Necessary and sufficient condition for existence of solution at eigenvalues of multiplicity two

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We begin by reviewing the Fredholm alternative, and the "nonlinear Fredholm alternative" of Landesman-Lazer.

Given a bounded domain $D \subset R^n$, with a smooth boundary, we denote by λ_k the eigenvalues of the Dirichlet problem

$$\Delta u + \lambda u = 0, \;\; x \in D, \;\; u = 0 \;\text{ on } \partial D\,,$$

and by $\varphi_k(x)$ the corresponding eigenfuctions. For the *resonant* problem

$$\Delta u + \lambda_k u = f(x), \;\; x \in D, \;\; u = 0 \;\text{ on } \partial D\,, \tag{0.1}$$

with a given $f(x) \in L^2(D)$, the following well-known *Fredholm alternative* holds: the problem (0.1) has a solution if and only if

$$\int_D f(x)\varphi_k(x)\,dx = 0\,. \tag{0.2}$$

One could expect things to be considerably harder for the nonlinear problem

$$\Delta u + \lambda_k u + g(u) = f(x), \;\; x \in D, \;\; u = 0 \;\text{ on } \partial D\,, \tag{0.3}$$

However, in the classical paper of E.M. Landesman and A.C. Lazer (1970) an interesting class of nonlinearities $g(u)$ was identified, for which one still has an analog of the Fredholm alternative. Namely, one assumes that the finite limits $g(-\infty)$ and $g(\infty)$ exist, and

$$g(-\infty) < g(u) < g(\infty), \;\; \text{for all } u \in R\,. \tag{0.4}$$

Let us assume for simplicity that $k = 1$. It is known that λ_1 is simple (1-d eigenspace) and $\varphi_1(x) > 0$. Multiply

$$\Delta u + \lambda_1 u + g(u) = f(x), \;\; x \in D, \;\; u = 0 \;\text{ on } \partial D$$

by $\varphi_1(x)$ and integrate to obtain

$$\int_D g(u)\varphi_1(x)\,dx = \int_D f(x)\varphi_1(x)\,dx\,, \tag{0.5}$$

which implies, in view of (0.4) that

$$g(-\infty)\int_D \varphi_1(x)\,dx < \int_D f(x)\varphi_1\,dx < g(\infty)\int_D \varphi_1(x)\,dx$$

This is a necessary condition for solvability. It was proved by E.M. Landesman and A.C. Lazer (1970) that this condition is also sufficient for solvability. For general (sign-changing) φ_k this condition takes the form:

$$\begin{aligned}(0.6)\qquad g(-\infty)\textstyle\int_{\varphi_k>0}\varphi_k\,dx + g(\infty)\int_{\varphi_k<0}\varphi_k\,dx &< \textstyle\int_D f(x)\varphi_k\,dx \\ &< g(\infty)\textstyle\int_{\varphi_k>0}\varphi_k\,dx + g(-\infty)\int_{\varphi_k<0}\varphi_k\,dx\,.\end{aligned}$$

However, one still needs to assume that λ_k is *simple*. E.M. Landesman and A.C. Lazer result is the following.

Theorem 0.1 *Assume that λ_k is a* **simple eigenvalue***, while $g(u) \in C(R)$ satisfies (0.4). Then for any $f(x) \in L^2(D)$ satisfying (0.6), the problem (0.3) has a solution $u(x) \in W^{2,2}(D) \cap W_0^{1,2}(D)$.*

What if λ_k is not simple. In the same year when Landesman-Lazer was published (1970), S.A. Williams proved:

Theorem 0.2 *Assume that $g(u)$ satisfies (0.4), $f(x) \in L^2(D)$, while for* **any** *$w(x)$ belonging to the eigenspace of λ_k*

$$(0.7)\qquad \int_D f(x)w(x)\,dx < g(\infty)\int_{w>0} w\,dx + g(-\infty)\int_{w<0} w\,dx\,.$$

Then the problem (0.3) has a solution $u(x) \in W^{2,2}(D)\cap W_0^{1,2}(D)$. Condition (0.7) is also necessary for the existence of solutions.

However, no examples for multiple eigenvalues were known for a while, until we observed in 2016 that another classical result of A.C. Lazer and D.E. Leach (1969) on periodic solutions of semilinear harmonic oscillator provides an example to Theorem 0.2 in case of double eigenvalues.

In this paper we prove a similar result for a disc in R^2, thus providing the first PDE example for Theorem 0.2 in case of a multiple dimensional eigenspace. Even for simple domains the eigenspace of a multiple eigenvalue can be very complicated, and multiplicity of eigenvalues may vary in non-obvious ways. So that verifying the inequality (0.7) for any element $w(x)$ of the eigenspace appears to be next to impossible for other domains (the integrals $\int_{w>0} w(x)\,dx$ and $\int_{w<0} w(x)\,dx$ are unlikely to remain constant over an eigenspace).

