Some special super-critical equations

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Consider the equation

$$u''(r) + \frac{n-1}{r} u'(r) + u^{\frac{n+2}{n-2}} = 0. \tag{0.1}$$

(Of course $u''(r) + \frac{n-1}{r} u'(r) = \Delta u$ for radially symmetric functions in R^n.) This equation involving the critical Sobolev power is prominent in both PDE and Differential Geometry. Let us set

$$u' = -aru^{\frac{n}{n-2}}, \quad \text{(with any } a > 0). \tag{0.2}$$

Then $u'' = -au^{\frac{n}{n-2}} - a\frac{n}{n-2} r u^{\frac{n}{n-2}-1} u' = -au^{\frac{n}{n-2}} - a^2 \frac{n}{n-2} r^2 u^{\frac{n+2}{n-2}}$. Using these expressions in (0.1) gives

$$u(r) = \left(\frac{a\,n}{1 + \frac{n}{n-2} a^2 r^2} \right)^{\frac{n-2}{2}}. \tag{0.3}$$

Such a "scheme" seems unlikely to work, but it does, producing the well known ground state solution. This solution satisfies the first order "ansatz" (0.2). The equation (0.1) is very special. If one tries any power other than $\frac{n+2}{n-2}$ in (0.1), this approach fails.

Similar approach works for another special equation: Gelfand's equation

$$u''(r) + \frac{n-1}{r} u'(r) + e^u = 0,$$

with applications to combustion.

The list of such equations is completed by the C.S. Lin and W.-M. Ni's equation

$$u''(r) + \frac{n-1}{r} u'(r) + u^p(r) + u^{2p-1}(r) = 0, \tag{0.4}$$

(with $\frac{n}{n-2} < p < \frac{n+2}{n-2} < 2p-1$, which appears to be very special as well, see K. *Quart. Appl. Math.* (2017) where it is shown that there are essentially no other such equations. In particular, its ground state solution

$$u(r) = \left(\frac{2p(np-n-2p)}{(np-n-2p)^2 + p(p-1)^2 r^2} \right)^{\frac{1}{p-1}}.$$

(discovered by C.S. Lin and W.-M. Ni) satisfies a first-order equation, similar to (0.2). Like the two equations above, I think the Lin-Ni equation "ought to have" some applications. More evidence that the Lin-Ni equation is special is given in I. Flores JDE (2004), who proved the existence of infinitely many ground state solutions approaching the Lin-Ni ground state.

We shall present some computational results on the Lin-Ni equation from the recent preprint of K. and D.S. Schmidt. But first we review the global bifurcation approach.

Consider the Dirichlet problem for semilinear Laplace equation on a unit ball in arbitrary space dimension, depending on a positive parameter λ :

$$\Delta u + \lambda f(u) = 0, \quad \text{for } |x| < 1, \; u = 0 \; \text{ for } |x| = 1 \tag{0.5}$$

in n dimensions. By the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg positive solutions are radially symmetric, so that $u = u(r)$, where $r = |x|$, and hence $u(r)$ satisfies an ODE

$$u''(r) + \frac{n-1}{r} u'(r) + \lambda f(u) = 0, \;\; u'(0) = 0, \; u(1) = 0. \tag{0.6}$$

Their theorem also asserts that $u_r < 0$ so that $u(0)$ gives the maximal value of the solution $u(r)$. It turns out that the value of $u(0)$ is a global parameter, uniquely identifying the solution pair $(\lambda, u(r))$. (If $u(0) = 5$, there is a unique $\lambda = \lambda_0$ and the corresponding solution $u_0(r)$, so that $u_0(0) = 5$.) It follows that two-dimensional curves in $(\lambda, u(0))$ plane provide a complete picture of the solution set of the PDE (0.5) (or of (0.6)). Computer program: K.-D.S. Schmidt https://homepages.uc.edu/∼kormanp/balls.pdf. Bifurcation theory approach was developed by Y. Li, T. Ouyang, J. Shi and P.K. It was summarized in the paper of T. Ouyang and J. Shi, JDE-1999, and in the book of P. Korman "Global Solution Curves for Semilinear Elliptic Equations", World Scientific, 2012.

Let us compute the solution curves of Lin-Ni with $n = 3$, $p = 4$:

$$u''(r) + \frac{2}{r} u'(r) + \lambda \left(u^4 + u^7\right) = 0, \;\; u'(0) = 0, \; u(1) = 0.$$

The Lin-Ni ground state solution $u(r) = \frac{2}{\sqrt[3]{36r^2+1}}$ has $u(0) = 2$.

