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Symmetry of Positive Solutions for Elliptic Problems in One Dimension

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PHILIP KORMAN
Department of Mathematical Sciences
University of Cincinnati
PO Box 210025
Cincinnati, Ohio 45221-0025
korman@ucbeh.san.uc.edu

Abstract. For semilinear elliptic problems in one space dimension we prove symmetry of positive solutions for both the bounded interval case and in R^1.

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1 INTRODUCTION

In their ground-breaking paper B. Gidas, W.-M. Ni and L. Nirenberg[1] showed that positive solutions of a Dirichlet problem in a ball in R^n,

$$\Delta u + f(r,u) = 0 \quad \text{for} \quad r = |x| < R, \quad u = 0 \ \text{when} \ |x| = R \tag{1.1}$$

are necessarily radially symmetric with $u'(r) < 0$, provided $f(r,u)$ is decreasing in r. In Gidas, Ni, and Nirenberg[2] a similar symmetry result is proved in R^n for two general classes of functions $f(u)$. This work was continued by several authors, particularly Y. Li and W.-M. Ni, see Li and Ni[6] for a review. It turned out that in the case $f(u) \leq 0$ for $u > 0$ one needs to place additional assumptions, either on $f(u)$ or on the behavior of the solution $u(x)$ at infinity, in order to prove symmetry.

In this paper for the R^1 case we prove symmetry for very general $f(x,u)$, without making any assumptions on the behavior of solutions. For a bounded interval we prove symmetry in a case when $f(x,u)$ is not decreasing in x. We also give symmetry results for a class of singular problems on an interval ($u(\pm 1) = \infty$), and an anti-symmetry result. While it is not particularly surprising that one can obtain stronger results in one dimension, we feel it might be of interest that we obtain our results by a technique different from the moving plane method of Gidas, Ni, and Nirenberg[1,2] and of later papers. Moreover, in all cases we prove that different positive solutions do not intersect. This property is useful in studying multiplicity of solutions, see Korman and Ouyang[3,4].

2 DIRICHLET PROBLEM ON AN INFINITE INTERVAL

Theorem 1 *Consider the boundary value problem*

$$u'' + f(x,u) = 0 \quad \textit{for} \quad x \in (-\infty,\infty), \quad u(-\infty) = u(\infty) = 0. \tag{2.1}$$

Assume that the function $f \in C^1(R \times R_+)$ *is such that*

$$f(-x,u) = f(x,u) \quad \textit{for all} \quad x \in (-\infty,\infty) \quad \textit{and} \quad u > 0, \tag{2.2}$$

$$x f_x(x,u) < 0 \quad \textit{for} \quad x \in (-\infty,\infty)\backslash\{0\} \quad \textit{and} \quad u > 0, \tag{2.3}$$

$$\textit{There is an } M > 0 \textit{ such that } |f(x,u)| \le M \tag{2.4}$$

$$\textit{for large } |x| \textit{ and small } u.$$

Then any positive solution of (2.1) is an even function with $u'(x) < 0$ *on* $(0,\infty)$. *Moreover, any two positive solutions of (2.1) cannot intersect on* $(-\infty,\infty)$ *(and hence they are strictly ordered).*

Proof. We begin by showing that a positive solution of (2.1) cannot have points of local minimum. Assume on the contrary that $\bar{x}$ is a point of local minimum of

$u(x)$. Let $\bar{y}$ be the smallest number greater than $\bar{x}$, such that $u(\bar{y}) = u(\bar{x})$. On the interval $(\bar{x}, \bar{y})$ let x_1 be the largest point of local minimum, and $y_1 > x_1$ be the smallest number with $u(y_1) = u(x_1)$ (it may happen that $x_1 = \bar{x}$, and then $y_1 = \bar{y}$). Let $\xi \in (x_1, y_1)$ be the unique point of local maximum of $u(x)$ on (x_1, y_1) and denote $u_1 = u(x_1) = u(y_1)$ and $\hat{u} = u(\xi)$. Multiply (2.1) by u' and integrate from x_1 to ξ. Obtain

