

Curves of Positive Solutions for Supercritical Problems

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For the Dirichlet problem on ball in R^n

$$\Delta u + \lambda |x|^s (u + \gamma)^p = 0 \quad \text{for } |x| < R, \quad u = 0 \quad \text{on } |x| = R,$$

with $p < (n+s)/(n-2)$, we show existence of a critical $\lambda_0 > 0$, such that there are two positive radial solutions for $\lambda \in (0, \lambda_0)$, one at $\lambda = \lambda_0$, and none for $\lambda > \lambda_0$.

In another direction we show that for the model problem

$$\Delta u + \lambda u^q + u^p = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D,$$

with $1 \leq q < (n+2)/(n-2) < p$, on a bounded star-shaped domain in R^n , there is a $\lambda^* > 0$ so that the problem has no positive solution for $\lambda < \lambda^*$.

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1. INTRODUCTION

Our first result deals with the exact multiplicity of positive radially symmetric solutions for the semilinear Dirichlet problem on a ball $|x| < R$ in R^n

$$\Delta u + \lambda |x|^s (u + \gamma)^p = 0 \quad \text{for } |x| < R, \quad u = 0 \quad \text{on } |x| = R. \quad (1.1)$$

Here λ is a positive parameter. We show that all positive radial solutions lie on a unique smooth curve of solutions, which makes exactly one turn at some $\lambda = \lambda_0$. In particular, we have an exact number of solutions: there are two solutions for $\lambda \in (0, \lambda_0)$, one when $\lambda = \lambda_0$, and none when $\lambda > \lambda_0$. In case $s > 2$ we cover in particular some supercritical non-linearities, i.e. the ones with $p > (n+2)/(n-2)$. While our result is limited to the problem (1.1), it is, to the best of our knowledge, the only known example of

a global fold for both equations with explicit dependence on x , and for supercritical equations.

We use bifurcation theory, as developed in [10]. Crucial to this approach is the study of the corresponding linearized equation. It turns out that at a singular solution of (1.1) one can solve explicitly the linearized equation, and then conclude that the solution is positive. Then it is easy to compute the direction of bifurcation. To complete the proof we need to show that the solution curve cannot go to infinity at a positive λ . Since our non-linearity vanishes when $x = 0$, we cannot use the well-known *a priori* estimates of Gidas and Spruck [8]. We derive an *a priori* estimate by reducing our problem to a Emden–Fowler equation.

In another direction we consider the problem

$$\Delta u + \lambda u^q + u^p = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D \quad (1.2)$$

on a bounded domain $D \subset \mathbb{R}^n$, star-shaped with respect to the origin. We assume that p is a supercritical constant, while q is subcritical, i.e.

$$1 \leq q < \frac{n+2}{n-2} < p.$$

We show that there is a $\lambda^* > 0$ so that the problem (1.2) has no non-trivial non-negative solution for $\lambda < \lambda^*$. We then generalize, replacing u^q and u^p by the sums of supercritical and subcritical powers respectively. In case $q = 1$ the proof can be found in [3]. If D is a ball this result appears in [1,11]. A related result is proved in [14].

2. EXACT MULTIPLICITY FOR A CLASS OF EQUATIONS

In this section we prove the following theorem.

THEOREM 2.1 *Assume that $s > 0$ and $\gamma > 0$ are constants, and $p > 1$ is any constant in case $n = 2$, and in case $n > 2$ it satisfies*

$$p < \frac{n+s}{n-2}. \quad (2.1)$$

Then there is a critical λ_0 , so that the problem (1.1) has exactly 2, 1 or 0 positive radial solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Moreover, all solutions lie on a unique smooth solution curve. This curve starts at $(\lambda = 0, u = 0)$, it makes exactly one turn at λ_0 , and then it tends to infinity as $\lambda \downarrow 0$.

Remark Notice that in case $s > 2$ the theorem allows supercritical p .

The radially symmetric solutions for the problem (1.1) satisfy (here $r = |x|$, $u = u(r)$)

$$u'' + \frac{n-1}{r} u' + \lambda f(r, u) = 0 \quad \text{for } r < R, \quad u'(0) = u(R) = 0, \quad (2.2)$$

where $f(r, u) = r^s(u + \gamma)^p$. The crucial role will be played by the linearized problem

$$w'' + \frac{n-1}{r} w' + \lambda f_u(r, u) w = 0 \quad \text{for } r < R, \quad w'(0) = w(R) = 0. \quad (2.3)$$

Recall that solution $u(r)$ of (2.2) is called singular if (2.3) admits a non-trivial solution. At a singular solution we are able to find $w(r)$ explicitly, and to prove that it is of one sign, i.e. we may assume it to be positive.

