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Global solution curves for self-similar equations

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Abstract

We consider positive solutions of a semilinear Dirichlet problem

$$\Delta u + \lambda f(u) = 0, \quad \text{for } |x| < 1, \qquad u = 0, \quad \text{when } |x| = 1$$

on a unit ball in R^n. For four classes of self-similar equations it is possible to parameterize the entire (global) solution curve through the solution of a single initial value problem. This allows us to derive results on the multiplicity of solutions, and on their Morse indices. In particular, we easily recover the classical results of D.D. Joseph and T.S. Lundgren [6] on the Gelfand problem. Surprisingly, the situation turns out to be different for the generalized Gelfand problem, where infinitely many turns are possible for any space dimension $n \geq 3$. We also derive detailed results for the equation modeling electrostatic micro-electromechanical systems (MEMS), in particular we easily recover the main result of Z. Guo and J. Wei [4], and we show that the Morse index of the solutions increases by one at each turn. We also consider the self-similar Henon's equation.

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1. Introduction

We consider radial solutions on a ball in R^n for four special classes of equations, the ones self-similar under scaling. For example, consider the so called Gelfand equation ($u = u(x)$, $x \in R^n$)

$$\Delta u + \lambda e^u = 0, \quad \text{for } |x| < 1, \qquad u = 0, \quad \text{when } |x| = 1. \tag{1.1}$$

Here λ is a positive parameter. By the maximum principle, solutions of (1.1) are positive, and then by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [2] they are radially symmetric, i.e., $u = u(r)$, $r = |x|$, and it satisfies

$$u'' + \frac{n-1}{r}u' + \lambda e^u = 0, \quad \text{for } 0 < r < 1, \qquad u'(0) = u(1) = 0. \tag{1.2}$$

This theorem also asserts that $u'(r) < 0$ for all $0 < r < 1$, which implies that the value of $u(0)$ gives the L^∞ norm of our solution. Moreover, $u(0)$ is a *global parameter*, i.e., it uniquely identifies the solution pair $(\lambda, u(r))$, see e.g., P. Korman [9]. It follows that a two-dimensional curve $(\lambda, u(0))$ completely describes the solution set of (1.1). The change of variables $v = u + a$, $\xi = br$, with constant a and b will transform the equation in (1.2) into the same equation if $e^a = b^2$. Here is what this self-similarity “buys” us. Let $w(t)$ be the solution of the following initial value problem

$$w'' + \frac{n-1}{t}w' + e^w = 0, \qquad w(0) = 0, \qquad w'(0) = 0 \quad (t > 0), \tag{1.3}$$

which is easily seen to be negative, and defined for all $t \in (0, \infty)$. It turns out that $w(t)$ gives us the entire solution curve of (1.2) (or of (1.1)):

$$\big(\lambda, u(0)\big) = \big(t^2 e^{w(t)}, -w(t)\big), \tag{1.4}$$

parameterized by $t \in (0, \infty)$. In particular, $\lambda = \lambda(t) = t^2 e^{w(t)}$, and

$$\lambda'(t) = te^w\big(2 + tw'\big),$$

so that the solution curve travels to the right (left) in the $(\lambda, u(0))$ plane if $2 + tw' > 0$ (< 0). This makes us interested in the roots of the function $2 + tw'$. If we set this function to zero

$$2 + tw' = 0,$$

then solution of this equation is of course $w(t) = a - 2\ln t$. Amazingly, if we choose $a = \ln(2n - 4)$, $n \geq 3$, then

$$w_0(t) = \ln(2n - 4) - 2\ln t$$

is also a solution of the equation in (1.3)! We show that $w(t)$ tends to $w_0(t)$ as $t \to \infty$, and the issue turns out to be how many times $w(t)$ and $w_0(t)$ cross as $t \to \infty$. We propose to call $w(t)$ *the generating solution*, and $w_0(t)$ *the guiding solution*. We show that the solution curve makes infinitely many turns if and only if $w(t)$ and $w_0(t)$ intersect infinitely many times. Then we prove that for $3 \le n \le 9$, $w(t)$ and $w_0(t)$ intersect infinitely many times, and hence the solution curve makes infinitely many turns, which is a part of the classical result of D.D. Joseph and T.S. Lundgren [6], see also J. Bebernes and D. Eberly [1] for an exposition. D.D. Joseph and T.S. Lundgren [6] also proved that for $n = 1, 2$ the solution curve makes exactly one turn, while for $n \ge 10$ there are no turns (we recover this result for $n \ge 10$ too). Our approach provides a remarkably short route to the classical result of D.D. Joseph and T.S. Lundgren [6], and some new results for other equations.

A similar approach works for the radially symmetric solutions of the generalized Gelfand's problem

$$\Delta u + \lambda |x|^\alpha e^u = 0, \quad \text{for } |x| < 1, \qquad u = 0, \quad \text{when } |x| = 1,$$

with a constant $\alpha > 0$. Remarkably, the picture here turns out to be different! We show that the solution curve makes infinitely many turns, provided that

$$3 \le n < 10 + 4\alpha.$$

We see that unlike the Gelfand equation, infinitely many turns occur for any space dimension $n \ge 3$, for large enough α. This result is sharp, because we prove that there are at most two turns if $n \ge 10 + 4\alpha$.

A similar approach works for three other classes of equations, notably

$$u'' + \frac{n-1}{r}u' + \lambda \frac{r^\alpha}{(1-u)^p} = 0, \qquad u'(0) = u(1) = 0, \quad 0 < u(r) < 1, \tag{1.5}$$

modeling the electrostatic micro-electromechanical systems (MEMS), see J.A. Pelesko [14], N. Ghoussoub and Y. Guo [3], and Z. Guo and J. Wei [4] for some of the active recent research. We give a much shorter and more elementary proof of the main result of Z. Guo and J. Wei [4], which states that the solution curve makes infinitely many turns, provided that

$$2 \le n < 2 + \frac{2(\alpha+2)}{p+1}\left(p + \sqrt{p^2 + p}\right). \tag{1.6}$$

(If $2 \le n \le 6$, this inequality holds for all $\alpha \ge 0$ and $p > 1$.) We also show that outside of this range the solution curve makes at most two turns, which is a new result.

For the Gelfand problem (1.1) it was shown by K. Nagasaki and T. Suzuki [13] that at each turn of the solution curve the Morse index of the solution increases by one. We recover this result, and then prove that the same thing is true for the MEMS problem (1.5), which is a new result. Finally, we use the self-similar nature of Henon's equation to discuss the exact multiplicity of the symmetry breaking solutions.