$$\int_{u_1}^{\hat{u}} f(x_1(u), u)du = 0, \tag{2.5}$$

where $x = x_1(u)$ is the inverse function of $u(x)$ on (x_1, ξ). Proceeding similarly on (ξ, y_1),

$$\frac{1}{2}u'^2(y_1) + \int_{\hat{u}}^{u_1} f(x_2(u), u)du = 0. \tag{2.6}$$

Adding (2.5) and (2.6),

$$\frac{1}{2}u'^2(y_1) + \int_{u_1}^{\hat{u}} [f(x_1(u), u) - f(x_2(u), u)]\, du = 0. \tag{2.7}$$

Since $x_2(u) > x_1(u)$ for all $u \in (u_1, \hat{u})$, we see that the integral term in (2.7) is positive, which obviously leads to a contradiction. It follows that $u(x)$ has no local minimums, and hence it has only one point of local maximum x_0, which is the point of global maximum.

Notice that $v(x) = u(-x)$ is also a solution of (2.1) with $u(0) = v(0) \equiv u_0$ and

$$|u'(0)| = |v'(0)|. \tag{2.8}$$

Assume that the solution $u(x)$ is not even, i.e., $v(x) \not\equiv u(x)$. If we now assume that the point of maximum $x_0 = 0$, then $u'(0) = v'(0) = 0$, and we get an immediate contradiction, since by uniqueness theorem for initial-value problems we would have $u(x) \equiv v(x)$. So we may assume that $x_0 > 0$.

Let ξ denote the point where $u(\xi) = u(0)$. Then arguing as in derivation of (2.7) (see also Korman and Ouyang[3]) we easily conclude that

$$|u'(0)| > |u'(\xi)|. \tag{2.9}$$

Assume we can find $\hat{u} < u_0$, with $u(\beta) = v(\alpha) = \hat{u}$ and

$$|u'(\beta)| \geq |v'(\alpha)| \quad (0 < \alpha \leq \beta). \tag{2.10}$$

In particular this will happen if $\beta = \alpha$, i.e. the graphs of $u(x)$ and $v(x)$ intersect. Multiply (2.1) by u' and integrate from ξ to β. Obtain

$$\frac{1}{2}u'^2(\beta) - \frac{1}{2}u'^2(\xi) + \int_{u_0}^{\hat{u}} f(x_4(u), u)du = 0, \tag{2.11}$$

where $x = x_4(u)$ is the inverse of $u(x)$ on (ξ, β). Similarly, integrating over $(0, \alpha)$,

$$\frac{1}{2}v'^2(\alpha) - \frac{1}{2}v'^2(0) + \int_{u_0}^{\hat{u}} f(x_3(v), v)dv = 0, \tag{2.12}$$

where $x = x_3(v)$ is the inverse of $v(x)$ on $(0, \alpha)$. From (2.11) subtract (2.12),

$$\begin{aligned} &\frac{1}{2}\left(v'^2(0) - u'^2(\xi)\right) + \frac{1}{2}\left(u'^2(\beta) - v'^2(\alpha)\right) \\ &\int_{\hat{u}}^{u_0} \left[f(x_3(u), u) - f(x_4(u), u)\right] du = 0. \end{aligned} \tag{2.13}$$

The first term in (2.14) is positive by (2.8) and (2.9), the second one is nonnegative by (2.10), and the integral term is positive because $x_4(u) > x_3(u)$ for all $u \in (\hat{u}, u_0)$. This is a contradiction.

It remains to handle the possibility that

$$|v'(u)| > |u'(u)| \quad \text{for all } u < \hat{u} \text{ and } x > 0. \tag{2.14}$$

We claim that $\lim_{x\to\infty} |u'(x)| = \lim_{x\to\infty} |v'(x)| = 0$. Assume on the contrary that $|u'(x_n)| \geq \epsilon > 0$ for some sequence $\{x_n\} \to \infty$. Since by (2.4) $u''(x)$ is uniformly bounded, we would have $|u'(x)| > \epsilon/2$ over $(x_n, x_n + \delta))$ for some $\delta > 0$. But this is clearly impossible, since $\lim_{x\to\infty} u(x) = 0$ and $u'(x) \le 0$ for $x > x_0$ ($u(x)$ drops by at least $\frac{\epsilon\delta}{2}$ over $(x_n, x_n + \delta)$). We now obtain a formula similar to (2.13) by integrating the equations for $u(x)$ and $v(x)$ over (ξ, ∞), and a similar contradiction. This completes the proof that $u(x)$ is even.