LEMMA 2.1 *Let $u(r)$ be singular solution of (2.2). Then*

$$w(r) = ru'(r) + \frac{2+s}{p-1}u(r) + \gamma \frac{2+s}{p-1}, \quad (2.4)$$

and $w(r) > 0$ for all $r \in [0, R)$.

Proof Let $v(r) = ru'(r) + \mu u + \alpha$, with the positive constants μ and α to be specified. One sees that $v(r)$ satisfies

$$\begin{aligned} v'' + \frac{n-1}{r}v' + \lambda f_u(r, u)v &= \lambda[-2f + \mu(f_u u - f) + \alpha f_u - r f_r] \\ &= \lambda r^s[(u + \gamma)^{p-1}(\mu p u + \alpha p) + (u + \gamma)^p(-2 - \mu - s)]. \end{aligned} \quad (2.5)$$

Setting

$$\alpha = \mu \gamma, \quad (2.6)$$

we rewrite the right hand side of (2.5) as

$$\lambda r^s(u + \gamma)^p(-2 - \mu - s + p\mu). \quad (2.7)$$

The quantity in (2.7) vanishes, provided we set $\mu = (2 + s)/(p - 1)$, and then from (2.6) $\alpha = \gamma(2 + s)/(p - 1)$. It follows that the function

$$v(r) = ru'(r) + \frac{2+s}{p-1}u(r) + \gamma \frac{2+s}{p-1}$$

satisfies the linearized equation in (2.3).

Since $v(0) = 0$, and $w(r)$ is defined up to a constant factor, it follows by the uniqueness results for the initial value problems of the type (2.3) (see [13]) that $w(r) = v(r)$. To see that $w(r)$ is positive, it suffices to show that it is a decreasing function, since $w(0) > 0$ and $w(R) = 0$. In view of the condition (2.1) we have

$$w'(r) = ru''(r) + \frac{p+1+s}{p-1}u'(r) \leq r(u''(r) + \frac{n-1}{r}u'(r)) = -\lambda r f(r, u) < 0,$$

completing the proof.

LEMMA 2.2 *Any positive solution of (2.2) is a strictly decreasing function, i.e.*

$$u'(r) < 0 \quad \text{for all } r \in [0, R).$$

Proof It is clear from the Eq. (2.2) that local maximum must occur at any critical point of $u(r)$.

LEMMA 2.3 Assume that $\lambda \in [\lambda_1, \lambda_2]$ for some $0 < \lambda_1 < \lambda_2$, and the condition (2.1) holds. Then there is a constant $c > 0$, so that any positive solution of the problem (2.2) satisfies

$$u(r) < c \quad \text{for all } r \in [0, R), \text{ and all } \lambda \in [\lambda_1, \lambda_2].$$

Proof Without loss of generality we assume that $R = 1$. In case $n > 2$ we set $t = r^{2-n}$ and $v = u + \gamma$, transforming (2.2) into (with $v = v(t)$)

$$\begin{aligned} v'' + a(t)v^p &= 0, & t \in (1, \infty) \\ v(1) &= \gamma, & v'(\infty) = 0 \\ v'(t) &> 0, & t \in (1, \infty), \end{aligned} \tag{2.8}$$

where

$$a(t) = \lambda \frac{1}{(n-2)^2} t^\alpha \quad \text{and} \quad \alpha = -2 - \frac{s+2}{n-2}.$$

This is a well studied Emden–Fowler equation. Since $\alpha + p + 1 < 0$, it follows by the Theorem 6 on p. 152 in [4], that $\lim_{t \rightarrow \infty} v(t) = v(\infty)$ is finite. In the rest of the proof (which is partly a repetition of the one in [4]) we clarify that $v(\infty)$ is uniformly bounded.

Denote $v'(1) = \delta$. Using shooting arguments, we show next that δ is uniformly bounded (i.e. the bound holds for all $\lambda \in [\lambda_1, \lambda_2]$ and all positive solutions of (2.8)). On say interval $[1, 2]$ we have

