Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 27, No. 9, pp. 1031-1047, 1996

0362-546X/96 $15.00+0.00

0362-546X(95)00108-5

SOLUTION CURVES FOR TWO CLASSES OF BOUNDARY-VALUE PROBLEMS

PHILIP KORMAN†‡ and TIANCHENG OUYANG§||

†Institute for Dynamics and Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.; and
§Department of Mathematics, Brigham Young University, Provo, UT 84602, U.S.A.

(*Received* 24 *September* 1994; *received in revised form* 17 *March* 1995; *received for publication* 1 *June* 1995)

Key words and phrases: Multiplicity results, bifurcation of solutions.

1. INTRODUCTION

We continue our studies, begun in [1, 2], of global solution curves for two point boundary-value problems of the type

$$u'' + \lambda f(x, u) = 0 \quad \text{on } (a, b), \qquad u(a) = u(b) = 0. \tag{1.1}$$

For a given value of a real parameter λ we are interested in multiplicity of solutions, and how solutions change with λ. Typically we prove that all solutions of (1.1) lie on a single solution curve. This fact is important for computation of solutions. It means that all solutions of (1.1) can be computed by very efficient continuation algorithms, and we can start the continuation of solutions somewhere on a stable branch, where it is easy to compute the solution. We also obtain exact multiplicity results for large λ, and sometimes for all λ. We recall our strategy in [1, 2]. We assumed the interval (a, b) to be symmetric about origin, taking $(a, b) = (-1, 1)$ without loss of generality, and $f(x, u)$ to be even in u. Under an additional condition $xf_x < 0$ for $x \neq 0$ (see Lemma 2.2 for the precise statement) we proved that any solution of (1.1) is even. This allowed us to prove that any nontrivial solution of the variational problem

$$w'' + \lambda f_u(x, u)w = 0 \quad \text{on } (a, b), \qquad u(a) = u(b) = 0 \tag{1.2}$$

does not change sign inside (a, b). The last fact was in turn used to show that at any singular solution $u(x)$ (i.e. where (1.2) has a nontrivial solution) a bifurcation theorem of Crandall and Rabinowitz [3] applies. This theorem was our basic tool, which, together with some variational arguments and implicit function theorem, allowed us to describe global solution curves.

In the present paper we study along the same lines the case where $f(x, u)$ is cubic in u with distinct positive roots. Here the condition $xf_x < 0$ for $x \neq 0$ is violated, which made the proof that the solution is even, and other steps in our analysis, much more complicated. In the case where f is independent of x, this problem was studied by Smoller and Wasserman [4], and Wang and Kazarinoff [5, 6], who obtained exact multiplicity results by a very involved phase plane analysis. In other direction we study the case where $f(x, u)$ is not assumed to be symmetric, and

‡Supported in part by the Taft Faculty Grant at the University of Cincinnati.
||Supported in part by a College of Science Research Grant, Brigham Young University.

present a condition which ensures that any nontrivial solution of (1.2) is one of sign. It is then possible to translate a number of results from [1, 2] and from the present paper to the nonsymmetric case, but we refrain from doing that.

Next we list some background results. Recall that a function $\phi(x) \in C^2(a, b) \cap C^0[a, b]$ is called a supersolution of (1.1) if

$$\phi'' + \lambda f(x, \phi) \leq 0 \quad \text{on } (a, b), \qquad \phi(a) \geq 0,\ \phi(b) \geq 0. \tag{1.3}$$

A subsolution $\psi(x)$ is defined by reversing the inequalities in (1.3). The following result is standard.

LEMMA 1.1. Let $\phi(x)$ and $\psi(x)$ be, respectively, super- and subsolutions of (1.1), and $\phi(x) \geq \psi(x)$ on (a, b) with $\phi(x) \not\equiv \psi(x)$, then $\phi(x) > \psi(x)$ on (a, b).

We shall often use this lemma with either $\phi(x)$ or $\psi(x)$ or both being solution of (1.1). The following lemma is a consequence of the first.

LEMMA 1.2. Let $u(x)$ be a nontrivial solution of (1.3) with $f(x, 0) \equiv 0$. If $u(x) \geq 0$ on (a, b) then $u > 0$ on (a, b).

Next we state a bifurcation theorem of Crandall and Rabinowitz [3].

