta+\epsilon_1)\overline{\|v\|}_n^2 + \rho_3^2\overline{\|v\|}_{n-1}^2 + \ldots +$$

$$+ \ldots + \rho_n^2\overline{\|v\|}_2^2] - c\int_t (g^{\alpha})^2 \quad .$$

And

$$(4.25)\quad |\mathrm{RMS}(4.23)| \leq \frac{1}{2}\int(f^{\alpha})^2 + \frac{1}{2}\int(v^{\alpha})^2$$

$$\leq \frac{1}{2}\int(f^{\alpha})^2 + \frac{1}{2}\int|\nabla v^{\alpha}|^2 \quad .$$

Combining (4.24) and (4.25) and summing in all α, $|\alpha| \leq n$ we get:

$$(4.26)\quad \sum_{|\alpha|\leq n}\int|\nabla v^{\alpha}|^2 \leq c[\epsilon\|v\|_{n+1}^2 + (\delta+\epsilon_1)\overline{\|v\|}_n^2$$

$$+ \rho_3^2\overline{\|v\|}_{n-1}^2 + \ldots + \rho_n^2\overline{\|v\|}_2^2 + \overline{\|F\|}_n^2] \quad .$$

In order to prove our estimate (4.6) we also need to estimate the derivatives of v which include more than one differentiation in y. By the definition of Sobolev norms we have:

$$(4.27)\quad \sum_{|\alpha|\leq n}\int|\nabla v^{\alpha}|^2 = \|v\|_{n+1}^2 - \sum_{|\alpha|\leq n-1}\|v_{yy}^{\alpha}\|_0^2 -$$

$$- \sum_{|\alpha|\leq n-2}\|v_{yyy}^{\alpha}\|_0^2 - \ldots - \|D_y^{n+1} v\|_0^2 \geq$$

$$\geq \|v\|_{n+1}^2 - c\|v_{yy}\|_{n-1}^2 \quad .$$

(In all sums α denotes derivates in x and z only.) To estimate $\|v_{yy}\|_{n-1}^2$ we use the equation (4.1b).

$$(4.28)\quad \|v_{yy}\|_{n-1}^2 \leq c(\|f\|_{n-1}^2 + \|v_{xx}\|_{n-1}^2 + \|v_{zz}\|_{n-1}^2 +$$

$$+ c_0^2\|v\|_{n+1}^2 + c_{n-1}^2\|v\|_2^2) \leq \mathrm{RHS}(4.26) \quad .$$

Using (4.27) and (4.28) in (4.26) we get:

$$(4.29)\quad \|v\|_{n+1}^2 \leq c[\epsilon\|v\|_{n+1}^2 + (\delta+\epsilon_1)\overline{\|v\|}_n^2 + \rho_3^2\overline{\|v\|}_{n-1}^2 + \ldots +$$

$$+ \ldots + \rho_n^2\overline{\|v\|}_2^2 + \overline{\|F\|}_n^2] \quad .$$

For $\epsilon, \epsilon_1, \delta$ sufficiently small using lemma (2.1) we can absorb the highest two terms of RHS(4.29) into LHS(4.29). So that

$$(4.30)\quad \|v\|_{n+1}+\overline{\|v\|}_n\le c\left[\overline{\|F\|}_n+\rho_3\overline{\|v\|}_{n-1}+\dots+\rho_n\overline{\|v\|}_2\right] .$$

Iterating the inequality (4.30) we get the desired inequality (4.6).

Remark. Examining the proof we see that for $n = 0, 1, 2$ the estimates are: $\|v\|_{n+1}\le c\overline{\|F\|}_n$.

Proposition (4.2). Consider the problem $(\sigma > 0)$

$$(4.31)\begin{cases} v_y-\sigma(v_{xx}+v_{zz})-r_1v_{xx}-r_2v_{xz}-r_3v_{zz}=g, & y=1\\ \Delta v-\epsilon\, c_{ij}v_{ij}-\epsilon\, c_iv_i-\epsilon\, c_0v=f , & 0<y<1\\ v=0, & y=0 .\end{cases}$$

Assume that all definitions and assumptions of Proposition 4.1 are true for (4.31). Then we have exactly the same a priori estimate (4.6) (for the problem 4.31) with the right hand side independent of σ .