That $u' < 0$ for $x > 0$ easily follows by differentiating (2.1). For the final claim, assume that two even solutions $u(x)$ and $v(x)$ intersect at $\xi > 0$, and $|u'(\xi)| < |v'(\xi)|$. By the previous arguments we can find $\xi < \alpha < \beta$ so that (2.10) holds. Integrating (2.1) times u' over (ξ, α) and over (ξ, β), we obtain a contradiction as before.

Remark. Clearly, it suffices for the inequality (2.3) to hold almost everywhere (and a similar generalization applies to all other results of this paper). In case $f = f(u)$ it is easy to prove that any positive solution of (2.1) is even about its point of global maximum, call it x_0, and $u'(x) < 0$ for $x > x_0$. To see this, we begin by remarking that if $u'(y_1) = 0$ then $u(x)$ is even with respect to y_1, which follows by observing that $v(x) \equiv u(2y_1 - x)$ is another solution of (2.1), satisfying the same initial condition. If now y_1 is the point defined in the proof above, then $u'^2(y_1) > 0$. Repeating the argument leading to (2.7), we conclude that $u(x)$ can have no points of local minimum. If x_0 is the unique point of global maximum of $u(x)$, then $u(2x_0 - x) \equiv u(x)$ by the uniqueness for the initial-value problems, completing the proof.

Remark. Examining the proof we see that under our conditions (2.2-2.4) any nonnegative solution of (2.1) is in fact positive, and hence all conclusions of the theorem hold for nonnegative solutions.

Remark. Under the additional assumption that $u'(-\infty) = u'(\infty)$ (homoclinic solutions) a similar result was proved in Korman and Lazer[5].

3 SYMMETRY FOR A CLASS OF SINGULAR PROBLEMS

We establish symmetry of solutions for a class of singular problems considered recently in a number of papers, see e.g. recent preprints of A.C. Lazer and P. McKenna and the references therein. Since the proof is similar to that of the theorem 1 we only sketch some of the steps. Notice that solution is not required to be positive, and that we write $u(1) = +\infty$ to signify that $\lim_{x \to 1^-} u(x) = +\infty$.

Theorem 2 *Consider the problem*

$$u'' + f(x,u) = 0 \quad on \quad (-1,1), \quad u(-1) = u(1) = +\infty. \tag{3.1}$$

Assume that the function $f \in C^1([-1,1] \times R)$ is such that

$$f(-x,u) = f(x,u) \quad for\ all \quad x \in (-1,1) \quad and\ all\ u, \tag{3.2}$$

$$f_x(x,u) > 0 \quad for \quad x \in (0,1) \quad and\ all\ u. \tag{3.3}$$

Then any solution of (3.1) is an even function with $u'(x) > 0$ for $x > 0$. Moreover, any two solutions of (3.1) cannot intersect.

Proof. As before we show that $u(x)$ has no local maximums on $(-1,1)$. This implies that $u(x)$ has only one (global) minimum, say at $x = x_0$. The function $v(x) \equiv u(-x)$ is also a solution of (3.1) with $v(0) = u(0) \equiv u_0$ and $v'(0) = -u'(0)$. If x_0, then $v(x) \equiv u(x)$ and solution is even, so assume that $x_0 > 0$. Define $\xi > 0$ to be the point where $u(0) = u(\xi) = u_0$. As before we show that

$$|u'(0)| > |u'(\xi)|. \tag{3.4}$$

Assume we can find $\hat{u} > u_0$ with $u(\beta) = u(\alpha) = \hat{u}$ and

$$u'(\beta) \geq v'(\alpha) \quad (\beta \geq \alpha). \tag{3.5}$$

In particular this will happen if $\beta = \alpha$, i.e. $u(x)$ and $v(x)$ intersect (notice that for small x the graph of $v(x)$ is above that of $u(x)$). Then multiplying (3.1) by u' and integrating over (ξ, β) and multiplying (3.1) by v' and integrating over $(0, \alpha)$, we obtain respectively

