Numerical computation of global solution curves using global parameters

Philip Korman and Dieter S. Schmidt
University of Cincinnati

The shoot-and-scale method

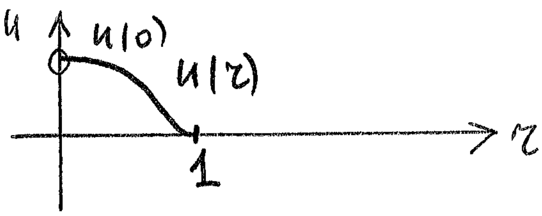
By classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg, any positive solution of the semilinear Dirichlet problem (here $u = u(x)$, $x \in R^n$, λ a positive parameter)

$$\Delta u + \lambda f(u) = 0 \text{ for } |x| < 1, \quad u = 0 \text{ when } |x| = 1 \tag{1}$$

is necessarily radially symmetric, i.e., $u = u(r)$, with $r = |x|$, and so the problem turns into an ODE

$$u''(r) + \frac{n-1}{r} u'(r) + \lambda f(u(r)) = 0 \text{ for } 0 < r < 1, \ u'(0) = u(1) = 0 . \tag{2}$$

Moreover, this theorem asserts that the function $u(r)$ is strictly decreasing, and hence $u(0)$ gives the maximum value of solution. A simple scaling argument shows that the value of $u(0) > 0$ is a global parameter, uniquely identifying the solution pair $(\lambda, u(r))$.



Lemma The value of $u(0)$ uniquely identifies the solution pair $(\lambda, u(r))$.

Proof: (Sketch) Assume that $(\mu, v(r))$ is another solution pair, $\mu \neq \lambda$, so that

$$v''(r) + \frac{n-1}{r} v'(r) + \mu f(v(r)) = 0 \text{ for } 0 < r < 1, \ v'(0) = v(1) = 0 , \tag{3}$$

and $u(0) = v(0)$. The substitution $r = \frac{1}{\sqrt{\lambda}} t$ takes (2) into

$$u''(t) + \frac{n-1}{t} u'(t) + f(u(t)) = 0 . \tag{4}$$

The first root of $u(t)$ is at $t = \sqrt{\lambda}$. Similarly, the substitution $r = \frac{1}{\sqrt{\mu}} t$ takes (3) into into

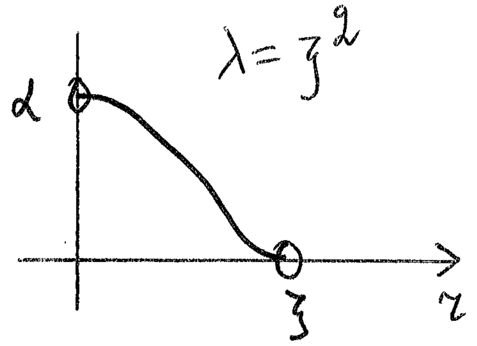
$$v''(t) + \frac{n-1}{t} v'(t) + f(v(t)) = 0 ,$$

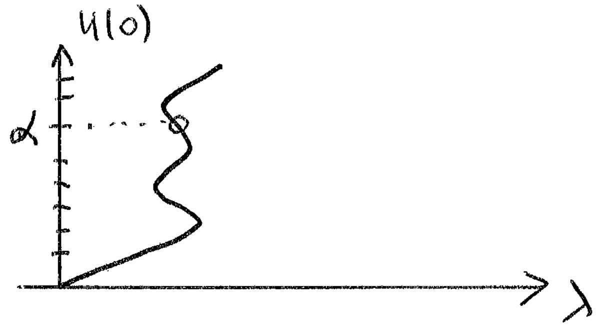
with the first root at $t = \sqrt{\mu}$. By uniqueness $u(r) = v(r)$, a contradiction.

Computations follow the same idea.

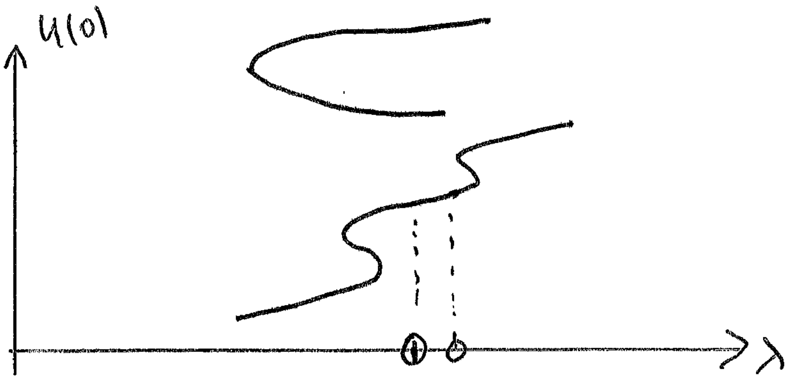
$$u''(t) + \frac{n-1}{t} u'(t) + f(u(t)) = 0\,, \ \ u(0) = \alpha\,, \ u'(0) = 0\,.$$

"Shooting". NDSolve. If ξ is the first root, then the corresponding $\lambda = \xi^2$. Plot many points (λ_n, α_n) to get a bifurcation diagram. The shoot-and-scale method.