THEOREM 1.1 [3]. Let X and Y be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda}, \bar{x})$ into Y. Let the null-space $N(F_x(\bar{\lambda}, \bar{x})) = \operatorname{span}\{x_0\}$ be one-dimensional and $\operatorname{codim} R(F_x(\bar{\lambda}, \bar{x})) = 1$. Let $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$. If Z is a complement of $\operatorname{span}\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.

2. POSITIVITY FOR THE LINEARIZED EQUATION

LEMMA 2.1. Consider the problem

$$u'' + f(x, u) = 0 \quad \text{on } (a, b), \qquad u(a) = u(b) = 0. \tag{2.1}$$

Assume that $f, f_x \in C^0([a, b] \times R_+)$, and that one of the following three conditions holds:

$$f_x(x, u) \leq 0 \quad \text{for all } x \in (a, b), \qquad u > 0; \tag{2.2}$$

$$f_x(x, u) \geq 0 \quad \text{for all } x \in (a, b), \qquad u > 0; \tag{2.3}$$

there is a point c, $a < c < b$, such that for all $u > 0$
$f_x(x, u) \geq 0$ for all $x \in (a, c)$ and $f_x(x, u) \leq 0$ for all $x \in (c, b)$. (2.4)

Moreover, $f_x(x, u) \not\equiv 0$ for almost all $x \in (a, b)$ and $u > 0$. Then any positive solution of (2.1) has only one maximum.

Proof. Assume that (2.2) holds. If a solution $u(x)$ has more than one maximum, it must have at least one local minimum on (a, b). Let x_1 be the largest point of local minimum, let $x_2 > x_1$ be such that $u(x_2) = u(x_1)$, and let $\bar{x}$ be the point of maximum of $u(x)$ on (x_1, x_2). Since $u(x)$ is increasing on $(x_1, \bar{x})$, we can represent its graph on that interval by a function $x = x_1(u)$. Similarly, on $(\bar{x}, x_2)$ we can represent $u(x)$ by $x = x_2(u)$, with $x_2(u) > x_1(u)$ for all $u \in (u(x_1), u(\bar{x}))$. Multiplying (2.1) by u' and integrating from x_1 to x_2, we obtain

$$\frac{u'^2(x_2)}{2} + \int_{x_1}^{\bar{x}} f(x, u)u' \, dx + \int_{\bar{x}}^{x_2} f(x, u)u' \, dx = 0. \tag{2.5}$$

Changing the variables in the integrals, we rewrite (2.5) as

$$\frac{u'^2(x_2)}{2} + \int_{u(x_1)}^{u(\bar{x})} [f(x_1(u), u) - f(x_2(u), u)] \, du = 0. \tag{2.6}$$

Since the integral in (2.6) is positive, we have a contradiction.

If now (2.3) holds, then assuming the lemma to be false, we denote by x_2 the smallest point of local minimum, by $x_1 < x_2$ the point where $u(x_1) = u(x_2)$, and by $\bar{x}$ the point of maximum of $u(x)$ on (x_1, x_2). As above we obtain

$$-\frac{u'^2(x_1)}{2} + \int_{u(x_1)}^{u(\bar{x})} [f(x_1(u), u) - f(x_2(u), u)] \, du = 0. \tag{2.7}$$

Since the integral in (2.7) is negative, we have a contradiction.

Finally, if the lemma is false in case of (2.4), the solution $u(x)$ will have a point of local minimum in either $[c, b)$ or $(a, c]$ (or both). In the first case we obtain a contradiction as in (2.6), and in the second one as in (2.7).

To prove positivity of any nontrivial solution of the linearized equation for (2.1), we shall need an extra condition, which will force the maximum of $u(x)$ to occur at $x = c$, under the assumption (2.5). We accomplish this by assuming $f(x, u)$ to be even in x with respect to (a, b).

After shifting and rescaling x, we may consider the problem

$$u'' + f(x, u) = 0 \quad \text{on } (-1, 1), \qquad u(-1) = u(1) = 0. \tag{2.8}$$

LEMMA 2.2 [1]. Consider the problem (2.8) and assume that $f(x, u) \in C^1([-1, 1] \times R_+)$ satisfies

(i) $f(-x, u) = f(x, u)$ for all $x \in (-1, 1)$ and $u > 0$;

(ii) $xf_x(x, u) < 0$ for all $x \in (-1, 1)\backslash\{0\}$ and $u > 0$.