$$\frac{1}{2}u'^2(\beta) - \frac{1}{2}u'^2(\xi) + \int_{u_0}^{\hat{u}} f(x_2(u), u)du = 0, \tag{3.6}$$

$$\frac{1}{2}v'^2(\alpha) - \frac{1}{2}v'^2(0) + \int_{u_0}^{\hat{u}} f(x_1(u), u)du = 0. \tag{3.7}$$

Here $x = x_1(u)$ is the inverse function of $u(x)$ on (ξ, β), and $x = x_2(u)$ is the inverse of $v(x)$ on $(0, \alpha)$. Subtracting

$$\begin{aligned} &\frac{1}{2}\left(v'^2(0) - u'^2(\xi)\right) + \frac{1}{2}\left(u'^2(\beta) - v'^2(\alpha)\right) \\ &\quad + \int_{u_0}^{\hat{u}} \left[f(x_2(u), u) - f(x_1(u), u)\right] du = 0. \end{aligned} \tag{3.8}$$

Since $x_2(u) > x_1(u)$ for all $u \in (u_0, \hat{u})$ it follows by (3.3-3.5) that the left hand side of (3.9) is positive, a contradiction.

To prove the evenness of solution it remains to exclude the possibility that (3.5) cannot be achieved, i.e. the possibility that for all $u > u_0$, $v'|_u > u'|_u$. In such a case we would have

$$\frac{dx_2}{du} = \frac{1}{u'} > \frac{1}{v'} = \frac{dx_1}{du} \quad \text{for all} \quad u > u_0.$$

Denoting $w(u) = x_2(u) - x_1(u)$, we have that for $u > u_0$, $w(u) > 0$, $w'(u) > 0$, while $\lim_{u\to\infty} w(u) = \lim_{u\to\infty} x_2(u) - \lim_{u\to\infty} x_1(u) = 1 - 1 = 0$, which is impossible.

The remaining claims follow as before.

4 SYMMETRY ON A BOUNDED INTERVAL

Consider the boundary value problem

$$u'' + f(x, u) = 0 \quad \text{on} \quad (-1, 1), \quad u(-1) = u(1) = 0. \tag{4.1}$$

We again assume that $f \in C^1([-1, 1] \times R_+)$ and

$$f(-x, u) = f(x, u) \quad \text{for all} \quad x \in (-1, 1) \quad \text{and} \quad u > 0. \tag{4.2}$$

If $f(x, u)$ is decreasing in x for $x > 0$ (or $xf_x < 0$ for all x) then the well-known theorem of B. Gidas, W.-M. Ni and L. Nirenberg[1] applies, and so any positive solution of (4.1) is even. Next we consider a class of nonlinearities where this condition ($xf_x < 0$) is violated. Namely we assume that $f(x, u) = (u - a)g(x, u)$, where a is a positive constant and the function $g(x, u) \in C^1([-1, 1] \times R_+)$ satisfies

$$g(-x, u) = g(x, u) \quad \text{for all} \quad x \in (-1, 1) \quad \text{and} \quad u > 0, \tag{4.3}$$

$$xg_x(x, u) < 0 \quad \text{for all} \quad x \in (-1, 1) \quad \text{and} \quad u > 0, \tag{4.4}$$

$$g(x, u) < 0 \quad \text{and} \quad g_u > 0 \quad \text{for} \quad 0 < u < a \quad \text{and} \quad x \in (-1, 1). \tag{4.5}$$

(Notice that $xf_x > 0$ for $0 < u < a$.)