Proof. Proceed exactly as in Proposition 4.1. Then the formula (4.23) will contain an extra term

$$-\sigma\int_t [(v_x^\alpha)^2+(v_z^\alpha)^2]$$

in the left hand side. This term is non-positive so that it drops out in the estimate (4.24). The rest of the proof is the same as in Proposition 4.1.

5. Existence of Solutions for the Linearized Problem

Proposition 5.1. Consider the linear problem (4.1). Suppose that conditions (4.4), (4.5) are satisfied; $g, f\in H^k (k\ge 5)$ and $c_k, r_k\le c$, $r_2<\delta$ (c_k, r_k were defined by (4.2)) . Then for ϵ and δ sufficiently small the problem (4.1) has a unique H^k solution.

Proof. The proof consists of the following steps.

(A). Show that the problem $(\sigma > 0)$

$$(5.1a)\quad v_y-\sigma(v_{xx}+v_{zz})-t(r_1v_{xx}+r_2v_{xz}+r_3v_{zz}) = g, \qquad y=1$$

$$(5.1b)\quad \Delta v-\epsilon c_{ij}v_{ij}-\epsilon c_iv_i-\epsilon c_0v=f\ (i\le j),\ 0<y<1$$

$$(5.1c)\quad v=0, \qquad y=0$$

is coercive for $0\le t\le 1$, $0\le\epsilon\le\epsilon_1$, ϵ_1 sufficiently small. This is the content of Lemma 5.1 which is proved at the end of the paragraph.

(B). Since the problem (5.1) is coercive its index is defined and is invariant of the homotopy transformations which do not take the problem out of the coercive class. Letting $t\to 0$, $\epsilon\to 0$ we see that the index of (5.1) is the same as that of (5.2):

$$(5.2)\quad \begin{cases} v_y-\sigma(v_{xx}+v_{zz})=g\\ \Delta v=f\\ v=0 .\end{cases}$$

(C). By elementary Fourier analysis we see that (5.2) is uniquely solvable, so that its index is 0. Hence the index of (5.1) is also 0. Since by Proposition 4.2 we have uniqueness for (5.1) we derive that (5.1) is solvable (uniquely). Denote its solution for $t = 1$ by v^{σ}.

(D). By the estimate (4.6) of Proposition (4.2) we have:

$$\| v^{\sigma} \|_{k+1} \leq c .$$

Hence $\{v^{\sigma}\}$ is compact in H^k and as $\delta \to 0$ there exists a subsequence converging to some $v \in H^k$. Passing to the limit in (5.1) as $\delta \to 0$ and $t = 1$, we see that v is solution of (4.1) which establishes the Proposition (5.1).

Lemma 5.1. The problem (5.1) is coercive for $0 \leq t \leq 1$, $\delta > 0$, $0 \leq \epsilon \leq \epsilon_1$, ϵ_1 sufficiently small.

Proof. Clearly it suffices to check coerciveness of the top boundary condition (5.1a). According to Agronovich [1] this is equivalent to checking that a certain ODE with an initial condition has a unique stable as $t \to \infty$ solution. The algorithm is as follows. Throw away the lower order terms in (5.1a) and (5.1b). Let $(x, 1, z)$ be an arbitrary point on the top boundary. Freeze r_1, r_2, r_3, c_{ij} at it. Take Fourier transforms of (5.1a), (5.1b) in x and z. We get:

$$(5.3)\quad (1 - \epsilon c_{22})v''(t) + i\epsilon(c_{12}\xi_1 + c_{23}\xi_3)v'(t)$$
$$+ [-(\xi_1^2 + \xi_3^2) + \epsilon(c_{11}\xi_1^2 + c_{13}\xi_1\xi_3 + c_{33}\xi_3^2)]v(t) = 0,\ t > 0 .$$

$$(5.4)\quad [\sigma(\xi_1^2 + \xi_3^2) + t(r_1\xi_1^2 + r_2\xi_1\xi_3 + r_3\xi_3^2)]v(0) = h ,\quad h = \text{const.}$$

Set

$$a = 1 - \epsilon c_{22}$$

$$b = i\epsilon(c_{12}\xi_1 + c_{23}\xi_3) \qquad (i = \sqrt{-1})$$

$$c = -(\xi_1^2 + \xi_3^2) + \epsilon(c_{11}\xi_1^2 + c_{13}\xi_1\xi_3 + c_{33}\xi_3^2) .$$

Characteristic exponents of (5.3) are:

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

For ϵ sufficiently small $a > 0$, $b^2 - 4ac > 0$ and for any $\xi' = (\xi_1, \xi_3)$ with $\xi_1^2 + \xi_3^2 \neq 0$, the unique stable as $t \to \infty$ solution of (5.3), (5.4) is

$$v(t) = \frac{h}{\sigma(\xi_1^2 + \xi_3^2) + t(r_1\xi_1^2 + r_2\xi_1\xi_3 + r_3\xi_3^2)} \cdot \exp\left[\frac{-b - \sqrt{b^2 - 4c}}{2a} t\right] .$$

This proves the lemma.