Then any positive solution of (2.8) is an even function with $u'(x) < 0$ on $(0, 1]$. Moreover, any two positive solutions cannot intersect on $(-1, 1)$.

Condition (ii) of the above lemma fails for a class of cubic nonlinearities considered in this paper. However, the corresponding result still holds.

We consider

$$u'' + \lambda(u - a)(u - b(x))(c(x) - u) = 0 \quad \text{on } (-1, 1), \qquad u(-1) = u(1) = 0. \tag{2.9}$$

Assume that the constant a and the even functions $b(x)$ and $c(x)$, of class $C^1(-1, 1) \cap C^0[-1, 1]$, satisfy the following conditions

$$0 < a < b(x) < c(x) \qquad \text{for all } x \in (-1, 1); \tag{2.10}$$

$$c''(x) < 0 \qquad \text{for all } x \in (-1, 1); \tag{2.11}$$

$$b'(x) + c'(x) \geq 0 \qquad \text{for } x \in (0, 1); \tag{2.12}$$

$$c'(x) < 0 \qquad \text{for } x \in (0, 1). \tag{2.13}$$

Notice that by the maximum principle any nontrivial solution of (2.9) is positive.

LEMMA 2.3. Consider the problem (2.9) under the conditions (2.10)–(2.13). Then any positive solution of (2.9) has only one maximum.

Proof. We begin by noticing that

$$u(x) < c(x) \qquad \text{for all } x \in (-1, 1). \tag{2.14}$$

Indeed, letting $v = c(x) - u$, we obtain

$$c'' - v'' + \lambda(c - a - v)(c - b - v)v = 0 \quad \text{on } (-1, 1), \qquad v(\pm 1) = c(\pm 1) > 0. \tag{2.15}$$

If (2.14) were violated, we would get a contradiction in (2.15) at the point of nonpositive minimum of $v(x)$.

Let x_0 be now a point of local maximum of $u(x)$. Since $u''(x_0) \leq 0$, it follows in view of (2.14) and Lemma 1.1 that either $b(x_0) \leq u(x_0) < c(x_0)$ or $u(x_0) < a$. In the first case we claim that in fact

$$b(x_0) < u(x_0) < c(x_0). \tag{2.16}$$

Indeed, if one assumes $u(x_0) = b(x_0)$ then $u''(x_0) = 0$, and differentiating the equation (2.9), we obtain

$$u'''(x_0) = \lambda b'(x_0)(u(x_0) - a)(c(x_0) - u(x_0)) \neq 0,$$

which is impossible at a point of maximum.

Letting

$$f(x, u) = (u - a)(u - b(x))(c(x) - u), \tag{2.17}$$

we obtain

$$f_x = (u - a)[-b'(c - u) + c'(u - b)]. \tag{2.18}$$

In case $u(x_0) < a$ we have $f > 0$ for all $x \in (-1, 1)$, and so $u(x)$ cannot have interior minima. It remains to consider the case when (2.16) holds. Examining the proof of Lemma 2.1, one sees that it suffices for the condition (2.4) to be satisfied when $x \in (x_1, x_2)$, the interval defined in the proof. In other words, it is enough to have $f_x > 0$ for $x \in (x_1, x_2)$ when $x < 0$, and $f_x < 0$ on (x_1, x_2) when $x > 0$. Since x_1 is a point of local minimum, $u''(x_1) \geq 0$, which implies that $a \leq u(x_1) \leq b(x_1)$, and in fact $u(x_1) > a$, since assuming $u(x_1) = a$, $u'(x_1) = 0$, we would have two solutions of the equation (2.9) satisfying the same initial conditions at x_1 ($u(x)$ and a), a contradiction. Using the conditions (2.12) and (2.13),

$$-b'(c-u)+c'(u-b) = -b'c-c'b+(b'+c')u \leq -b'c-c'b+(b'+c')c = c'(c-b).$$

This implies by (2.18) that $f_x < 0$ (>0) when $x > 0$ (<0) over the interval (x_1, x_2), and so the Lemma 2.1 applies.