Theorem 3 *Under the assumptions (4.3-4.5), any positive solution of (4.1) is an even function with $u'(x) < 0$ for $x \in (0, 1)$. Moreover, any two positive solutions of (4.1) cannot intersect.*

Proof. Notice that (4.5) implies that

$$f(x, u) > 0 \quad \text{and} \quad f_u(x, u) < 0 \quad \text{for} \quad u \in (0, a) \quad \text{and} \quad x \in (-1, 1). \tag{4.6}$$

We need the following lemma. It says that among two monotone solutions near $x = 1$, the larger one is steeper.

Lemma. Assume that $u(x)$ and $v(x)$ are two positive solutions of (4.1), such that for some $\gamma \in (-1,1)$

(i) $\quad v(x) < u(x) \leq a \quad \text{on} \quad (\gamma, 1),$

(ii) $\quad u'(x) \leq 0 \quad \text{and} \quad v'(x) \leq 0 \quad \text{on} \quad (\gamma, 1).$

(iii) Assume $\bar{\alpha}, \bar{\beta} \in (\gamma, 1)$ are such that $(\bar{\alpha} < \bar{\beta})$ $v(\bar{\alpha}) = u(\bar{\beta})$.

Then

$$u'(\bar{\beta}) < v'(\bar{\alpha}) \quad (\text{i.e. } |u'(\bar{\beta})| > |v'(\bar{\alpha})|). \tag{4.7}$$

Proof. On $(\bar{\beta}, 1)$ we consider $w = u - v > 0$. Then

$$w'' + c(x)w = 0 \quad \text{on} \quad (\bar{\beta}, 1), \quad w(1) = 0, \tag{4.8}$$

where, using (4.6)

$$c(x) = \int_0^1 f_u(x, \theta u + (1-\theta)v)d\theta < 0.$$

Multiplying (4.8) by w and integrate from $\bar{\beta}$ to 1,

$$-w(\bar{\beta})w'(\bar{\beta}) - \int_{\bar{\beta}}^1 w'^2 dx + \int_{\bar{\beta}}^1 c(x)w^2 dx = 0. \tag{4.9}$$

Since the integral terms in (4.9) are negative, it follows that $w'(\bar{\beta}) < 0$, i.e. $u'(\bar{\beta}) < v'(\bar{\beta})$. But by (4.6) $v'' = -f(x, v) < 0$ when $v(x) < a$, and hence

$$u'(\bar{\beta}) < v'(\bar{\beta}) < v'(\bar{\alpha}),$$

concluding the proof of the lemma.

Corollary. Let $u(x)$ and $v(x)$ be two positive solutions of (4.1) with $v(x) < u(x)$ near $x = 1$. Then these solutions cannot intersect so long as $u \leq a$ and they are both decreasing.

Proof. If $u(\delta) = v(\delta) \leq a$, then $|u'(\delta)| < |v'(\delta)|$. On the other hand, passing to the limit in (4.7) as $\bar{\alpha}, \bar{\beta} \to \delta$, $|u'(\delta)| \geq |v'(\delta)|$.

Returning to the proof of the theorem, we claim that any positive solution of (4.1) cannot have local minimums, and hence it has only one (global) maximum. Indeed, if x_1 is a point of local minimum, then $f(x_1, u(x_1)) = -u''(x_1) \leq 0$, i.e. $u(x_1) \geq a$. But then $u(x) > a$ on the interval (x_1, y_1) defined in the proof of the theorem 1, and we obtain the same contradiction as we did there.

Assume now that a solution $u(x)$ is not even. As in theorem 1 we may assume that it takes its global maximum at $\bar{x} > 0$, and $v(x) \equiv u(-x)$ is a different solution with $u(0) = v(0) \equiv u_0$, and $|u'(0)| = |v'(0)|$. We can assume that

$$\bar{u} > a, \tag{4.10}$$

for otherwise we would have two different solutions of (4.1) in the region where $f_u < 0$, which easily leads to a contradiction. We claim that

$$v(x) < u(x) \quad \text{for all} \quad x \in (0,1). \tag{4.11}$$

Assume (4.11) to be violated. Then there are three cases.