2. Parameterization of the global solution curves

Consider the problem

$$u'' + \frac{n-1}{r}u' + \lambda \frac{r^\alpha}{(1-u)^p} = 0 \quad \text{for } 0 < r < 1,$$
$$u'(0) = u(1) = 0, \quad 0 < u(r) < 1, \tag{2.1}$$

which arises in modeling of electrostatic micro-electromechanical systems (MEMS), see [14,3,4]. Here λ is a positive parameter, $\alpha > 0$ and $p > 1$ are constants. Any solution $u(r)$ of (2.1) is a positive and decreasing function (by the maximum principle), so that $u(0)$ gives its maximum value. It is known, see e.g., P. Korman [10], that $u(0)$ is a *global parameter*, i.e., it uniquely identifies the solution pair $(\lambda, u(r))$ (the proof is by scaling). It follows that a two-dimensional curve $(\lambda, u(0))$ completely describes the solution set of (2.1). Our goal is to compute the global solution curve $(\lambda, u(0))$. Let $1 - u = v$. Then $v(r)$ satisfies

$$v'' + \frac{n-1}{r}v' = \lambda \frac{r^\alpha}{v^p} \quad \text{for } 0 < r < 1, \qquad v'(0) = 0, \qquad v(1) = 1. \tag{2.2}$$

Assume that $v(0) = a$. We scale $v = aw$, and $t = br$. The constants a and b are assumed to satisfy

$$\lambda = a^{p+1} b^{\alpha+2}. \tag{2.3}$$

Then (2.2) becomes

$$w'' + \frac{n-1}{t}w' = \frac{t^\alpha}{w^p}, \qquad w(0) = 1, \qquad w'(0) = 0. \tag{2.4}$$

It is easy to see that the solution $w(t)$ of (2.4) is an increasing function, defined for all $t > 0$. We can compute it numerically (over a large interval). It turns out that this particular solution $w(t)$ gives us the entire solution curve of (2.1)! We have

$$1 = v(1) = aw(b),$$

and so $a = \frac{1}{w(b)}$, and then $\lambda = \frac{b^{\alpha+2}}{w^{p+1}(b)}$. The global solution curve is

$$\big(\lambda, u(0)\big) = \left(\frac{b^{\alpha+2}}{w^{p+1}(b)}, 1 - \frac{1}{w(b)}\right), \tag{2.5}$$

parameterized by $b \in (0, \infty)$. This parameterization was pointed out previously in [14], and was then used in [3].

Example. Using *Mathematica*, we have solved the problem (2.1) with $p = 2$, $n = 2$ and $\alpha = 0.2$. The global solution curve, obtained through the parameterization (2.5), is given in Fig. 1.

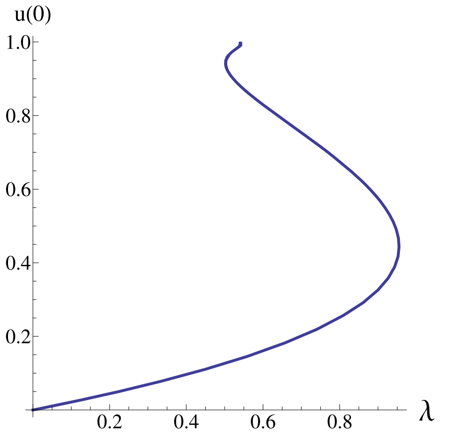

Fig. 1. Solution curve for the problem (2.1).

We now show that the situation is similar for three other important classes of equations. Consider the problem

$$u'' + \frac{n-1}{r}u' + \lambda r^{\alpha}(1+u)^{p} = 0 \quad \text{for } 0 < r < 1, \qquad u'(0) = 0, \qquad u(1) = 0. \tag{2.6}$$

We set $v = 1 + u$, followed by $v = aw$, and $t = br$, where $a = v(0) = 1 + u(0)$. The constants a and b are assumed to satisfy

$$\lambda = \frac{b^{\alpha+2}}{a^{p-1}}. \tag{2.7}$$

Then (2.6) becomes

$$w'' + \frac{n-1}{t}w' + t^{\alpha}w^{p} = 0, \qquad w(0) = 1, \qquad w'(0) = 0. \tag{2.8}$$

The solution of this problem is easily seen to be a decreasing function, which, for sub-critical p, vanishes at some $t_0 > 0$. (If $p \geq \frac{n+2+2\alpha}{n-2}$, then $w(t)$ has no roots on $(0, \infty)$, see e.g., T. Kusano and M. Naito [11], or E. Yanagida and S. Yotsutani [18].) As before, $a = \frac{1}{w(b)}$, and then $\lambda = b^{\alpha+2}w^{p-1}(b)$. The global solution curve is

$$\big(\lambda, u(0)\big) = \left(b^{\alpha+2}w^{p-1}(b), -1 + \frac{1}{w(b)}\right),$$

parameterized by $b \in (0, t_0)$.

Next, we consider the generalized Gelfand's equation

$$u'' + \frac{n-1}{r}u' + \lambda r^{\alpha}e^{u} = 0 \quad \text{for } 0 < r < 1, \qquad u'(0) = 0, \qquad u(1) = 0. \tag{2.9}$$

We set $u = w + a$, $t = br$, with $a = u(0)$. The constants a and b are assumed to satisfy

$$\lambda = b^{\alpha+2} e^{-a}.$$

Then (2.9) becomes

$$w'' + \frac{n-1}{t} w' + t^\alpha e^w = 0, \qquad w(0) = 0, \qquad w'(0) = 0.$$

We compute numerically the solution of this problem $w(t)$, which is a negative decreasing function, defined for all $t > 0$. We have

$$0 = u(1) = a + w(b),$$

i.e., $a = -w(b)$, and then $\lambda = b^{\alpha+2} e^{w(b)}$. The global solution curve for (2.9) is

$$\big(\lambda, u(0)\big) = \big(b^{\alpha+2} e^{w(b)}, -w(b)\big),$$

parameterized by $b \in (0, \infty)$. Moreover, $u(r) = -w(b) + w(br)$ is the solution of (2.9) at $\lambda = b^{\alpha+2} e^{w(b)}$.

Finally, we consider

$$u'' + \frac{n-1}{r} u' + \lambda r^\alpha e^{-u} = 0 \quad \text{for } 0 < r < 1, \qquad u'(0) = 0, \qquad u(1) = 0. \tag{2.10}$$

We set $u = w + a$, $t = br$, with $a = u(0)$. The constants a and b are assumed to satisfy

$$\lambda = b^{\alpha+2} e^{a}.$$

Then (2.10) becomes

$$w'' + \frac{n-1}{t} w' + t^\alpha e^{-w} = 0, \qquad w(0) = 0, \qquad w'(0) = 0.$$

We compute numerically the solution of this problem $w(t)$, which is a negative decreasing function, tending to $-\infty$ at some $t_1 > 0$. We have

$$0 = u(1) = a + w(b),$$

i.e., $a = -w(b)$, and then $\lambda = b^{\alpha+2} e^{-w(b)}$. The global solution curve for (2.10) is

$$\big(\lambda, u(0)\big) = \big(b^{\alpha+2} e^{-w(b)}, -w(b)\big),$$

parameterized by $b \in (0, t_1)$.

3. A generalization of Joseph and Lundgren's result

As we saw above, for Gelfand's problem

$$u'' + \frac{n-1}{r}u' + \lambda e^u = 0 \quad \text{for } 0 < r < 1, \qquad u'(0) = 0, \qquad u(1) = 0 \tag{3.1}$$

the solution curve $(\lambda, u(0))$ is given by $(t^2 e^{w(t)}, -w(t))$, parameterized by $t \in (0, \infty)$, where $w(t)$ is the solution of

$$w'' + \frac{n-1}{t}w' + e^w = 0, \qquad w(0) = 0, \qquad w'(0) = 0 \quad (t > 0). \tag{3.2}$$

In particular, $\lambda = t^2 e^{w(t)}$, and the issue is how many times this function changes its direction of monotonicity for $t \in (0, \infty)$. Compute

$$\lambda'(t) = te^w\left(2 + tw'\right), \tag{3.3}$$

so that we are interested in the roots of the function $2 + tw'$. If we set this function to zero

$$2 + tw' = 0,$$

then solution of this equation is of course $w(t) = a - 2\ln t$. Amazingly, if we choose $a = \ln(2n - 4)$, $n \geq 3$, then

$$w_0(t) = \ln(2n - 4) - 2\ln t$$

is a solution of the equation in (3.2)! It turns out that the solution of (3.2) tends to $w_0(t)$ as $t \to \infty$, and the issue is how many times $w(t)$ and $w_0(t)$ cross as $t \to \infty$. We shall only consider $n \geq 3$, since for $n = 1, 2$ the problem (3.1) can be explicitly solved, see e.g., [1].