Example Let $D = (0, \pi) \times (0, \pi)$ in R^2. The eigenvalues of

$$\Delta u + \lambda u = 0\,, \quad \text{in } D \quad u = 0 \text{ on } \partial D$$

are $\lambda_{nm} = n^2 + m^2$ with positive integers n and m, corresponding to the eigenfunctions $\sin nx \sin my$. These eigenfunctions are obtained by separation of variables, and there are no other eigenfunctions since these eigenfunctions form a complete set in $L^2(D)$. The principal eigenvalue $\lambda_1 = 2$ is simple, with the corresponding eigenfunction $\sin x \sin y > 0$. The eigenvalue $\lambda_2 = 5 = 1^2 + 2^2$ has multiplicity two, with the eigenspace spanned by $\sin x \sin 2y, \sin 2x \sin y$. The eigenvalue $\lambda_3 = 8 = 2^2 + 2^2$ is simple, with the eigenspace spanned by $\sin 2x \sin 2y$. The eigenvalue $50 = 1^2 + 7^2 = 5^2 + 5^2$ is triple, with the eigenspace

$$w = c_1 \sin x \sin 7y + c_2 \sin 7x \sin y + c_3 \sin 5x \sin 5y\,.$$

The set where $w > 0$ appears to to be complicated, and $\int_{w>0} w\, dxdy$ will depend on a particular choice of eigenvector w.

Remarkably, the eigenvalues of Laplacian on a disc $B : x^2+y^2 < a^2$ in two dimensions, with zero boundary condition, all have multiplicity two, except for the principal eigenvalue, which is simple. The eigenvalues are $\lambda_{n,m} = \dfrac{\alpha_{n,m}^2}{a^2}$ $(n = 0, 1, 2, \ldots;\ m = 1, 2, \ldots)$, with the corresponding eigenfunctions

$$J_n\left(\frac{\alpha_{n,m}}{a} r\right)(A \cos n\theta + B \sin n\theta)\ ,$$

where $\alpha_{n,m}$ is the m-th root of $J_n(x)$, the n-th Bessel function of the first kind, $r = \sqrt{x^2 + y^2}$ (A and B are arbitrary constants). The principal eigenvalue is simple, while all other eigenvalues have multiplicity two, because any two Bessel functions with indices different by an integer do not have any roots in common, see G.N. Watson, p. 484 for the following result.

Proposition 1 *For any integers $n \geq 0$ and $m \geq 1$, the functions $J_n(x)$ and $J_{n+m}(x)$ have no common zeros other than the one at $x = 0$.*

This result was apparently once a long standing conjecture, known in the 19-th century as Bourget's hypothesis (after the 19th-century French mathematician), until it was proved in 1929 by C.L. Siegel, see a very informative Wikipedia article on the Bessel functions. The name "hypothesis" suggests that it was used to prove other results. It immediately implies the following result that we need.

We show that the following integrals are independent of the choice of eigenfunction w:

$$\int_{w>0} w(r,\theta)\, r dr\, d\theta \equiv J_{n,m}\,,$$

$$\int_{w<0} w(r,\theta)\, r dr\, d\theta = -J_{n,m}\,.$$

Then the theorem of Williams above applies, giving the result described next. Let $\lambda_k = \lambda_{n,m}$, and φ_k, ψ_k corresponding eigenfunctions. Denote

$$\varphi_k = J_n\left(\frac{\alpha_{n,m}}{a} r\right) \cos n\theta\,,$$

$$\psi_k = J_n\left(\frac{\alpha_{n,m}}{a} r\right) \sin n\theta\,,$$

$$A_k = \int_B f(x,y)\varphi_k\, dxdy\,,$$

$$B_k = \int_B f(x,y)\psi_k\, dxdy\,.$$

The numbers A_k and B_k can be easily approximated by *Mathematica* for any $f(x,y)$ and k.

Theorem 0.3 *Assume that $g(u)$ satisfies the condition (0.4). Then the condition*

$$\sqrt{A_k^2 + B_k^2} < J_{n,m}\left(g(\infty) - g(-\infty)\right) \tag{0.8}$$

is both necessary and sufficient for the existence of solution $u(x) \in W^{2,2}(D) \cap W_0^{1,2}(D)$ of (0.3).

As an application, we get an unboundness result for the corresponding semilinear heat equation, when the inequality sign in (0.8) is reversed.