Remark. The original linearized problem (4.1) is non-coercive in general. Indeed in this case instead of (5.4) we have

$$(5.5)\qquad (r_1\xi_1^2 + r_2\xi_1\xi_3 + r_3\xi_3^2)v(0) = h .$$

If ξ_1, ξ_3 are such that $\xi_1^2 + \xi_3^2 \neq 0$ and $r_1\xi_1^2 + r_2\xi_1\xi_3 + r_3\xi_3^2 = 0$, then the problem (5.3), (5.5) has no nontrivial stable as $t \to \infty$ solution. So that (4.1) is non-coercive.

6. Proof of the Main Theorem

As was indicated in §3 we consider an operator $F(u) : B_1^m \to B_2^{m-\alpha}$ defined by

$$F(u) = \begin{pmatrix} u_y - \rho(u_{xx}, u_{xz}, u_{zz}) \\ \\ \Delta u - \epsilon f(x, y, z, u, u_i, u_{ij}) \end{pmatrix}. \tag{6.1}$$

The Banach scales B_1^m, B_2^m were defined by (3.2); the constants $\alpha \geq 5$ and $m \geq 2\alpha$ will be specified later. We shall get solution of our problem by solving

$$F(u) = 0 \quad . \tag{6.2}$$

This will be done by verifying the conditions of IFT.

Denote

$$r_1 = \rho_{u_{xx}}(u_{xx}, u_{xz}, u_{zz}), \; r_2 = \rho_{u_{xz}}(u_{xx}, u_{xz}, u_{zz}),$$
$$r_3 = \rho_{u_{zz}}(u_{xx}, u_{xz}, u_{zz});$$

(6.3)

$$c_0 = f_u(x, y, z, u, u_i, u_{ij}), \; c_{u_\nu} = f_{u_\nu}(x, y, z, u, u_i, u_{ij}),$$
$$c_{\mu\tau} = f_{u_{\mu\tau}}(x, y, z, u, u_i, u_{ij}), \; 1 \leq \nu, \mu, \tau \leq 3 \; .$$

We shall also consider constants r_k, c_k, ρ_k as defined by (4.2). These are estimated in the following lemma.

<u>Lemma 6.1</u>. Suppose that all the functions defined by (6.3) possess continuous derivatives in all variables up to order $k \geq 1$. Assume also

$$\| u \|_5 \leq \delta \quad . \tag{6.4}$$

Then for δ sufficiently small we have:

$$\begin{aligned} r_k, c_k, \rho_k &\leq c(\| u \|_{k+5} + 1), \\ \overline{\| F(u) \|_k} &\leq c(\| u \|_{k+3} + 1) \; . \end{aligned} \tag{6.5}$$

<u>Proof</u>. Condition (6.4) guarantees that u and all its derivatives up to order two are bounded by 1 in $\overline{V}$. Then the estimates (6.5) follow from the lemma 2.3.

<u>Remark</u>. Similar estimates hold for higher order derivatives of the functions $r_1, r_2, r_3, c_0, c_i, c_{ij}$.

We now proceed to verify the conditions of the IFT.

Condition (i). Compute $(\rho = \rho(\alpha, \beta, \gamma))$

$$F'(u)v = \begin{bmatrix} v_y - \rho_\alpha(u_{xx}, u_{xz}, u_{zz})v_{xx} - \rho_\beta(u_{xx}, u_{xz}, u_{zz})v_{xz} - \\ \quad - \rho_\gamma(u_{xx}, u_{xz}, u_{zz})v_{zz} \\ \\ \Delta v - \epsilon f_u(x, y, z, u, u_i, u_{ij})v - \epsilon f_{u_i} v_i - \epsilon f_{u_{ij}} v_{ij} \end{bmatrix}$$