Lemma 3.1. *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (3.1) makes infinitely many turns. On the other hand, assume that $w(t)$ and $w_0(t)$ intersect only a finite number of times, and $(w - w_0)'(t)$ is of one sign for $t > T$, with a point T coming after the last point of intersection. Then $\lambda(t)$ is monotone for $t > T$.*

Proof. Indeed, assuming that $w(t)$ and $w_0(t)$ intersect infinitely many times, let $\{t_n\}$ denote the points of intersection. At $\{t_n\}$'s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial value problems). Since $2 + t_n w_0'(t_n) = 0$, it follows that $2 + t_n w'(t_n) > 0$ (< 0) if $w(t)$ intersects $w_0(t)$ from below (above) at t_n. Hence, on any interval (t_n, t_{n+1}) there is a point t_0, where $2 + t_0 w'(t_0) = 0$, i.e., $\lambda'(t_0) = 0$, and t_0 is a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}).

On the other hand, assume that $w(t)$ and $w_0(t)$ intersect only a finite number of times, and $(w - w_0)'(t)$ is of one sign for $t > T$, say $w'(t) > w_0'(t)$. Then $2 + tw'(t) > 2 + tw_0'(t) = 0$, i.e., $\lambda'(t) > 0$, and the solution curve is monotone for $t > T$. $\square$

The linearized equation for (3.2) is

$$z'' + \frac{n-1}{t} z' + e^{w} z = 0.$$

At the solution $w = w_0(t)$, this becomes

$$z'' + \frac{n-1}{t} z' + \frac{2n-4}{t^2} z = 0, \tag{3.4}$$

which is Euler's equation! Its characteristic equation has the roots

$$r = \frac{-n+2 \pm \sqrt{(n-2)(n-10)}}{2}. \tag{3.5}$$

When $3 \le n \le 9$, the roots are complex, and hence $z(t)$ changes sign infinitely many times. We shall show that $w(t)$ tends to $w_0(t)$, and oscillates infinitely many times around $w_0(t)$, which implies infinitely many turns of the solution curve. For other n, the solution curve turns at most once. We obtain a remarkably short route to the classical result of D.D. Joseph and T.S. Lundgren [6].

We shall present the details for the more general problem ($\alpha > 0$)

$$u'' + \frac{n-1}{r} u' + \lambda r^{\alpha} e^{u} = 0 \quad \text{for } 0 < r < 1, \qquad u'(0) = 0, \qquad u(1) = 0. \tag{3.6}$$

The solution curve $(\lambda, u(0))$ is now given by $(t^{2+\alpha} e^{w(t)}, -w(t))$, parameterized by $t \in (0, \infty)$, where $w(t)$ is the solution of

$$w'' + \frac{n-1}{t} w' + t^{\alpha} e^{w} = 0, \qquad w(0) = 0, \qquad w'(0) = 0 \quad (t > 0). \tag{3.7}$$

In particular, $\lambda = t^{2+\alpha} e^{w(t)}$, and the issue is how many times this function changes the direction of monotonicity for $t \in (0, \infty)$. Compute

$$\lambda'(t) = t^{\alpha+1} e^{w} \left(2 + \alpha + t w'\right),$$

so that we are interested in the roots of the function $2 + \alpha + t w'$. If we set this function to zero

$$2 + \alpha + t w' = 0,$$

then the general solution of this equation is $w(t) = a - (2 + \alpha) \ln t$. If we choose $a = \ln(2 + \alpha)(n - 2)$, $n \ge 3$, then

$$w_0(t) = \ln(2 + \alpha)(n - 2) - (2 + \alpha) \ln t$$

is a solution of the equation in (3.7). It turns out that the solution $w(t)$ of (3.7) tends to $w_0(t)$ as $t \to \infty$, and the issue is how many times $w(t)$ and $w_0(t)$ cross as $t \to \infty$.

The linearized equation for (3.7) is

$$z'' + \frac{n-1}{t} z' + t^{\alpha} e^{w} z = 0.$$

At the solution $w = w_0(t)$, this becomes

$$z'' + \frac{n-1}{t} z' + \frac{(2+\alpha)(n-2)}{t^2} z = 0, \tag{3.8}$$

which is Euler's equation. Its characteristic equation has the roots

$$r = \frac{-n+2 \pm \sqrt{(n-2)(n-10-4\alpha)}}{2}.$$

When $3 \le n < 10 + 4\alpha$, the roots are complex. Hence, $z(t)$ changes sign infinitely many times. We shall show that this implies infinitely many turns of the solution curve, while for other n at most two turns of the solution curve is possible.

We shall need the following version of Sturm's comparison theorem.

Lemma 3.2. *Consider the following two equations*

$$y'' + \frac{n-1}{t} y' + \frac{a(t)}{t^2} y = 0, \tag{3.9}$$

$$v'' + \frac{n-1}{t} v' + \frac{b(t)}{t^2} v = 0. \tag{3.10}$$

Assume that $b(t) > a(t)$ for all $t \in R$. Then $v(t)$ has a root between any two consecutive roots of $y(t)$.

Proof. Assume that $y(t_1) = y(t_2) = 0$, $y(t) > 0$ on (t_1, t_2), while on the contrary $v(t) > 0$ on (t_1, t_2). From Eqs. (3.9) and (3.10)

$$\left[t^{n-1}\left(y'v - yv'\right)\right]' = t^{n-3}\left(b(t) - a(t)\right)y(t)v(t) > 0 \quad \text{on } (t_1, t_2).$$

Integrating over (t_1, t_2),

$$t_2^{n-1} y'(t_2) v(t_2) - t_1^{n-1} y'(t_1) v(t_1) > 0,$$

which is a contradiction, since both terms on the left are non-positive. □

Lemma 3.3. *Consider the equation (here a, a_1, a_2 are constants)*

$$y'' + \frac{n-1}{t} y' + \frac{a + f(t)}{t^2} y = 0. \tag{3.11}$$

Assume that Eq. (3.9), with $a(t) = a$, has infinitely many roots for all $a \in [a_1, a_2]$, while $f(t) \to 0$, as $t \to \infty$. Then Eq. (3.11) has infinitely many roots for all $a \in (a_1, a_2)$.