Case i. $v(x) > u(x)$ near $x = 1$. Let $0 < \alpha, \beta < 1$ be such that $u(\beta) = v(\alpha) = a$. Since $v(x)$ has the same maximum as $u(x)$, $v(x)$ must intersect $u(x)$ on $(\bar{x}, 1)$. Since these functions cannot intersect where $u \leq a$, it follows that $\beta < \alpha$ and there is $\eta \in (\bar{x}, \beta)$ such that

$$u(\eta) = v(\eta) \equiv u_1 > a, \quad |u'(\eta)| > |v'(\eta)|. \tag{4.12}$$

By lemma, reversing the roles of u and v,

$$|v'(\alpha)| > |u'(\beta)|. \tag{4.13}$$

Multiplying (4.1) by u' and integrate over (η, β),

$$\frac{1}{2}u'^2(\beta) - \frac{1}{2}u'^2(\eta) + \int_{u_1}^{a} f(x_1(u), u)du = 0, \tag{4.14}$$

where $x = x_1(u)$ is the inverse function of $u(x)$ on (η, β). Similarly, integrating over (η, α)

$$\frac{1}{2}v'^2(\alpha) - \frac{1}{2}v'^2(\eta) + \int_{u_1}^{a} f(x_2(u), u)du = 0, \tag{4.15}$$

with $x_2(u) > x_1(u)$ for $u \in (a, u_1)$. Subtracting (4.15) from (4.14), and using (4.12) and (4.13), we obtain the same contradiction as previously.

Case ii. $v(x) < u(x)$ near $x = 1$, but $u(x)$ and $v(x)$ intersect somewhere on $(\bar{x}, 1)$. Reversing the roles of u and v, we obtain the same contradiction as in case i.

Case iii. $v(x) < u(x)$ near $x = 1$, but $u(x)$ and $v(x)$ intersect on $(0, \bar{x})$. Let $\xi \in (0, \bar{x})$ be the point of intersection. Since $v(\xi) > v(0)$ and $v(-x_0) > v(0)$, it follows that $v(x)$ has point(s) of local minimum, which is impossible. The claim (4.11) is proved.

In view of (4.10) there are two possibilities.

(i) $\quad u_0 < a < \bar{u},$

(ii) $\quad a < u_0 < \bar{u}.$

Assume (i) holds. We can find $\gamma < 0 < p < q < 1$, such that $u(p) = u(q) = u(\gamma) = a$. By the definition of $v(x)$, $\gamma = -p$ and $|v'(\gamma)| = |u'(p)|$. By lemma,

$$|u'(p)| = |v'(\gamma)| < |u'(q)|.$$

On the other hand, arguing as in the theorem 1,

$$|u'(p)| > |u'(q)|,$$

a contradiction.

Assume now (ii) $a < u_0 < \bar{u}$. Define $\xi \in (0,1)$ by $u(\xi) = u(0) = u_0$, and $0 < \alpha < \beta < 1$ by $v(\alpha) = u(\beta) = a$. By the argument from the theorem 1, just mentioned,

$$|v'(0)| = |u'(0)| > |u'(\xi)|. \tag{4.16}$$

We now multiply (4.1) by u' and integrate over (ξ, β), then multiply the corresponding equation for v by v' and integrate over $(0, \alpha)$, then subtract,

$$\begin{aligned} &\frac{1}{2}\left(u'^2(\beta) - v'^2(\alpha)\right) + \frac{1}{2}\left(v'^2(0) - u'^2(\xi)\right) \\ &+ \int_a^{u_0} \left[f(x_1(u), u) - f(x_2(u), u)\right] du = 0, \end{aligned} \tag{4.17}$$

where $x_1(u)$ and $x_2(u)$ are inverse functions of $u(x)$ and $v(x)$ on the intervals (ξ, β) and $(0, \alpha)$ respectively. The first term in (4.18) is positive by lemma, the second one by (4.16), and the third one is positive, since $x_2(u) > x_1(u)$ for all $u \in (a, u_0)$. We have a contradiction, proving that any solution of (4.1) is even.

That any two even and positive solutions of (4.1) cannot intersect follows by essentially the same argument as the one leading to (4.15).