Using lemmae 2.2 and 6.1 we estimate:

$$\|F'(u)\| = \sup_{\|v\|_m \le 1} \overline{\|F'(u)v\|}_{m-\alpha}$$

$$= \sup_{\|v\|_m \le 1} \Big(\overline{\|v_y - r_1 v_{xx} - r_2 v_{xz} - r_3 v_{zz}\|}_{m-\alpha} + \|\Delta v - \epsilon c_{ij} v_{ij} - \epsilon c_i v_i - \epsilon c_0 v\|_{m-\alpha} \Big)$$

$$\le c \sup_{\|v\|_m \le 1} \Big(\overline{\|v\|}_{m-\alpha+1} + r_0 \overline{\|v\|}_{m-\alpha+2} + r_{m-\alpha} \overline{\|v\|}_2 + \|v\|_{m-\alpha+2} + c_0 \|v\|_{m-\alpha+2} + c_{m-\alpha} \|v\|_2 \Big)$$

$$\le c \sup_{\|v\|_m \le 1} \Big[\|v\|_{m-\alpha+3} + (1 + \|u\|_{m-\alpha+5}) \|v\|_3 \Big]$$

$$\le c \quad (\text{as } \alpha \ge 5 \, , \; m \ge 2\alpha) \; .$$

The boundness of $F''(u)$ is proved similarly.

Condition (iia). Solvability of the linearized problem, i.e., existence of the operator $L(u)$ follows directly from the Proposition 5.1. Indeed since $\|u\|_m < \delta$ we get by lemmae 6.1 and 2.3

$$r_{m-\alpha} \, , \; c_{m-\alpha} \, , \; \rho_{m-\alpha} \le c(\|u\|_{m-\alpha+5} + 1) \le c \; , \; \rho_2 = o(\delta) \; ,$$

fulfilling the conditions of Proposition 5.1.

Condition (iib). Let $h \in B_2^n$, $m - \alpha \le n \le m$.

By the Proposition 4.1 we have:

$$\|L(u)h\|_{n-\alpha} \le c\Big[\overline{\|h\|}_{n-\alpha} + p_3 \overline{\|h\|}_{n-\alpha-1} + \dots + p_{n-\alpha} \overline{\|h\|}_2\Big] \; .$$

By lemma 6.1

$$\rho_k \le c(\|u\|_{k+5} + 1) \le c\|u\|_m \le c \; , \; k = 3, 4, \dots, n-\alpha \; ,$$

which implies $p_k \le c$, $k = 3, 4, \dots, n-\alpha$ and hence

$$\|L(u)h\|_{n-\alpha} \le c\overline{\|h\|}_{n-\alpha} \le c\overline{\|h\|}_n \; ,$$

which verifies (iib).

Condition (iic). By the Proposition 4.1 we have:

$$\|L(u)F(u)\|_{m+9\alpha} \le c\Big[\overline{\|F(u)\|}_{m+9\alpha} + p_3 \overline{\|F(u)\|}_{m+9\alpha-1} + \dots + p_{m+9\alpha} \overline{\|F(u)\|}_2\Big] \; . \tag{6.6}$$

Notice that $\|u\|_7 < \delta < 1$, since $m \ge 10$. If we now denote $\tau = \|u\|_{m+10\alpha}^{\frac{1}{m+10\alpha-7}}$, then by lemma 2.4 we have:

$$\|u\|_k \le c\|u\|_{m+10\alpha}^{\frac{k-7}{m+10\alpha-7}} \|u\|_7^{1-\frac{1}{m+10\alpha-7}} \le c\,\tau^{k-7} \, ,$$

$$k = 8, \dots, m+10\alpha - 1 \; .$$

Then by lemma 6.1

$$\rho_k \le c(\|u\|_{k+5} + 1) \le c(\tau^{k-2} + 1) \, , \; k = 3, 4, \dots, m+9\alpha \; .$$

Hence

$$p_3 = \rho_3 \le c(\tau + 1)$$

$$(6.7)\qquad \begin{aligned} p_4 &= \rho_3 p_3 + \rho_4 \le c(\tau^2+1) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{m+9\alpha} &\le c(\tau^{m+9\alpha-2}+1) \; . \end{aligned}$$

Similarly by lemma 6.1 we estimate

$$\overline{\| F(u) \|}_{m+9\alpha} \le c(\| u \|_{m+9\alpha+3} + 1) \le c(\tau^{m+9\alpha-4} + 1)$$

$$(6.8)\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overline{\| F(u) \|}_5 \le c(\| u \|_8 + 1) \le c(\tau + 1)$$

$$\overline{\| F(u) \|}_k \le c \quad , \quad k = 4 \; , \; 3 \; , \; 2 \quad .$$

(Here we used that $\rho \; , \; f \in c^{m+9\alpha}$) .