Traditional approach: curve following, $\lambda_n \to \lambda_{n+1}$, using Newton's method.



Very short *Mathematica* program. Also, have a program for Neumann B.C.

The p-Laplace case

(5) $$\varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda f(u) = 0, \ \ u'(0) = u(1) = 0\,,$$

where $\varphi(v) = v|v|^{p-2}$, $p > 1$. As in case $p = 2$, the value of $u(0)$ uniquely identifies the solution pair $(\lambda, u(r))$, so that the solution curves can be drawn in the $(\lambda, u(0))$ plane. Write (5) as

$$u'' + \frac{n-1}{(p-1)r} u' + \lambda \frac{f(u)}{(p-1)|u'|^{p-2}} = 0\,.$$

If $p > 2$, then $u''(0)$ does not exist, $u(r) \neq C^2$. The singularity is also a problem with computations. *Regularizing transformation:*

$$z = r^{\frac{p}{2(p-1)}} .$$

Obtain

$$av''(z) + \frac{A}{z}u'(z) + \frac{z^{p-2}}{(p-1)|v'(z)|^{p-2}} f(v) = 0 ,$$

with some positive constants a and A. Conclude $v(z) \in C^2$. So that solutions of p-Laplace equation (5) is of the form $u = v\left(r^{\frac{p}{2(p-1)}}\right)$, with $v \in C^2$. (Say, $p = 3$. Then $u = v(r^{3/4})$.) Have a *Mathematica* program.

Non-homogeneous problems

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda f(r, u(r)) = 0 \text{ for } 0 < r < 1, \;\; u'(0) = u(1) = 0 .$$

Can "shoot", but not scale. In some cases $u(0)$ is still a global parameter. Have a *Mathematica* program, using Newton-like method.

Solution curves for the elastic beam equation

Displacements of an elastic beam, clamped at both end points,

$$\begin{aligned} u''''(x) &= \lambda f(u(x)), \;\; \text{for } x \in (-1, 1) \\ u(-1) &= u'(-1) = u(1) = u'(1) = 0 . \end{aligned} \tag{6}$$

For positive solutions $u(0)$ is a global parameter, Korman (2004) (if $f(t) > 0$ and $f'(t) > 0$ for $t > 0$). Simple "shooting" does not work here. (The knowledge of $u(0) = \alpha$ alone is not enough to shoot.)

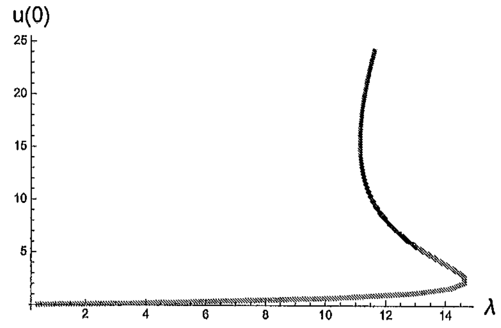


Figure 1: The global solution curve for the problem (6), with $f(u) = e^{\frac{5u}{5+u}}$

What other quantities can be used a global parameter?

Continuation in First Harmonic

We describe numerical computation of solutions for the problem

$$u'' + f(u) = \mu \sin x + e(x), \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0\,, \tag{7}$$

although similar results hold for the corresponding PDE's on general domains. Here $\int_0^\pi e(x) \sin x \, dx = 0$, μ a real parameter. Writing $u(x) = \xi \sin x + U(x)$, with $\int_0^\pi U(x) \sin x \, dx = 0$, we shall compute the solution curve of (7): $(u(\xi), \mu(\xi))$, by using Newton's method to perform continuation in ξ. If $f'(u) < \lambda_2 = 4$, the value of ξ is a global parameter, uniquely identifying the solution pair $(\mu, u(x))$.

Example We solved

$$u'' + \sin u = \mu \sin x + x - \frac{\pi}{2}\,, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0\,. \tag{8}$$

Observe that $\int_0^\pi \left(x - \frac{\pi}{2}\right) \sin x \, dx = 0$.

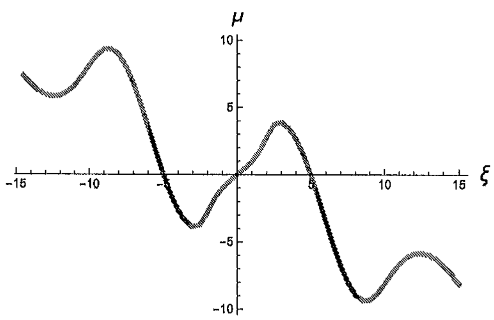


Figure 2: The global solution curve for the problem (8)

Look at the points of intersection with the ξ-axis, where $\mu = 0$. The picture suggests that the problem

$$u'' + \sin u = x - \frac{\pi}{2}\,, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0$$

has exactly three solutions, one of which has zero first harmonic.