Proof. Choose an $\epsilon > 0$, so that $a - \epsilon > a_1$. Since $a + f(t) > a - \epsilon > a_1$ for t large, the proof follows by Lemma 3.2. $\square$

Lemma 3.4. *Consider Eq. (3.11), with $n \geq 3$, $a > 0$, and $\lim_{t\to\infty} f(t) = 0$. Assume that its solution $y(t)$ is bounded on some interval (t_0, ∞). Then $\lim_{t\to\infty} y(t) = 0$.*

Proof. Letting $t = e^s$, we transform (3.11) to

$$y_{ss} + (n-2)y_s + ay = g(s), \tag{3.12}$$

with $g(s) \equiv -f(e^s)y \to 0$, as $s \to \infty$. The roots of the corresponding homogeneous equation are either $\alpha \pm i\beta$, with $\alpha < 0$, or both roots are negative. Let us assume it is the former case, and the other case is similar. The general solution of (3.12) is

$$y(s) = c_1 e^{\alpha s} \cos \beta s + c_2 e^{\alpha s} \sin \beta s + \frac{1}{\beta} \int_0^s e^{\alpha(s-\xi)} \sin \beta(s-\xi) g(\xi)\, d\xi,$$

and

$$\left| \int_0^s e^{\alpha(s-\xi)} \sin \beta(s-\xi) g(\xi)\, d\xi \right| \leq e^{\alpha s} \int_0^s e^{-\alpha \xi} \left|g(\xi)\right| d\xi \to 0, \quad \text{as } s \to \infty,$$

concluding the proof. $\square$

Theorem 3.1. *Assume that*

$$3 \leq n < 10 + 4\alpha.$$

Then the solution curve of (3.6) makes infinitely many turns.

Proof. In view of Lemma 3.1, we need to show that $w(t)$ oscillates infinitely many times around $w_0(t)$. We claim that these functions get close to each other, as t increases. Denote $p(t) = w_0(t) - w(t)$. It satisfies

$$p'' + \frac{n-1}{t} p' + t^\alpha a(t) p = 0, \tag{3.13}$$

where $a(t) = \int_0^1 e^{s w_0(t) + (1-s) w(t)}\, ds > 0$. We have $p(\epsilon) > 0$ and $p'(\epsilon) < 0$, for $\epsilon > 0$ small. From (3.13),

$$\left(t^{n-1} p'\right)' = -t^{n+\alpha-1} a(t) p < 0, \quad \text{while } p(t) > 0.$$

Hence, $p'(t) < 0$, while $p(t) > 0$. So either $p(t)$ becomes zero at some t_1, or else $p(t)$ remains positive, and $\lim_{t\to\infty} p(t) = b \geq 0$. In the latter case, $w(t) = w_0(t) + b + o(1)$, and then $t^\alpha a(t) = \frac{a_0 + f(t)}{t^2}$, with $a_0 = (2+\alpha)(n-2) \int_0^1 e^{(1-s)b}\, ds > 0$, and $\lim_{t\to\infty} f(t) = 0$. Since $p(t)$

is bounded, by Lemma 3.4, $p(t) \to 0$ as $t \to \infty$, i.e., $b = 0$, and so $w(t) - w_0(t) \to 0$, as $t \to \infty$. In case $p(t_1) = 0$, we show by the same argument that the linear equation (3.13) has either the second root at some $t_2 > t_1$, or else $p(t)$ remains negative, and $\lim_{t\to\infty} p(t) = 0$. (We have $(t^{n-1}p')' > 0$, when $p < 0$, from which it is easy to deduce that $p(t)$ remains bounded.)

Next, we rule out the possibility of $p(t)$ keeping the same sign and tending to zero over an infinite interval (t_k, ∞). We have

$$t^\alpha a(t) = t^\alpha e^{w_0} \int_0^1 e^{(1-s)(w(t)-w_0(t))}\, ds = \frac{(2+\alpha)(n-2)}{t^2}\bigl(1 + o(1)\bigr), \quad \text{as } t \to \infty.$$

Since Euler's equation (3.8) has infinitely many roots on (t_k, ∞), we conclude by Lemma 3.3 that $p(t)$ must vanish on that interval too. It follows that $p(t)$ changes sign infinitely many times. ($p(t)$ cannot remain positive, so that it has its first root, after that $p(t)$ cannot remain negative, so that it has its second root, and so on.) □

We see that unlike the Gelfand equation ($\alpha = 0$), infinitely many turns are possible for any $n \ge 3$, for large enough α.

We now turn to the case when $n \ge 10 + 4\alpha$. In that case both roots of the characteristic equation of Euler's equation (3.8) are negative, and so any solution of (3.8) may have at most one root. By a simple comparison argument we shall show that $w(t)$ and $w_0(t)$ intersect at most twice, and then (in the case $\alpha = 0$) we will show that $w(t)$ and $w_0(t)$ do not intersect at all. We have

$$e^{w(t)} - e^{w_0(t)} > e^{w_0(t)}\bigl(w(t) - w_0(t)\bigr) = \frac{(2+\alpha)(n-2)}{t^{2+\alpha}}\bigl(w(t) - w_0(t)\bigr).$$

Denoting $p(t) = w(t) - w_0(t)$, we then have from (3.7)

$$p'' + \frac{n-1}{t}p' + \frac{(2+\alpha)(n-2)}{t^2}p < 0. \tag{3.14}$$

This inequality implies that $p(t)$ oscillates slower (faster) than $z(t)$, the solution of Euler's equation (3.8), provided that $p(t) < 0$ (> 0). The following lemma makes this observation precise.

Lemma 3.5. *Assume that $z(t)$ is a solution of (3.8), such that $z(t_0) = p(t_0)$ and $z'(t_0) = p'(t_0)$ at some $t_0 \in (0, \infty)$, and $z(t) < 0$ on (t_0, ∞). Then $p(t) < 0$ on (t_0, ∞).*

Proof. Assume, on the contrary, that $p(\xi) = 0$ at some $\xi \in (t_0, \infty)$, while $p(t) < 0$ on (t_0, ξ). From Eqs. (3.8) and (3.14) (keep in mind that $z(t) < 0$)

$$\bigl(pz' - p'z\bigr)' + \frac{n-1}{t}\bigl(pz' - p'z\bigr) < 0 \quad \text{on } (t_0, \xi), \tag{3.15}$$

and so the function $Q(t) \equiv t^{n-1}(pz' - p'z)$ is decreasing on (t_0, ξ). But $Q(t_0) = 0$, while $Q(\xi) = -\xi^{n-1}p'(\xi)z(\xi) \ge 0$, a contradiction. □

Theorem 3.2. *In case $n \geq 10 + 4\alpha$ the solution curve of the generalized Gelfand's equation (3.6) admits at most two turns.*

Proof. Let again $p(t) = w(t) - w_0(t)$. In view of Lemma 3.1, we need to show that $p'(t)$ changes its sign at most twice, i.e., $p(t)$ changes its monotonicity at most twice. Since $p(t)$ satisfies the linear equation (3.13), $p(t)$ cannot have points of positive local minimum, and of negative local maximum, and hence $p(t)$ changes its monotonicity once between two consecutive roots, and once after its last root (since $p(t)$ tends to zero as $t \to \infty$, which follows by Lemma 3.4, the same way as in the proof of Theorem 3.1), and no other changes of monotonicity are possible. We will show that $p(t)$ has at most two roots, which will imply that $p(t)$ changes its monotonicity at most twice.