Using (6.7) and (6.8) in (6.6) we get for $\tau > 1$.

$$(6.9)\quad \begin{aligned} \| L(u)F(u) \|_{m+9\alpha} &\le c[\tau^{m+9\alpha-4} + \tau \cdot \tau^{m+9\alpha-5} + \dots + \tau^{m+9\alpha-5} \cdot \tau \\ &\quad + \tau^{m+9\alpha-4} + \tau^{m+9\alpha-3} + \tau^{m+9\alpha-2} + 1] \\ &\le c(\tau^{m+9\alpha-2} + 1) = c(\| u \|_{m+10\alpha}^{\frac{m+9\alpha-2}{m+10\alpha-7}} + 1) \\ &\le c(\| u \|_{m+10\alpha} + 1) \quad , \end{aligned}$$

since $m + 9\alpha - 2 \le m + 10\alpha - 7$ (as $\alpha \ge 5$) .

In the case $\tau < 1$ (6.9) holds by choosing c sufficiently large, and (iic) is verified.

Finally, by fixing $\alpha = 5$, $m = 2\alpha = 10$ and δ sufficiently small we complete the proof.

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REFERENCES

1. M. S. Agronovich, Singular elliptic integro-differential operators, Uspekhi Math. Nauk, 20 (1965), 5, 1-120 (Russian).
2. J. R. Cannon and G. H. Meyer, On diffusion in a fractured medium, SIAM J. Appl. Math. 20, No. 3, (1971), 434-448.
3. W. Craig, A Bifurcation theory for periodic dissipative wave equations, Ph.D. Dissertation, New York University, 1981
4. A. Friedman, Partial Differential Equations, Robert E. Krieger, New York, 1976.
5. S. Klainerman, Global existence for nonlinear wave equations, Ph.D. Dissertation, New York University, 1978.
6. Ph. Korman, Existence of solutions for a nonlinear boundary value problem associated with water waves, Ph.D. Dissertation, New York University, 1981.
7. J. Moser, A new technique for the construction of solutions of non-linear differential equations, Proc. Nat. Acad. Sci, USA, 47 (1961), 1824-1831.

8. J. T. Schwartz, Non-linear Functional Analysis, Gordon and Breach, New York, 1969.

9. M. Shinbrot, Water waves over periodic bottoms in three dimensions, J. Inst. Math. Applic., 25 (1980) 367-385.

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PERTURBED ELLIPTIC PROBLEMS WITH ESSENTIAL NONLINEARITIES

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1. Introduction. Our goal is to study the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the Dirichlet problem for the elliptic differential equation

$$(1.1)\qquad \varepsilon\Delta u = \underset{\sim}{A}(\underset{\sim}{x},u)\cdot\nabla u + h(\underset{\sim}{x},u),\ \underset{\sim}{x} \text{ in } \Omega \subset \mathbb{R}^N,$$

when the vector $\underset{\sim}{A} \equiv (a_1(\underset{\sim}{x},u),\dots,a_N(\underset{\sim}{x},u))$ depends strongly on u in a sense that we will make precise later. Here $\underset{\sim}{x} \equiv (x_1,\dots,x_N)$, $\nabla \equiv (\partial/\partial x_1,\dots,\partial/\partial x_N)$, $\cdot$ is the usual Euclidean inner product, $\Delta \equiv \nabla\cdot\nabla$ is the Laplacian, and Ω is a bounded region in $\mathbb{R}^N$ whose boundary Γ is an $(N-1)$-dimensional manifold. Many results are known if $\underset{\sim}{A}$ is independent of u or if $\underset{\sim}{A}$ depends weakly on u, provided the boundary Γ is noncharacteristic, in that $\underset{\sim}{A}(\underset{\sim}{x},u)\cdot\underset{\sim}{n}(\underset{\sim}{x}) \neq 0$ for all u of interest, where $\underset{\sim}{n}$ is the unit outer normal at the point $\underset{\sim}{x}$ on Γ. The reader can consult [16], [18], [5] and [11] for details and for further references to the literature. Recently the author [12, 14] has ex-

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