LEMMA 2.4. Assume that $u(x)$ and $v(x)$ are two solutions of (2.9) such that for some $\gamma \in (-1, 1)$,
(i) $v(x) < u(x) \leq a$ on $(\gamma, 1)$,
(ii) $u'(x) \leq 0$, $v'(x) \leq 0$ on $(\gamma, 1)$,
Assume $\bar{\alpha}, \bar{\beta} \in (\gamma, 1)$ are such that $\bar{\alpha} < \bar{\beta}$ and
(iii) $u(\bar{\beta}) = v(\bar{\alpha})$.
Then

$$u'(\bar{\beta}) < v'(\bar{\alpha}) \qquad (\text{i.e. } |u'(\bar{\beta})| > |v'(\bar{\alpha})|). \tag{2.19}$$

Proof. We begin by proving that, with $f(x, u)$ as defined in (2.17),

$$f_u < 0 \quad \text{for } u < a \qquad \text{and} \qquad x \in (-1, 1). \tag{2.20}$$

Indeed,

$$\begin{aligned} f_u(x, u) &= -3u^2 + 2(a+b+c)u - (ab+ac+bc), \\ f_{uu}(x, u) &= -6u + 2(a+b+c), \\ f_u(x, a) &= -a^2 + ab + ac - bc = (b-a)(a-c) < 0. \end{aligned} \tag{2.21}$$

When $u < a$,

$$f_{uu} > -6a + 2(a+b+c) = 4\left(\frac{b+c}{2} - a\right) > 0. \tag{2.22}$$

From (2.21) and (2.22) the claim (2.20) follows. On $(\bar{\beta}, 1)$ we consider $w = u - v > 0$. Then

$$w'' + c(x)w = 0 \quad \text{on } (\bar{\beta}, 1), \qquad w(1) = 0, \tag{2.23}$$

where, in view of (2.20),

$$c(x) = \lambda \int_0^1 f_u\,(x, \theta u + (1-\theta)v)\,\mathrm{d}\theta < 0. \tag{2.24}$$

Multiply (2.23) by w, and integrate from $\bar{\beta}$ to 1 to obtain

$$-w(\bar{\beta})w'(\bar{\beta}) - \int_{\bar{\beta}}^1 w'^2\,\mathrm{d}x + \int_{\bar{\beta}}^1 c(x)w^2\,\mathrm{d}x = 0. \tag{2.25}$$

Since the last two terms in (2.25) are negative, it follows that $w'(\bar{\beta}) < 0$, i.e.

$$u'(\bar{\beta}) < v'(\bar{\beta}). \tag{2.26}$$

Since $f(x, v) > 0$ for all $x \in (-1, 1)$ when $v < a$, it follows that

$$v'(\bar{\beta}) = v'(\bar{\alpha}) + \int_{\bar{\alpha}}^{\bar{\beta}} v''\,\mathrm{d}x = v'(\bar{\alpha}) - \lambda \int_{\bar{\alpha}}^{\bar{\beta}} f(x, v)\,\mathrm{d}x < v'(\bar{\alpha}).$$

Combining this with (2.26), we conclude the proof of the lemma.

COROLLARY 1. Let $u(x)$ and $v(x)$ be two different solutions of (2.9) and $v(x) < u(x)$ near $x = 1$. Then these solutions cannot intersect so long as $u \leq a$, and they are both decreasing.

Proof. If $u(\delta) = v(\delta) \leq a$, then $|u'(\delta)| < |v'(\delta)|$. On the other hand, passing to the limit in (2.19) as $\bar{\alpha}, \bar{\beta} \to \delta$, $|u'(\delta)| \geq |v'(\delta)|$.

THEOREM 2.1. Assuming the conditions (2.10)–(2.13), any (positive) solution of (2.9) is an even function with $u'(x) < 0$ on $(0, 1]$. Moreover, two different solutions of (2.9) cannot intersect.

Proof. Assume a solution $u(x)$ is not even. Then $v(x) = u(-x)$ is another solution of (2.9) with $u(0) = v(0) = u_0$, $u'(0) = -v'(0)$, and $v(x) \not\equiv u(x)$. Clearly, $u(x)$ cannot take its maximum at $x = 0$, since otherwise $u'(0) = v'(0) = 0$, and we would have two different solutions satisfying the same initial conditions at $x = 0$, which is impossible. So assume $u(x)$ takes its maximum at some $\bar{x} > 0$, $u(\bar{x}) = \bar{u}$. We can assume that

$$\bar{u} > a, \tag{2.27}$$

for otherwise we would have two different solutions of (2.9) in the region where $f_u \leq 0$, which easily leads to a contradiction. We claim that

$$v(x) < u(x) \qquad \text{for all } x \in (0, 1). \tag{2.28}$$

Assume (2.28) to be violated. There are two cases.