$$0 < a_1 \leq a(t) < a_2, \tag{2.9}$$

with some constants a_1 and a_2 . Let $z(t)$ be solution of

$$z'' + a_2 z^p = 0, \quad z(1) = \gamma, \quad z'(1) = \delta. \tag{2.10}$$

By direct integration one sees that $z(t)$ will reach its maximum value at some $t_1 > 1$, and then decrease and become zero at some $t_2 > t_1$. Also one sees that $z(t_1) \sim c\delta^{2/(p+1)}$, and that t_2 gets small for δ large. Since $v'(1) > z'(1)$, it follows that $v(t)$ is larger than $z(t)$ for t close to 1. We claim that this inequality is preserved so long as both functions are increasing. Indeed, assume on the contrary that $v(\xi) = z(\xi)$ at some $\xi > 1$, with $z'(\xi) > v'(\xi) > 0$. Multiplying (2.10) by z' and integrating, we have

$$\frac{1}{2} z'^2(\xi) - \frac{1}{2} \delta^2 + \frac{a_2}{p+1} z^{p+1}(\xi) - \frac{a_2}{p+1} \gamma^{p+1} = 0. \tag{2.11}$$

Since $v''(t) + a_2 v^p > 0$, we derive similarly

$$\frac{1}{2} v'^2(\xi) - \frac{1}{2} \delta^2 + \frac{a_2}{p+1} v^{p+1}(\xi) - \frac{a_2}{p+1} \gamma^{p+1} > 0. \tag{2.12}$$

But the left hand side of (2.12) is smaller than that of (2.11), a contradiction.

Since $v(t)$ rides on top of $z(t)$, we will have $v(t_1) > c\delta^{(2/(p+1))}$, with t_1 close to 1 for δ large. Since $v(t)$ is concave, we have $v'(t_1) < \delta$. From the Eq. (2.8), we have $v''(t) \leq -c\delta^{(2p/(p+1))}$, for all $t > t_1$, so long as $v(t)$ is increasing. So either $v(t)$ will eventually become a decreasing function, contradicting (2.8), or else

$$v'(t) = v'(t_1) + \int_{t_1}^t v''(\tau) d\tau \leq \delta - c\delta^{(2p/(p+1))}(t - t_1).$$

It follows that for δ large $v'(t)$ will become negative for some $t \in (1, 2)$, again contradicting (2.8). We conclude that $v'(1) = \delta$ is uniformly bounded.

Integrating (2.8)

$$\frac{1}{2} v'^2(t) + \int_1^t a(t) v^p v' dt = \frac{1}{2} \delta^2. \quad (2.13)$$

Integrating by parts in (2.13)

$$\frac{1}{2} v'^2(t) + a(t) \frac{v^{p+1}}{p+1} - \int_1^t a'(t) \frac{v^{p+1}}{p+1} dt = \frac{1}{2} \delta^2 + a(1) \frac{v^{p+1}}{p+1}. \quad (2.14)$$

Since $a'(t) < 0$, we conclude

$$v(t) \leq c_1 t^{-(\alpha/(p+1))}, \quad (2.15)$$

where c_1 , and later c_2, \dots will denote positive constants independent of λ and $\delta = v'(1)$. Integrating the Eq. (2.8) over (t, ∞) , and using (2.14)

$$\begin{aligned} v'(t) &= \int_t^\infty a(t) v^p dt \leq c_2 \int_t^\infty t^\alpha t^{-(\alpha p/(p+1))} dt \\ &\leq c_3 t^{(\alpha/(p+1))+1} = c_3 t^\theta, \end{aligned} \quad (2.16)$$

where $\theta = (\alpha + p + 1)/(p + 1) < 0$. Hence

$$v(t) = \gamma + \int_1^t v'(t) dt \leq \gamma + c_4 t^{1+\theta} \leq c_5 t^{1+\theta}. \quad (2.17)$$

If $\theta \leq -1$, we are done. So assume that $-1 < \theta < 0$. Using the estimate (2.17) in (2.16)

$$v'(t) \leq c_6 \int_t^\infty t^\alpha t^{p+p\theta} dt \leq c_7 t^{\alpha+p+1+p\theta} \leq c_7 t^{p\theta}. \quad (2.18)$$

The power of t in (2.18) is less than it was in (2.16). Using (2.18) in (2.17), we obtain an improved bound for $v(t)$, which we will use in (2.16). Eventually we shall obtain an estimate like (2.16), with $\theta < -1$. Then (2.17) will give us a uniform bound.

In case $n = 2$, we set $t = -\ln r$, and $v(t) = u(t) + \gamma$, and proceed similarly (the proof is easier in this case).

Proof of the Theorem 2.1 For small λ the problem (2.2) has a positive radial solution, since it has a subsolution $\psi = (\lambda\gamma^n/(s+2)(s+n))(R^{s+2} - r^{s+2})$ and a supersolution $\phi = (1/2n)(R^2 - r^2)$, with $\psi < \phi$ for all $r \in [0, R)$. We now continue this solution for increasing λ . This can be done using the implicit function theorem in case the solution is non-singular. In case the solution is singular, we have in view of Lemma 2.1

$$\int_0^R f(r, u)w(r)r^{n-1} dr > 0,$$

which means that the Crandall–Rabinowitz [5] bifurcation theorem applies, see [10] or [12] for more details.