No a priori estimates are possible here! All solutions with sufficiently large $u(0)$ lie on a unique top curve. On the top curve solutions tend to the singular solution, with $u(0) = \infty$. There are infinitely many curves

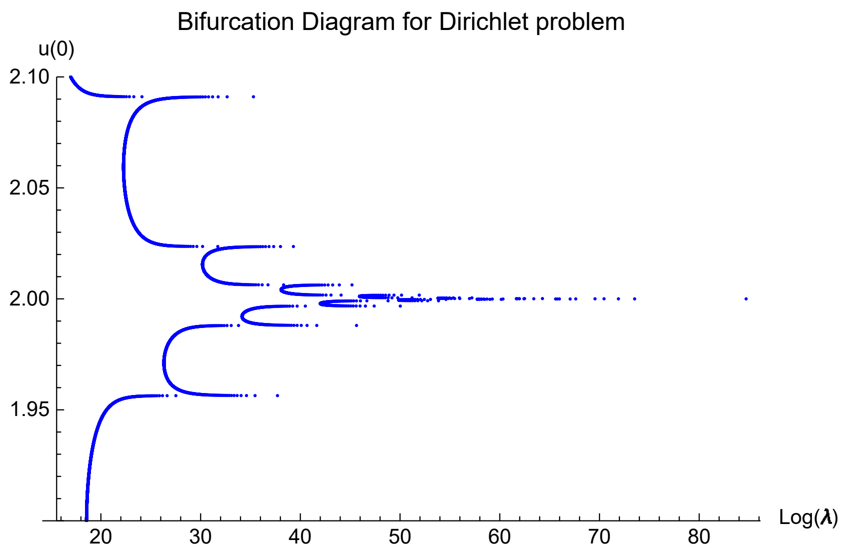

Figure 1: Infinitely many Dirichlet curves and infinitely many ground state solutions on both sides of $u(0) = 2$

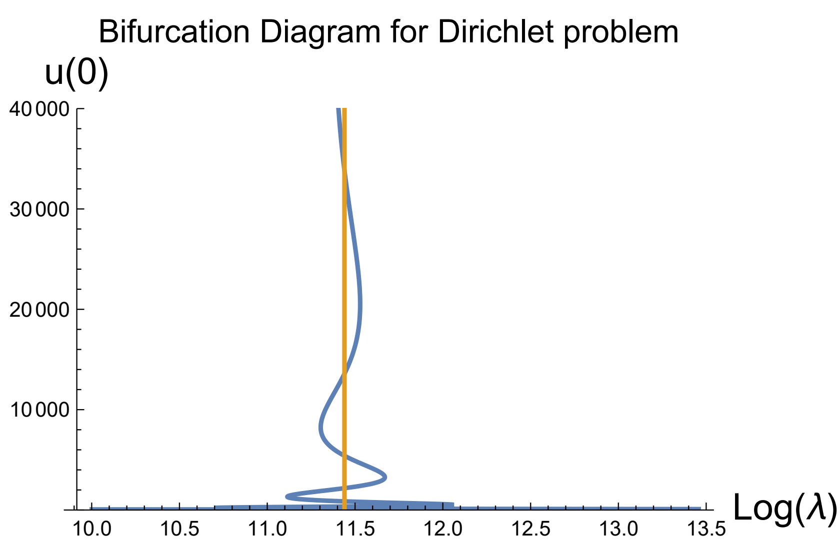

Figure 2: The curve approaching infinity

of solutions approaching from below and above the Lin-Ni ground state solution, where $u(0) = 2$.

Consider now

$$(0.7)\qquad u''(r)+\frac{n-1}{r}u'(r)+\lambda\left(u^p+u^q\right)=0,\;\; u'(0)=0,\; u(1)=0,$$

with $1 \le p < \frac{n+2}{n-2} < q$ (the case $p = 1$ and $q = \frac{n+2}{n-2}$ was studied in the classical paper of Brezis and Nirenberg, 1983). Strong results on multiplicity of large solutions are given by Y. Miyamoto 2014, and in later papers.

It follows from I. Flores JDE (2004): the number of solution curves is finite for all equations other than Lin-Ni. K. and D.S. Schmidt, preprint: the singular solution of (0.7) has the form $u(r) = r^{-\frac{2}{q-1}}v(r^\gamma)$, with $\gamma = 2\frac{q-p}{q-1}$. For the Lin-Ni equation, $\gamma = 1$. Both results suggest that the Lin-Ni equation is special.

Another type of special equations

Consider positive solutions of the super-critical Dirichlet boundary value problem

$$(0.8)\qquad u''+\frac{n-1}{r}u'+\lambda(1+u)^p=0\,,\;\; u'(0)=0\,,\; u(1)=0,$$

depending on a parameter λ, with $p > \frac{n+2}{n-2}$. Studied by D.D. Joseph and T.S. Lundgren in 1972.

Lemma 0.1 *Define* $B = \frac{2}{(p-1)^2}\left[-2+(p-1)(n-2)\right]$. *Assume that* $p > \frac{n}{n-2}$. *Then the singular solution of (0.8) occurs at* $\lambda \equiv \lambda_\infty = B$, *and it is* $w_0(r) = r^{-\frac{2}{p-1}} - 1$.