Assume that $p(t)$ has at least two roots, and let t_1 and t_2 denote the first two roots (if there are less than two roots, there are less than two turns). Then $p(t)$ is negative on $(0, t_1)$, positive on (t_1, t_2), and again negative after t_2. Pick any point $t_0 \in (0, t_1)$, and let $Z(t)$ be the solution of Euler's equation (3.8), such that $Z(t_0) = p(t_0) < 0$, $Z'(t_0) = p'(t_0)$. We claim that $Z(t)$ vanishes on $(0, t_1)$. Indeed, assuming that $Z(t) < 0$ on $(0, t_1)$, we argue as in Lemma 3.5, and conclude that the function $Q(t) \equiv t^{n-1}(pZ' - p'Z)$ is decreasing on (t_0, t_1), with $Q(t_0) = 0$, while $Q(t_1) = -t_1^{n-1} p'(t_1) Z(t_1) \geq 0$, a contradiction. At its root $Z(t)$ changes to being positive, and it stays positive after its root, since $Z(t)$ is a solution of Euler's equation with two negative characteristic roots (the roots coincide when $n = 10 + 4\alpha$). In particular,

$$Z(t) > 0 \quad \text{for } t > t_1. \tag{3.16}$$

We now return to $p(t)$. After its second root t_2, it will either have the third root at some t_3, or stay negative and tend to zero as $t \to \infty$. We now consider these cases in turn.

Case 1. $p(t_2) = p(t_3) = 0$, $p(t) < 0$ on (t_2, t_3). In place (3.15), we now have (because of (3.16))

$$\left(pZ' - p'Z\right)' + \frac{n-1}{t}\left(pZ' - p'Z\right) > 0 \quad \text{on } (t_1, \infty). \tag{3.17}$$

Integrating this over (t_2, t_3), we get

$$-t_3^{n-1} p'(t_3) Z(t_3) + t_2^{n-1} p'(t_2) Z(t_2) > 0,$$

which is a contradiction, since (using (3.16)) both terms on the left are non-positive.

Case 2. We have $p(t) > 0$ on (t_2, ∞), and $p(t) \to 0$, as $t \to \infty$. This case is possible (or rather, we are unable to rule this case out). Then $p(t)$ changes its monotonicity twice, and the solution curve has two turns. □

It is natural to expect that in the case $n \geq 10 + 4\alpha$ the solution curve does not turn at all (if $\alpha = 0$, this is part of D.D. Joseph and T.S. Lundgren's result [6]). Surprisingly, we found this hard to prove. We shall prove this only if $\alpha = 0$ (with a computer assistance at one step), so that we recover the classical result of D.D. Joseph and T.S. Lundgren [6]. By a different method, the case $n \geq 10 + 4\alpha$ was covered by J. Jacobsen and K. Scmitt [5], who proved that for $0 < \lambda < (n-2)(\alpha + 2)$, the problem (3.6) has a unique solution, and no solution exists for $\lambda \geq (n-2)(\alpha + 2)$.

So we consider now the case $\alpha = 0$. The linearized equation at $w_0(t)$ is then Euler's equation (3.4), whose characteristic exponents are given by (3.5). In case $n \geq 10$, both characteristic exponents are negative and hence $z(t) \to 0$ as $t \to \infty$. This solution will either vanish once or keep the same sign depending on the initial conditions. Assume that the initial conditions are given at some $A > 0$. By scaling we may assume that $z(A) = 1$. The following lemma says that in order for $z(t)$ to vanish, $z'(A)$ must be negative and sufficiently large in absolute value.

Lemma 3.6. *For $n \geq 10$ consider the problem*

$$z'' + \frac{n-1}{t} z' + \frac{2n-4}{t^2} z = 0, \qquad z(A) = 1, \qquad z'(A) = q \quad (\text{for } t > A > 0). \tag{3.18}$$

Assume that

$$qA > \frac{-n+2-\sqrt{(n-2)(n-10)}}{2}. \tag{3.19}$$

Then $z(t) > 0$ on (A, ∞).

Proof. If $n = 10$, the general solution of the equation in (3.18) is

$$z(t) = t^{-4}(c_1 + c_2 \ln t).$$

From the initial conditions

$$c_2 = A^4(qA + 4) > 0,$$

since (3.19) reads: $qA > -4$. If $c_1 \geq 0$, then $z(t) > 0$ for all $t > 0$. If $c_1 < 0$, then the function $c_1 + c_2 \ln t$ has a root, but it is smaller than A (observe that $c_1 + c_2 \ln A = A^4 > 0$). So that $z(t) > 0$ on (A, ∞). If $n > 10$, the general solution of (3.18) is

$$z(t) = c_1 \left(\frac{t}{A}\right)^r + c_2 \left(\frac{t}{A}\right)^s = \left(\frac{t}{A}\right)^r \left(c_1 + c_2 \left(\frac{t}{A}\right)^{s-r}\right),$$

where $r = \frac{-n+2-\sqrt{(n-2)(n-10)}}{2}$ and $s = \frac{-n+2+\sqrt{(n-2)(n-10)}}{2} > r$. From the initial conditions

$$c_2 = \frac{Aq - r}{s - r} > 0,$$

in view of (3.19). If $c_1 \geq 0$, then $z(t) > 0$ for all $t > 0$. If $c_1 < 0$, then the function $c_1 + c_2 (\frac{t}{A})^{s-r}$ has a root, but it is smaller than A (at A this function equals 1). So that $z(t) > 0$ on (A, ∞). $\square$

Let now t_0 denote the root of $w_0(t)$, i.e., $t_0 = \sqrt{2n-4}$. With $p(t) = w(t) - w_0(t)$, we have $p(t_0) = w(t_0) < 0$. By Lemma 3.5 we shall have $p(t) < 0$ on (t_0, ∞), provided the solution of the linearized equation (3.18) with the initial conditions $z(t_0) = p(t_0)$ and $z'(t_0) = p'(t_0)$ satisfies $z(t) < 0$ on (t_0, ∞). By Lemma 3.6 (with $A = t_0$) this will happen if

$$\frac{t_0 w'(t_0)+2}{w(t_0)} > \frac{-n+2-\sqrt{(n-2)(n-10)}}{2}. \tag{3.20}$$

When $n=10$, a numerical computation shows that the quantity on the left in (3.20) is approximately -1.72324, while the one on the right is -4. When one increases n, numerical computations show that the quantity on the left in (3.20) is monotone increasing, while the one on the right is decreasing rapidly. It follows that $p(t)<0$, i.e., $w(t)<w_0(t)$ for all $t>0$. We conclude that $p'(t)>0$ for $t>0$, since we would get a contradiction in (3.12) at any point of local maximum. From (3.3), $\lambda'(t)>0$ for all $t>0$, i.e., the solution curve always travels to the right in the $(\lambda, u(0))$ plane.