Case 1: $v(x) > u(x)$ near $x = 1$ (since $u'(1) \neq v'(1)$, it is either this inequality or (2.28) near $x = 1$). Let $0 < \alpha$, $\beta < 1$ be such that $u(\beta) = v(\alpha) = a$. Since $v(x)$ has the same maximum as $u(x)$, $v(x)$ must intersect $u(x)$ on $(\bar{x}, 1)$. By Corollary 1 these functions cannot intersect where $u \leq a$, hence $\beta < \alpha$, and there is $\eta \in (0, \beta)$, such that

$$u(\eta) = v(\eta) = u_1, \qquad |u'(\eta)| > |v'(\eta)|. \tag{2.29}$$

From Lemma 2.4, reversing the roles of u and v,

$$|v'(\alpha)| > |u'(\beta)|. \tag{2.30}$$

We rewrite (2.9) in the form

$$u'' + f(x, u) = 0 \quad \text{on } (-1, 1), \qquad u(-1) = u(1) = 0. \tag{2.31}$$

We know from the proof of Lemma 2.3 that

$$f_x(x, u) < 0 \quad \text{for } x > 0, \qquad \text{when } u > a. \tag{2.32}$$

We now multiply (2.31) by u', and integrate from η to β,

$$\tfrac{1}{2}u'^2(\beta) - \tfrac{1}{2}u'^2(\eta) + \int_{u_1}^{a} f(x_1(u), u)\, du = 0, \tag{2.33}$$

where $x = x_1(u)$ is the inverse function of $u(x)$ on the interval (η, β). Multiplying the equation (2.31) for $v(x)$ by v', and integrating from η to α,

$$\tfrac{1}{2}v'^2(\alpha) - \tfrac{1}{2}v'^2(\eta) + \int_{u_1}^{a} f(x_2(v), v)\,\mathrm{d}v = 0, \tag{2.34}$$

where $x = x_2(v)$ is the inverse function of $v(x)$ on (η, α). Clearly,

$$x_2(u) > x_1(u) \qquad \text{for } a < u < u_1. \tag{2.35}$$

We subtract (2.34) from (2.33), to obtain

$$\tfrac{1}{2}(u'^2(\beta) - v'^2(\alpha)) + \tfrac{1}{2}(v'^2(\eta) - u'^2(\eta)) + \int_{a}^{u_1} [f(x_2(u), u) - f(x_1(u), u)]\,\mathrm{d}u = 0. \tag{2.36}$$

In view of (2.29), (2.30), (2.32) and (2.35), all three terms in (2.36) are negative, a contradiction.

Case 2: $v(x) < u(x)$ near $x = 1$, but $v(x) > u(x)$ inside (0, 1) somewhere. Reversing the roles of u and v, we get the same contradiction as in Case 1. The claim (2.28) is proved.

In view of (2.27), only two cases are possible:
(i) $u_0 < a < \bar{u}$,
(ii) $a < u_0 < \bar{u}$.
Assume (i) $u_0 < a$. In view of (2.28), $v(x)$ is decreasing on (0, 1), and so we can find $\gamma < 0 < p < q < 1$, such that (see Fig. 1), $u(p) = u(q) = v(\gamma) = a$. By the definition of $v(x)$,

$$|v'(\gamma)| = |u'(p)|. \tag{2.37}$$

By Lemma 2.4,

$$|v'(\gamma)| < |u'(q)|. \tag{2.38}$$

Combining (2.37) and (2.38),

$$|u'(p)| < |u'(q)|. \tag{2.39}$$

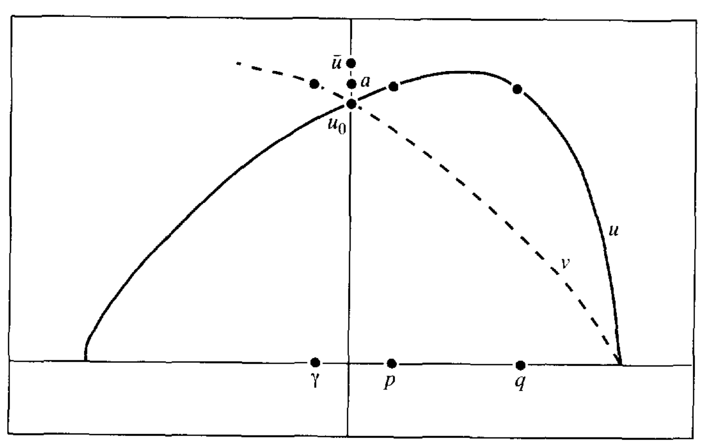

Fig. 1.