Example. $u'' + (u-a)(u-b(x))(c(x)-u) = 0$ on $(-1,1)$, $u(\pm 1) = 0$. Here $b(x)$ and $c(x)$ are even functions, a is a positive constant. We assume that $c'(x) < 0$, $b'(x) + c'(x) \geq 0$ and $c''(x) < 0$ for $x \in (0,1)$, and $a < b(x) < c(x)$ for all $x \in (-1,1)$. Then the theorem applies. Indeed, using maximum principle, we easily conclude that $u(x) < c(x)$ for all x, and then

$$g_x = -b'(c-u) + c'(u-b) = -b'c - c'b + (b'+c')u$$

$$\leq \quad -b'c - bc' + (b' + c')c = c'(c - b) < 0,$$

for $x \in (0, 1)$. This example was considered in Korman and Ouyang[4].

5 ODD SOLUTIONS

With a constant $A > 0$, we consider the problem

$$u'' + f(x, u) = 0 \quad \text{on} \quad (-1, 1), \quad u(-1) = -A, u(1) = A. \tag{5.1}$$

We assume that $f \in C^1([-1, 1] \times [-A, A])$ and

$$f(-x, u) = -f(x, u) \quad \text{for all} \quad x \in (-1, 1) \quad \text{and} \quad u \in (-A, A), \tag{5.2}$$

$$f(x, -u) = f(x, u) \quad \text{for all} \quad x \in (-1, 1) \quad \text{and} \quad u \in (-A, A) \tag{5.3}$$

$$f(x, u) \geq 0 \quad \text{for} \quad x \in (0, 1) \quad \text{and} \quad u \in (-A, A), \tag{5.4}$$

$$f_x(x, u) > 0 \quad \text{for} \quad x \in (-1, 1) \quad \text{and} \quad u \in (-A, A). \tag{5.5}$$

We are interested in solutions satisfying

$$-A < u(x) < A \quad \text{for all} \quad x \in (-1, 1). \tag{5.6}$$

Theorem 4 *Under the conditions (5.2-5.5) the problem (5.1) has at most one solution satisfying (5.6). This solution is an odd strictly increasing function.*

Proof. Arguing as in theorem 3, we use the conditions (5.3) and (5.5) to show that $u(x)$ has no local maximums. It follows that $u(x)$ is an increasing function on $(-1, 1)$. The function $v(x) = -u(-x)$ is also a solution of (5.1) with $v(0) = -u(0)$ and $v'(0) = u'(0)$. If $u(0) = 0$ then $u(x) \equiv v(x)$ proving that $u(x)$ is odd, so assume that $u(0) > 0$. Let $\xi \in (0, 1)$ be such that $v(\xi) = u(0) \equiv u_0$. By (5.4)

$$v'(\xi) \leq v'(0) = u'(0). \tag{5.7}$$

Multiply (5.1) by u' and integrate over $(0,1)$,

$$\frac{1}{2}u'^2(1) - \frac{1}{2}u'^2(0) + \int_{u_0}^{A} f(x_1(u), u)du = 0, \tag{5.8}$$

where $x = x_1(u)$ is the inverse of $u(x)$ on $(0,1)$. Similarly,

$$\frac{1}{2}v'^2(1) - \frac{1}{2}v'^2(\xi) + \int_{u_0}^{A} f(x_2(u), u)du = 0, \tag{5.9}$$

where $x = x_2(u)$ is the inverse of $v(x)$ on $(\xi, 1)$. Subtracting (5.9) from (5.8), and using (5.5) and (5.7) we obtain a contradiction, as we did before.

Uniqueness of the odd solution follows by essentially the same argument (integrating over $(0,1)$).

Remark. H. Berestycki and L. Nirenberg[7] have a more general result: they consider the PDE case in cylindrical domains. Our Theorem 4 provides a very simple proof in one-dimensional case.

Example. $u'' + \sin x u^p = 0$ on $(-1,1), u(\pm 1) = \pm 1$. If p is an even integer, the theorem applies. We computed the odd solutions satisfying $-1 < u(x) < 1$ for $p = 2$ and 4 (they are stable). For $p = 3$ we computed a monotone solution with $-1 < u(x) < 1$, which is however not odd.

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