4. Micro-electromechanical systems (MEMS)

Recall that for the problem (with constants $\alpha \ge 0$, and $p>1$)

$$u'' + \frac{n-1}{r}u' + \lambda\frac{r^\alpha}{(1-u)^p} = 0 \quad \text{for } 0<r<1, \qquad u'(0)=0, \qquad u(1)=0 \tag{4.1}$$

the solution curve $(\lambda, u(0))$ is given by $(\frac{t^{\alpha+2}}{w^{p+1}(t)}, 1-\frac{1}{w(t)})$, parameterized by $t \in (0,\infty)$, where $w(t)$ is the solution of

$$w'' + \frac{n-1}{t}w' = \frac{t^\alpha}{w^p}, \qquad w(0)=1, \qquad w'(0)=0 \quad (t>0), \tag{4.2}$$

with $w(t)>0$ and $w'(t)>0$ for all $t>0$. In particular, $\lambda(t) = \frac{t^{\alpha+2}}{w^{p+1}(t)}$. Compute

$$\lambda'(t) = \frac{t^{\alpha+1}w^p[(\alpha+2)w-(p+1)tw']}{w^{2(p+1)}}.$$

To study the direction of the solution curve we are interested in the sign of $\lambda'(t)$, or in the roots of the function $(\alpha+2)w(t)-(p+1)tw'(t)$. If we set this function to zero

$$(\alpha+2)w - t(p+1)w' = 0,$$

then the general solution of this equation is $w(t)=ct^\beta$, with $\beta = \frac{\alpha+2}{p+1}$. One verifies that

$$w_0(t) = c_0 t^\beta, \quad \text{with } c_0 = \frac{1}{[\beta(\beta+n-2)]^{\frac{1}{p+1}}}$$

is a solution of the equation in (4.2) (the guiding solution). The linearized equation for (4.2) is

$$z'' + \frac{n-1}{t}z' = -pt^\alpha w^{-p-1}z.$$

At the solution $w=w_0(t)$, this becomes

$$z'' + \frac{n-1}{t}z' + \frac{p\beta(\beta+n-2)}{t^2}z = 0, \tag{4.3}$$

which is again Euler's equation. Its characteristic equation has the roots

$$r=\frac{-n+2\pm\sqrt{(n-2)^2-4p\beta(\beta+n-2)}}{2}.$$

The roots are complex, if β satisfies

$$4p\beta^2+4p(n-2)\beta-(n-2)^2>0. \tag{4.4}$$

When $n=1$, this happens when $\beta>\frac{p+\sqrt{p^2+p}}{2p}$, the larger root for the quadratic on the left, i.e., when $\alpha>\frac{(p+1)(p+\sqrt{p^2+p})}{2p}-2$. If $p=2$, this becomes $\alpha>\frac{6+3\sqrt{6}}{4}-2$. This is the same inequality as $\alpha>-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{27}{2}}$, obtained on p. 900 in J.A. Pelesko [14]. When $n=2$, the inequality (4.4) holds for all $\alpha\geq 0$, which will imply (as we show below) infinitely many turns of the solution curve, as was previously proved for the case $p=2$ in [14], and for any $p>1$ in [4].

We now assume that $n\geq 3$. The inequality (4.4) holds if β is greater than the larger root for the quadratic on the left, i.e., for

$$\beta>\frac{n-2}{2}\frac{1}{p+\sqrt{p^2+p}}. \tag{4.5}$$

This inequality is equivalent to the condition (1.6).

Remark. Since $\beta=\frac{\alpha+2}{p+1}>\frac{2}{p+1}$, we shall see that in the lower dimensions (4.5) holds automatically, i.e., without restricting α. Indeed, we rewrite the inequality

$$\frac{2}{p+1}>\frac{n-2}{2}\frac{1}{p+\sqrt{p^2+p}}$$

as

$$\frac{p+1}{p+\sqrt{p^2+p}}<\frac{4}{n-2}. \tag{4.6}$$

On the left we have a decreasing function, which takes its maximum at $p=1$. So that (4.6) will follow if

$$\frac{2}{1+\sqrt{2}}<\frac{4}{n-2},$$

which happens for $3\leq n\leq 6$.

Similarly to Lemma 3.1 we prove the following lemma.

Lemma 4.1. *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (4.1) makes infinitely many turns.*

Theorem 4.1. *Assume that $\beta = \frac{\alpha+2}{p+1}$ satisfies (4.4) (i.e., either $n = 2$, or $n \geq 3$, and the condition (1.6) holds). Then the solution curve of (4.1) makes infinitely many turns.*

Proof. In view of Lemma 4.1, we need to show that $w(t)$ oscillates infinitely many times around $w_0(t)$. We show first that these functions get close to each other, as t increases. Denote $P(t) = w(t) - w_0(t)$. It satisfies

$$P'' + \frac{n-1}{t} P' + a(t) P = 0, \tag{4.7}$$

where $a(t) = pt^\alpha \int_0^1 \frac{1}{[sw(t)+(1-s)w_0(t)]^{p+1}} \, ds > 0$. As in the proof of the Theorem 3.1, we see that either $P(t)$ has infinitely many roots, or else $P(t)$ keeps the same sign over some infinite interval (t_k, ∞), and tends to a constant as $t \to \infty$. We now rule out the latter possibility. Write

$$a(t) = p\, t^\alpha \frac{1}{w_0^{p+1}} \int_0^1 \frac{1}{[s\frac{w(t)}{w_0(t)} + (1-s)]^{p+1}} \, ds = \frac{p\beta(\beta + n - 2)}{t^2} \big(1 + o(1)\big),$$

as $t \to \infty$. (Observe that $\frac{w(t)}{w_0(t)} = 1 + \frac{P(t)}{w_0(t)} \to 1$, as $t \to \infty$.) Since Euler's equation (4.3) has infinitely many roots on (t_k, ∞), we conclude by Lemma 3.3 that $P(t)$ must vanish on that interval too. It follows that $P(t)$ changes sign infinitely many times. □

Theorem 4.2. *Assume that $n \geq 2 + \frac{2(\alpha+2)}{p+1}(p + \sqrt{p^2 + p})$. Then the solution curve of (4.1) admits at most two turns.*

Proof. We follow the proof of Theorem 3.2. With $P(t) = w(t) - w_0(t)$, we need to show that $P(t)$ changes its monotonicity at most twice. Since $P(t)$ satisfies the linear equation (4.7), $P(t)$ cannot have points of positive local minimum, and of negative local maximum, and hence $P(t)$ changes its monotonicity once between two consecutive roots, and once after its last root (since $z(t)$ tends to zero as $t \to \infty$), and no other changes of monotonicity are possible. We will show that $P(t)$ has at most two roots, which will imply that $P(t)$ changes its monotonicity at most once. Under our conditions Euler's equation (4.3) has at most one root, while $P(t)$ satisfies

$$P'' + \frac{n-1}{t} P' + \frac{p\beta(\beta + n - 2)}{t^2} P < 0,$$

since the nonlinearity in (4.1) is convex in u. The rest of the proof is identical to that of Theorem 3.2. □

5. Morse index of solutions to the Gelfand and MEMS problems

We now use the generating solution to show that all turning points of the Gelfand problem are non-degenerate, and that the Morse index of solutions increases by one at each turning point, thus recovering a result of K. Nagasaki and T. Suzuki [13]. Recall that positive solutions of the Gelfand problem

$$\Delta u + \lambda e^u = 0, \quad \text{for } |x| < 1, \qquad u = 0, \quad \text{when } |x| = 1 \tag{5.1}$$

are radially symmetric, i.e., $u = u(r)$, $r = |x|$, with $u'(r) < 0$, and that the solution curve in the $(\lambda, u(0))$ plane is given by (1.4) and (1.3). In particular, $\lambda(t) = t^2 e^{w(t)}$, where $w(t)$ is the solution of (1.3), the generating solution.