This curve of positive solutions cannot be continued indefinitely. Indeed, as in [10] we can show that solutions are increasing in λ , and so by Lemma 2.2 the solutions are bounded away from zero in any interval $[0, \rho)$, with $\rho < R$. If the curve continued for all $\lambda > 0$, the quantity $\lambda(f(r, u)/u)$ would get large say in the interval $(0.1R, 0.5R)$, and then by the Sturm comparison theorem the solution would have to vanish in this interval, contradicting the fact that it is positive. Since by Lemma 2.3 solutions stay bounded, away from $\lambda = 0$, arguing as in [10] (or [12]), we see that the solution curve must reach a critical solution, at which a turn to the left will occur.

After the turn the solution curve continues to the left with no more turns. Indeed, since $f(r, u)$ is convex in u , it follows as in [10] (or [12]) that only turns to the left are possible. As $\lambda \downarrow 0$, infinity is the only place for the solution to go. (It cannot go to zero, since the implicit function theorem holds at $(\lambda = 0, u = 0)$, which implies uniqueness of solutions in a neighbourhood.) Finally, we claim that the solution curve just described exhausts all positive solutions. Indeed, any other solution would have to lie on a similar curve, which will produce multiple solutions in a neighbourhood of $(\lambda = 0, u = 0)$, contradicting the implicit function theorem.

Remark It is natural to expect that any positive solution of (1.1) is radially symmetric. However, since here $f_r(r, u) > 0$, this equation is not covered by the well-known results of Gidas *et al.* [7].

3. ON THE EXPONENTIAL NON-LINEARITIES

For the problem (1.1) we were able to compute explicitly the solution of the linearized equation at a turning point. It is natural to ask if the same is possible for other $f(u)$. It turns out that there is only one more such case, albeit an important one, when $f(u) = e^u$ (the Gelfand problem). Indeed consider positive solutions of the Dirichlet problem

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < R, \quad u = 0 \quad \text{on } |x| = R.$$

In view of [7] positive solutions are radially symmetric, i.e. $u = u(r)$, and hence they satisfy

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0 \quad \text{for } r < R, \quad u'(0) = u(R) = 0. \quad (3.1)$$

Consider the test function $v = ru_r + \mu u + \alpha$. Then

$$L[v] = v'' + \frac{n-1}{r}v' + \lambda f'(u)v = -2f(u) + \mu(uf'(u) - f(u)) + \alpha f'(u).$$

Setting the right hand side to zero, we see that either $f(u) = (u + \gamma)^p$ or $f(u) = e^{au}$, with some constant a . In particular for the Gelfand problem, when $f(u) = e^u$, we see that at any turning point, $w = ru_r + 2$ is the solution of the corresponding linearized equation.

Remarks

1. For the dimensions $3 \leq n \leq 9$ it was shown by Joseph and Lundgren [9] that in case $f(u) = e^u$ the solution curve for (3.1) makes infinitely many turns. At each of these turns the "tangent direction" is given by $w = ru_r + 2$. Using rather involved phase plane analysis, Joseph and Lundgren [9] gave a complete picture for $f(u) = e^u$ and $f(u) = (u + \gamma)^p$, exactly the same two equations that we found to be special for the bifurcation approach.
2. The test function $v(r)$ above is a linear combination of three basic test functions: $v_1 = ru_r$, $v_2 = u$ and $v_3 = 1$. In all cases $L[v_i]$ does not contain derivatives of u . It is natural to wonder if there are any other test functions. New test functions would probably imply new exact multiplicity results. We suspect that the answer is "no".
3. Similarly to the Theorem 2.1, we could give an exact multiplicity result for the two-dimensional problem

$$u'' + \frac{1}{r}u' + \lambda r^s e^u = 0 \quad \text{for } r < R, \quad u'(0) = u(R) = 0. \quad (3.2)$$

We did not pursue this, since the problem (3.2) can also be explicitly integrated (following [2]) by setting $t = -\ln r$, followed by $v = u - (s+2)t$.

4. NON-EXISTENCE OF POSITIVE SOLUTION FOR A SUPERCRITICAL PROBLEM

We begin with a model problem, then generalize.