On the other hand, multiplying the equation (2.31) by u' and integrating, first from p to $\bar{x}$, and then from $\bar{x}$ to q, and adding the resulting identities,

$$\tfrac{1}{2}(u'^2(q) - u'^2(p)) + \int_a^{\bar{u}} [f(x_1(u), u) - f(x_2(u), u)]\,du = 0, \tag{2.40}$$

with $x_2(u) > x_1(u)$ for all $u \in (a, \bar{u})$. The integral term in (2.40) is positive in view of (2.32). Hence,

$$|u'(p)| > |u'(q)|,$$

contradicting (2.39).

Assume now (ii) $a < u_0 < \bar{u}$. Again $v(x)$ is decreasing on (0, 1). We define $0 < \alpha < \beta < 1$ and ξ by $u(\xi) = u(0) = u_0$, $v(\alpha) = u(\beta) = a$ (see Fig. 2). Repeating the argument leading to (2.40), we see that

$$|v'(0)| = |u'(0)| > |u'(\xi)|. \tag{2.41}$$

Multiply (2.31) by u', and integrate from ξ to β,

$$\tfrac{1}{2}(u'^2(\beta) - u'^2(\xi)) + \int_{u_0}^{a} f(x_2(u), u)\,du = 0, \tag{2.42}$$

where $x_2(u)$ is the inverse function of $u(x)$ on (ξ, β).

Multiplying the equation (2.31) for v by v', and integrating from 0 to α,

$$\tfrac{1}{2}(v'^2(\alpha) - v'^2(0)) + \int_{u_0}^{a} f(x_1(v), v)\,dv = 0, \tag{2.43}$$

where $x_1(v)$ is the inverse function of $v(x)$ on $(0, \alpha)$, with $x_1(u) < x_2(u)$ for all $u \in (a, u_0)$. From (2.42) subtract (2.43),

$$\tfrac{1}{2}(u'^2(\beta) - v'^2(\alpha)) + \tfrac{1}{2}(v'^2(0) - u'^2(\xi)) + \int_a^{u_0} [f(x_1(u), u) - f(x_2(u), u)]\,du = 0. \tag{2.44}$$

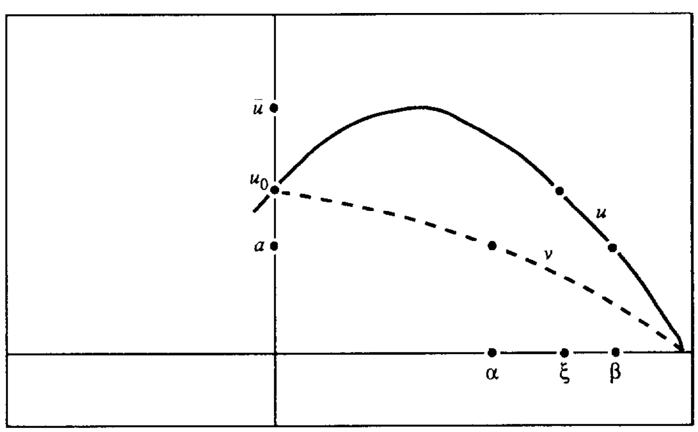

Fig. 2.

The first term in (2.44) is positive by Lemma 2.4, the second is positive by (2.41), and the third is positive by (2.32). We have a contradiction, which finishes the proof that $u(x)$ is even.

Finally, since any solution is decreasing on (0, 1), it follows that any two solutions of (2.9) cannot intersect in the region where $u < a$ by Corollary 1, and in the region where $u > a$ by repeating the argument leading to (2.36).

We consider next the linearized equation for (2.9),

$$w'' + \lambda[-3u^2 + 2u(a + b + c) - ab - ac - bc]w = 0 \text{ on } (-1, 1),\ w(-1) = w(1) = 0. \tag{2.45}$$

LEMMA 2.5. If (2.45) has a nontrivial solution, we can choose it so that $w(x) > 0$ on $(-1, 1)$.