Theorem 5.1. *Let $u(t_n)$ be a singular solution of (5.1), i.e., $\lambda'(t_n) = 0$. Then $u(t_n)$ is non-degenerate, i.e., $\lambda''(t_n) \neq 0$.*

Proof. Compute $\lambda'(t) = te^{w(t)}(2 + tw'(t))$. Since $\lambda'(t_n) = 0$, we have $2 + t_n w'(t_n) = 0$. Then

$$\lambda''(t_n) = t_n e^{w(t_n)}\bigl(w'(t_n) + t_n w''(t_n)\bigr), \tag{5.2}$$

and we need to show that $w'(t_n) + t_n w''(t_n) \neq 0$. For the guiding solution $w_0(t) = \ln(2n - 4) - 2\ln t$ we have

$$w_0'(t_n) + t_n w_0''(t_n) = 0.$$

Since $w'(t_n) = w_0'(t_n)$ $(= -\frac{2}{t_n})$, it suffices to show that

$$w''(t_n) \neq w_0''(t_n). \tag{5.3}$$

The function $p(t) = w_0(t) - w(t)$ satisfies the linear equation (3.13), with $p'(t_n) = 0$. It follows that $w(t_n) \neq w_0(t_n)$, since otherwise $p(t) \equiv 0$, which is impossible. Since both $w(t)$ and $w_0(t)$ satisfy the same equation (1.3), we conclude (5.3) by expressing the second derivatives from the corresponding equations. □

By C.S. Lin and W.-M. Ni [12], any solution of the linearized problem for (5.1) is radially symmetric, and hence it satisfies

$$\omega'' + \frac{n-1}{r}\omega' + \lambda e^u \omega = 0, \quad \text{for } 0 < r < 1, \qquad \omega'(0) = \omega(1) = 0. \tag{5.4}$$

We call $u(r)$ a singular solution of (5.1) if the problem (5.4) has a non-trivial solution. (Differentiating (5.1) in t, and setting $t = t_n$, it is easy to see that a solution is singular iff $\lambda'(t_n) = 0$.) The following lemma was proved in P. Korman [8].

Lemma 5.1. *Let $u(r)$ be a singular solution of (5.1). Then*

$$\omega(r) = ru'(r) + 2$$

gives a solution of (5.4).

We now recover the following result of K. Nagasaki and T. Suzuki [13].

Theorem 5.2. *As one follows the solution curve of (5.1) in the direction of increasing $u(0)$, the Morse index of solution increases by one at each turn.*

Proof. The Morse index of solution is the number of negative eigenvalues μ of

$$\Delta\omega + \lambda e^u \omega + \mu\omega = 0, \quad \text{for } |x| < 1, \qquad \omega = 0, \quad \text{when } |x| = 1.$$

By [12] solutions of this problem are radially symmetric. At a singular solution $\mu = 0$, and then $\omega(r) = ru'(r) + 2$ by Lemma 5.1. Assume that at a singular solution $u(t_n)$, $\mu(t_n) = 0$ is the k-th eigenvalue. Following [13], we will show that $\mu'(t_n) < 0$, which means that for $t < t_n$ ($t > t_n$) the k-th eigenvalue is positive (negative), i.e., the Morse index increases by one through $t = t_n$. We shall show that the sign of $\mu'(t_n)$ is the same as that of $-(\lambda''(t_n))^2$, which is negative by Theorem 5.1. Recall the following known formulas (here $u = u(t_n)$, ω is a solution of (5.4), and B is the unit ball around the origin in R^n):

$$\mu'(t_n) \int_B \omega^2\, dx = -\lambda(t_n) \int_B e^u \omega^3\, dx \quad \text{(p. 11 in [9])},$$

$$-\lambda(t_n) \int_B e^u \omega^3\, dx = \lambda''(t_n) \int_B f(u)\omega\, dx \quad \text{(p. 3 in [9])},$$

$$\int_B f(u)\omega\, dx = \frac{1}{2\lambda(t_n)} u'(1)\omega'(1) \quad \text{(p. 5 in [9])}.$$

Since $u'(1) < 0$, it follows from these formulas that the sign of $\mu'(t_n)$ is opposite to that of $\lambda''(t_n)\omega'(1)$. Using Lemma 5.1, we have $\omega'(r) = u' + ru''$, $\omega'(1) = u'(1) + u''(1)$. Recall that $u(r) = w(t) + a$, with $t = br$ (where $w(t)$ is the generating solution). Observing that $b = t_n$ for $r = 1$, we have $u(r) = u(0) + w(t_n r)$, and then

$$\omega'(1) = t_n\big(w'(t_n) + t_n w''(t_n)\big),$$

which by (5.2) has the same sign as $\lambda''(t_n)$. It follows that the sign of $\mu'(t_n)$ is the same as that of $-(\lambda''(t_n))^2 < 0$. $\square$

For the MEMS problem ($p > 0$)

$$\Delta u + \lambda \frac{1}{(1-u)^p} = 0, \quad \text{for } |x| < 1, \qquad u = 0, \quad \text{when } |x| = 1 \tag{5.5}$$

the situation is similar, which is a new result.

Theorem 5.3. *Let $u(t_n)$ be a singular solution of (5.5), i.e., $\lambda'(t_n) = 0$. Then $u(t_n)$ is non-degenerate, i.e., $\lambda''(t_n) \neq 0$. Moreover, when one follows the solution curve of (5.5) in the direction of increasing $u(0)$, the Morse index of solution increases by one at each turn.*

As before, by [12] solutions of the linearized problem corresponding to (5.5) are radially symmetric, and hence they satisfy

$$\omega'' + \frac{n-1}{r}\omega' + \lambda \frac{p}{(1-u)^{p+1}}\omega = 0, \quad 0 < r < 1, \qquad \omega'(0) = \omega(1) = 0. \tag{5.6}$$

Lemma 5.2. *Let $u(r)$ be a singular solution of (5.5). Then*

$$\omega(r) = ru'(r) - \frac{2}{p+1}u(r) + \frac{2}{p+1}$$

gives a solution of (5.6).

Proof. The function $v(r) \equiv ru'(r) - \frac{2}{p+1}u(r) + \frac{2}{p+1}$ solves (5.6), and we have $v'(0) = 0$, $v(0) > 0$ (since solutions of (5.5) are smaller than 1). By scaling of $\omega(r)$, we may assume that $\omega(0) = v(0)$, and then by the uniqueness result for this type of initial value problems (see [15]), it follows that $\omega(r) \equiv v(r)$. $\square$

Proof of Theorem 5.3. The proof is similar to that of Theorem 5.2. This time $\lambda(t) = t^2 w^{-p-1}(t)$. Compute

$$\lambda'(t) = t w^{-p-2}(t)\big(2w(t) - (p+1)t w'(t)\big).$$

Since $\lambda'(t_n) = 0$, we have

$$2w(t_n) - (p+1)t_n w'(t_n) = 0. \tag{5.7}$$

Then

$$\lambda''(t_n) = -t_n w^{-p-2}(t_n)\big((p-1)w'(t_n) + (p+1)t_n w''(t_n)\big), \tag{5.8}$$

and we need to show that

$$S \equiv (p-1)w'(t_n) + (p+1)t_n w''(t_n) \neq 0,$$

to conclude that $\lambda''(t_n) \neq 0$. Using Eq. (4.2), and then (5.7), we express

$$\begin{aligned} S &= -\big[(p+1)(n-1) - (p-1)\big]w'(t_n) + \frac{(p+1)t_n}{w^p(t_n)} \\ &= -\frac{2[(p+1)(n-1) - (p-1)]w(t_n)}{(p+1)t_n} + \frac{(p+1)t_n}{w^p(t_n)}. \end{aligned}$$