Using the self-similar nature of the equation in (0.8) it is possible to describe the entire solution curve of (0.8), which we do next.

Let $w(t)$ be the solution of

$$(0.9)\qquad w''+\frac{n-1}{t}w'+w^p=0\,,\;\; w'(0)=0\,,\; w(0)=1.$$

It is well-known (by Pohozhaev's identity) that for super-critical $p > \frac{n+2}{n+2}$, $w(t)$ remains positive for all $t \in (0,\infty)$, it is decreasing for all t, and tends to zero as $t \to \infty$.

Lemma 0.2 *Assume that $p > \frac{n+2}{n-2}$. Let $w(t)$ be the solution of (0.9). Then the entire solution curve of (0.8) is given by*

$$(\lambda, u(0)) = \left(t^2 w^{p-1}(t), \frac{1}{w(t)} - 1\right), \quad \textit{with } t \in (0, \infty). \tag{0.10}$$

The solution of (0.9) at the parameter value of t is

$$u(r) = \frac{1}{w(t)} w(tr) - 1.$$

The direction in which the solution curve travels is given by the sign of

$$\lambda'(t) = 2tw^{p-1} + (p-1)t^2 w^{p-2} w' = (p-1)tw^{p-2}\left[tw' + \frac{2}{p-1}w\right].$$

In K. JDE-2014, and in Proc. Roy. Soc. Edinburgh Sect. A-2018, $w(t)$ was called *the generating solution* for its role in (0.10), while $w_0(t)$ was called *the guiding solution*. The following lemma explains why.

Lemma 0.3 *Let $t_1 < t_2$ be two consecutive points of intersection of $w(t)$ and $w_0(t)$. Then $\lambda'(t_1)\lambda'(t_2) < 0$, and there is a turning point of (0.8) $t_0 \in (t_1, t_2)$, with $\lambda'(t_0) = 0$.*

Theorem 0.1 *Define $p_{_{JL}} = \infty$ for $3 \le n \le 10$ and $p_{_{JL}} = 1 + \frac{4}{n-2\sqrt{n-1}-4}$ for $n \ge 11$. If $\frac{n+2}{n-2} < p < p_{_{JL}}$ the solution curve makes infinitely many turns.*

In addition to a shorter proof (than in Joseph-Lundgren), there is considerable extra information provided with our approach. In particular all turning points are non-degenerate ($\lambda''(t_0) \neq 0$), and hence the entire picture persists under small perturbations of the equation. The Morse index of solutions increases by one after each turn.

Other equations for which a similar approach works: Gelfand equation

$$u'' + \frac{n-1}{r}u' + \lambda e^u = 0\,, \;\; u'(0) = 0\,,\; u(1) = 0,$$

and the equation modeling MEMS (micro-electronic-mechanical systems)

$$u'' + \frac{n-1}{r}u' + \lambda \frac{r^a}{(1-u)^p} = 0\,, \;\; u'(0) = 0\,,\; u(1) = 0.$$

The idea to parametrize the solution curves as in (0.10) was introduced by J.A. Pelesko, SIAM J. Appl. Math.-2002 in his work on MEMS. The results also cover the p-Laplace case.

Actually we had an extra term r^a in all three of the equations above:

$$u'' + \frac{n-1}{r}u' + \lambda r^a f(u) = 0\,, \;\; u'(0) = 0\,, \; u(1) = 0, \tag{0.11}$$

and the results were similar (the extra term r^a scales: $(cr)^a = c^a r^a$, and the equation is still self-similar), except that the conditions on the spatial dimensions were altered by the number a. This raised a Question: can one remove the r^a term by a change of variables?

Answer: The change of variables $t = \frac{r^{1+a/2}}{1+a/2}$ transforms the problem (0.11) into

$$u''(t) + \frac{m}{t}u'(t) + \lambda f(u(t)) = 0, \;\; u(0) = \alpha, \; \frac{du}{dt}(0) = 0, \tag{0.12}$$

with $m = \frac{n-1+a/2}{1+a/2}$. Observe that both the Laplacian in the equation, and the initial conditions are preserved! Only the dimension of the space is changed (if $n = 2$, the new dimension is $m + 1 = 2$, no change). K., *Quart. Appl. Math.* (2017).

There is a similar change of variables transforming radial k-Hessian equation to radial p-Laplace equation. K., Commun. Pure Appl. Anal. 19 (2020), and infinitely many turns for p-Laplace equations were established in Proc. Roy. Soc. Edinburgh Sect. A-2018.

Recently we studied the curves of solutions for the equation

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda\,(u^p + u^q) = 0, \;\; u'(0) = 0, \; u(1) = 0 \tag{0.13}$$

by comparing solutions with large $u(0)$ to solutions of

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda\,(1+u)^q = 0, \;\; u'(0) = 0, \; u(1) = 0,$$

to confirm the computational results of K-D.S. Schmidt, preprint. The equation (0.13) can in turn serve as a prototype for more general equations.