Proof. We can assume that $w(x) > 0$ near $x = 1$. Assuming that $w(x)$ vanishes on (0, 1), let $0 \le \xi < 1$ be the largest point where $w(\xi) = 0$ (the case of $w(x)$ vanishing on $(-1, 0)$ is similar). Denote by $0 < \eta < 1$ the point where $u(\eta) = a$. Assume first that $\xi \ge \eta$. Notice that the square bracket in (2.45), which is $f_u(x, u)$, is then negative over $(\xi, 1)$, since $u(x) < a$ over that interval. By the maximum principle it follows that $w(x) \equiv 0$ over $(\xi, 1)$, a contradiction.

Next, we consider the case $\xi < \eta$. We claim that

$$w'(\eta) < 0. \tag{2.46}$$

Indeed, by the above we know that $w > 0$ and $w'' > 0$ over $(\eta, 1)$. Then

$$w'(\eta) = -\int_\eta^1 w''(x)\,\mathrm{d}x + w'(1) < 0.$$

Differentiate (2.9),

$$u_x'' + \lambda f_u u_x + \lambda f_x = 0. \tag{2.47}$$

Next, we multiply the equation (2.45) by u_x, (2.47) by w, integrate from ξ to η and subtract, obtaining

$$w'u'|_\xi^\eta - \lambda \int_\xi^\eta f_x w\,\mathrm{d}x = 0. \tag{2.48}$$

From above we know that $f_x \le 0$ over (ξ, η) (where $u > a$). Using (2.46) and $w'(\xi) > 0$, we see that the left-hand side of (2.48) is positive, a contradiction.

Next, we consider equations where nonlinearity is not even in x. To simplify the presentation we consider a problem

$$u'' + \alpha(x)f(u) = 0 \quad \text{on } (a, b), \qquad u(a) = u(b) = 0, \tag{2.49}$$

although similar results can be given for more general nonlinearities.

We assume that $\alpha(x) \in C^2(a, b) \cap C^0[a, b]$ satisfies

$$\alpha(x) > 0 \qquad \text{on } [a, b]. \tag{2.50}$$

The function $f(u) \in C^1(R_+)$ satisfies

$$f(u) > 0 \qquad \text{for } u > 0. \tag{2.51}$$

Since any positive solution $u(x)$ of (2.49) is concave, it follows that $u(x)$ has only one point of maximum.

LEMMA 2.6. In addition to the conditions (2.50) and (2.51), assume that

$$\frac{3}{2}\frac{\alpha'^2}{\alpha} - \alpha'' < 0 \qquad \text{for all } x \in (a, b). \tag{2.52}$$

Then any nontrivial solution of the linearized problem

$$w'' + \alpha(x)f'(u)w = 0 \quad \text{on } (a, b), \qquad w(a) = w(b) = 0, \tag{2.53}$$

can be chosen to be positive (where u is a solution of (2.49)).

Proof. Let x_0 be the point of maximum of $u(x)$. If $w(x)$ has zero points on (a, b), then it either has a zero on $(a, x_0]$, or on $[x_0, b)$, or both. Assume for definiteness that $w(\gamma) = 0$ with $a < \gamma \leq x_0$; the other case is similar. Differentiate (2.49),

$$u_x'' + \alpha(x)f_u u_x + \alpha'(x)f(u) = 0. \tag{2.54}$$

Multiply the equation (2.54) by $g(x)w$ and subtract from it the equation (2.53) multiplied by $g(x)u_x$, with $g(x) > 0$ to be specified. Then integrate over (a, γ),

$$-gu_x w'|_a^\gamma - \int_a^\gamma g'wu_x' \,dx + \int_a^\gamma g'u_x w' \,dx + \int_a^\gamma \alpha' gfw \,dx = 0. \tag{2.55}$$

The nonintegral terms on the left in (2.55) are positive. Integrating by parts in the second integral on the left, and using the equation (2.49), we combine all the integral terms in (2.55) as

$$-\int_a^\gamma g''u_x w \,dx + \int_a^\gamma (2g'\alpha + \alpha' g)fw \,dx = 0.$$

We shall obtain a contradiction in (2.55) if $g(x)$ satisfies $g'' < 0$ and $2g'\alpha + \alpha' g = 0$ on (a, b). Integrating the last relation, we see, in view of (2.52), that $g(x) = \alpha^{-1/2}(x)$ is a suitable choice.