For the guiding solution $w_0(t) = c_0 t^\beta$, $\beta = \frac{2}{p+1}$, we have

$$-\frac{2[(p+1)(n-1) - (p-1)]w_0(t_n)}{(p+1)t_n} + \frac{(p+1)t_n}{w_0^p(t_n)} = 0.$$

Observing that S is a decreasing function of $w(t_n)$ we conclude that $S \neq 0$, once we show that

$$w(t_n) \neq w_0(t_n). \tag{5.9}$$

The function $p(t) = w_0(t) - w(t)$ satisfies the linear equation (4.7), with $p'(t_n) = 0$. Then (5.9) is true, since otherwise $p(t) \equiv 0$, which is impossible.

We see as before that the sign of $\mu'(t_n)$ is opposite to that of $\lambda''(t_n)\omega'(1)$. By Lemma 5.2,

$$\omega'(1) = \frac{p-1}{p+1}u'(1) + u''(1).$$

In terms of the generating solution $w(t)$, we have $u(r) = 1 - aw(t)$, $t = br$, with $b = t_n$ at the singular solutions. Then $u(r) = 1 - aw(t_n r)$, and we have

$$\omega'(1) = -\frac{at_n}{p+1}\big[(p-1)w'(t_n) + (p+1)t_n w''(t_n)\big].$$

Comparing this with (5.8), we conclude that $\omega'(1)$ has the same sign as $\lambda''(t_n)$, and then the sign of $\mu'(t_n)$ is the same as that of $-(\lambda''(t_n))^2 < 0$, concluding the proof as before. □

6. The Henon equation

We study positive solutions of the Dirichlet problem for the Henon equation

$$u'' + \lambda|x|^\alpha u^p = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{6.1}$$

Here $\alpha > 0$ and $p > 1$ are constants, $\lambda > 0$ is a parameter. This problem has both symmetric (even) and non-symmetric positive solutions, see D. Smets et al. [16], R. Kajikiya [7]. The exact multiplicity of the positive solutions is not known. We shall approach this problem by using "shooting", scaling and numerical computations.

For a variable $\xi > 0$ we consider the initial value problem

$$z'' + |x|^\alpha z^p = 0, \qquad z(\xi) = 1, \qquad z'(\xi) = 0. \tag{6.2}$$

Let $a(\xi)$ denote the first root of $z(x)$ which is greater than ξ, and $b(\xi)$ the first root of $z(x)$ which is to the left of ξ.

Theorem 6.1. *Assume that the equation*

$$a(\xi) = -b(\xi) \quad (\xi > 0) \tag{6.3}$$

has a unique solution $\xi_0 > 0$. Then for any positive λ the problem (6.1) has exactly three positive solutions: $u_1(x)$ which is an even function, $u_2(x)$ which has its point of maximum at $\xi = \frac{\xi_0}{a(\xi_0)}$, and $u_3(x) = u_2(-x)$. Moreover, if b denotes the maximum value of $u_2(x)$ (i.e., $b = u_2(\xi)$), then $\lambda = \frac{a(\xi_0)^{\alpha+2}}{b^{p-1}}$.

Proof. Denote $\eta = a(\xi_0)$. In (6.2) we let $x = \eta t$, $z = \frac{1}{b}v$, obtaining

$$v'' + \frac{\eta^{\alpha+2}}{b^{p-1}}|t|^\alpha v^p = 0 = 0, \qquad v(\pm 1) = 0,$$

i.e., $v(t)$ is a solution of (6.1), with $\lambda = \frac{\eta^{\alpha+2}}{b^{p-1}}$. The maximum value of this solution is equal to b, and it occurs at $\xi = \frac{\xi_0}{\eta}$. Solutions of (6.1) at other λ's are obtained by scaling of u.

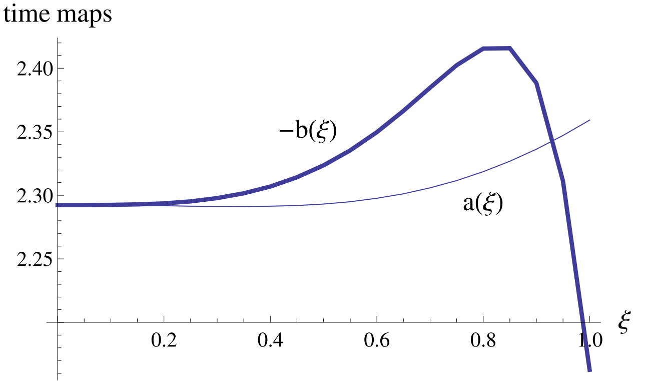

Fig. 2. The functions $a(\xi)$ and $-b(\xi)$.

We now show that there is exactly one positive solution, which takes its maximum value at a positive x. Let $u_3(x)$ be another solution of (6.1), with the maximum value achieved at $\xi_1 > 0$. Assume first that $\xi_1 = \xi_0$. By scaling of u and x, we obtain from $u_3(x)$ a solution of (6.1) (at the same λ), which at ξ_0 has the same initial data as $u_1(x)$ (and so is identical to $u_1(x)$), but it has its first root at some $x \neq 1$, a contradiction. Next, assume that $\xi_1 \neq \xi_0$. Again, we scale $u = Av$, $x = Bt$. Choose $A = u(\xi_1)$, then $v(\xi_1) = 1$. Now choose B, so that $\lambda B^{2+\alpha} A^{p-1} = 1$. Then we get a solution of (6.2), for which $a(\xi_1) = \frac{1}{B}$, $b(\xi_1) = -\frac{1}{B}$, a contradiction with the uniqueness of solution of (6.3). $\square$

For particular α and p one can verify computationally that (6.3) has a unique solution.

Example. $\alpha = 2$, $p = 3$. The graphs of $a(\xi)$ and $-b(\xi)$, computed using *Mathematica*, are given in Fig. 2. For $\xi > 0$ and small, we have $-b(\xi) > a(\xi)$, as a graph on a smaller scale shows. We see that the graphs of $a(\xi)$ and $-b(\xi)$ intersect exactly once for $\xi > 0$. This computation provides a computer assisted proof that the problem

$$u'' + \lambda x^2 u^3 = 0, \quad -1 < x < 1, \qquad u(-1) = u(1) = 0$$

has exactly three positive solutions for any $\lambda > 0$.

We have seen a similar picture for many other $\alpha > 0$ and $p > 1$ that we tried. For particular α and p one can produce a computer assisted proof, based on these calculations. However, some restrictions on α and p appear to be necessary. For example for $\alpha = p = 2$ the graph of $-b(\xi)$ is below that of $a(\xi)$, which indicates that there are no symmetry breaking solutions. In a recent paper S. Tanaka [17] proved that symmetry breaking solutions exist, provided that $\alpha(p-1) \geq 4$. For symmetric (even) solutions existence and uniqueness is known for all $\alpha > 0$ and $p > 1$, see e.g., [9].

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