THEOREM 4.1 *Consider the problem*

$$\Delta u + \lambda u^q + u^p = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D. \quad (4.1)$$

Here D is a bounded domain in R^n , $n > 2$, with a smooth boundary, and λ is a positive parameter. Assume that p is a supercritical constant, while q is subcritical, i.e.

$$1 \leq q < \frac{n+2}{n-2} < p.$$

Assume also that D is star-shaped, i.e.

$$x \cdot n(x) \geq 0 \quad \text{for all } x \text{ on } \partial D,$$

where $n(x)$ denotes the unit normal vector at $x \in \partial D$, pointing outside. Then that there is a $\lambda^* > 0$ so that the problem (4.1) has no non-trivial non-negative solutions for $\lambda < \lambda^*$.

Proof Assume on the contrary that there is a sequence $\lambda_k \rightarrow 0$ for which the problem (4.1) is solvable. Recall that any solution of the problem

$$\Delta u + f(u) = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D$$

satisfies the Pohozaev's identity, see e.g. [3] (here $F(u) = \int_0^u f(t) dt$)

$$\left(\frac{n}{2} - 1\right) \int_D |\nabla u|^2 dx + \frac{1}{2} \int_{\partial D} x \cdot n(x) |\nabla u|^2 dS = n \int_D F(u) dx. \quad (4.2)$$

Since $x \cdot n(x) \geq 0$ on ∂D , for our problem (4.1) we have

$$\left(\frac{n}{2} - 1\right) \int_D |\nabla u|^2 dx \leq n \int_D \left(\lambda \frac{u^{q+1}}{q+1} + \frac{u^{p+1}}{p+1} \right) dx. \quad (4.3)$$

By the Sobolev inequality

$$\int_D |\nabla u|^2 dx \geq c_1 \left(\int_D u^{q+1} dx \right)^{2/(q+1)}, \quad (4.4)$$

where c_1 is a positive constant independent of u . From the Eq. (4.1)

$$\int_D |\nabla u|^2 dx = \int_D (\lambda u^{q+1} + u^{p+1}) dx. \quad (4.5)$$

Combining (4.3) with (4.5)

$$\int_D (\lambda u^{q+1} + u^{p+1}) dx \leq \frac{2n}{n-2} \int_D \left(\lambda \frac{u^{q+1}}{q+1} + \frac{u^{p+1}}{p+1} \right) dx,$$

or

$$c_2 \int_D u^{p+1} dx < c_3 \lambda \int_D u^{q+1} dx, \quad (4.6)$$

where $c_2 = 1 - 2n/((n-2)(p+1)) > 0$ and $c_3 = 2n/((n-2)(q+1)) - 1 > 0$. Using (4.4)–(4.6)

$$c_1 \left(\int_D u^{q+1} dx \right)^{2/(q+1)} \leq \lambda \int_D u^{q+1} dx + \lambda \frac{c_3}{c_2} \int_D u^{q+1} dx. \quad (4.7)$$

If $q = 1$, we have a contradiction in (4.7), as $\lambda \rightarrow 0$, and the proof is complete. (Actually for $q = 1$ this proves non-existence of any non-trivial solutions, not just non-negative ones. Up to this point we followed the argument of Brezis and Nirenberg [3].)

So assume that $q > 1$. Then (4.7) implies

$$\left(\int_D u^{q+1} dx \right)^{(q-1)/(q+1)} \geq \frac{c_4}{\lambda},$$

i.e.

$$\int_D u^{q+1} dx \rightarrow \infty \quad \text{as } \lambda \rightarrow 0. \quad (4.8)$$

By the Holder inequality

$$\int_D u^{q+1} dx \leq c_5 \left(\int_D u^{p+1} dx \right)^{(q+1)/(p+1)}. \quad (4.9)$$

Using this in (4.6), we conclude that $\int_D u^{p+1} dx$ is bounded as $\lambda \rightarrow 0$. But then from (4.9) $\int_D u^{q+1} dx$ is also bounded, contradicting (4.8).

Examining the proof, we see that in the same way we can prove the following more general result.

THEOREM 4.2 *Consider the problem*

$$\Delta u + \lambda \sum a_i u^{q_i} + \sum b_i u^{p_i} = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D \quad (4.10)$$

on a bounded domain in R^n , $n > 2$, with a smooth boundary. Assume that p_i are supercritical constants, while q_i are subcritical, i.e.

$$1 \leq q_i < \frac{n+2}{n-2} < p_i,$$

and a_i, b_i are positive constants.

Then there is a $\lambda^ > 0$ so that the problem (4.10) has no non-trivial non-negative solutions for $\lambda < \lambda^*$.*

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