Remark. The class of functions satisfying (2.52) includes $\alpha(x) = \beta(x) + c$, with $\beta(x)$ a convex function, and c a large constant.

3. A CLASS OF CUBIC NONLINEARITIES

In this section we consider the problem

$$u'' + \lambda(u - a)(u - b(x))(c(x) - u) = 0 \quad \text{on } (-1, 1), \qquad u(-1) = u(1) = 0, \tag{3.1}$$

with a, $b(x)$ and $c(x)$ satisfying the conditions (2.10)–(2.13) ($b(x)$ and $c(x)$ being even). Using maximum principle and (2.14) we see that any solution of (3.1) satisfies

$$0 < u(x) < c(x) \qquad \text{for all } x \in (-1, 1). \tag{3.2}$$

LEMMA 3.1. Let $u(x, \lambda)$ be a continuous in λ branch of solutions of (3.1). Then one of the following three possibilities holds:

(i) $\lim_{\lambda\to\infty} u(x, \lambda) = c(x)$ for all $x \in (-1, 1)$;

(ii) $\lim_{\lambda\to\infty} u(x, \lambda) = a$ for all $x \neq 0$;

(iii) $\lim_{\lambda\to\infty} u(x, \lambda) = c(x)$ for $x \in (-\alpha, \alpha)$, $\lim_{\lambda\to\infty} u(x, \lambda) = a$ on $(-1, 1)\setminus[-\alpha, \alpha]$ for some $0 < \alpha < 1$. (No claim is made about the behavior at $x = \pm\alpha$.)

Remark. In Theorem 3.3 we will be able to exclude the possibility (iii).

Proof. Since a bounded unimodular function $u(x)$ cannot have a large second order derivative on any subinterval of $(-1, 1)$, we see that at almost all $x_0 \in (-1, 1)$ $u(x_0, \lambda)$ tends to either a, $b(x_0)$ or $c(x_0)$. Since the shapes of $u(x, \lambda)$ and $b(x)$ are different, $u(x, \lambda)$ cannot tend to $b(x)$ over any subinterval of $(-1, 1)$. Recalling the shape of $u(x, \lambda)$, we see similarly that $u(x_0, \lambda)$ cannot tend to $b(x_0)$ at any $x_0 \neq 0$, unless $u(x, \lambda)$ tends to $c(x)$ on $(-|x_0|, |x_0|)$, and to a on $(-1, 1)\setminus[-|x_0|, |x_0|]$, which finishes the proof.

Using the implicit function theorem, one sees that for sufficiently small λ there is a curve of solutions emanating from $u = 0$, $\lambda = 0$. Under some conditions there are no other solutions of (3.1). It appears that even for constant a, b and c this case has not been sufficiently clarified in the literature.

THEOREM 3.1. Assume that a, $b(x)$ and $c(x)$ satisfy the conditions (2.10)–(2.13). Let $u(x)$ be any decreasing function on $[0, 1]$ with the range $[0, \alpha]$, and $a < \alpha < c(0)$. Let $x(u)$ denote the inverse function of $u(x)$. Assume that for any such $x(u)$

$$\int_a^\alpha (u - a)(u - b(x(u)))(c(x(u)) - u)\, du \leq 0. \tag{3.3}$$

Then for any $\lambda > 0$ the problem (3.1) has exactly one (positive) solution. The solutions $u(x, \lambda)$ lie on a smooth in λ curve, which is strictly increasing in λ, and $\lim_{\lambda\to\infty} u(x, \lambda) = a$ for all $x \in (-1, 1)$.

Remark. In case of constant a, b and c this condition says that the area of the negative hump of the function $f = (u - a)(u - b)(c - u)$ is larger than the area of the positive hump.

Proof. There are two possibilities for $u_m = u(0)$, the maximum value of $u(x)$. In view of (2.16) either $0 < u_m < a$ or $b(0) < u_m < c(0)$. We shall exclude the second possibility. Define $0 < \eta < 1$ by $u(\eta) = a$. Multiply (3.1) by u' and integrate from 0 to η,

$$\frac{u'^2(\eta)}{2} - \lambda \int_a^{u_m} (u - a)(u - b(x(u)))(c(x(u)) - u)\, du = 0,$$

which is a contradiction since $u'(\eta) < 0$ (if $u'(\eta) = 0$ then two solutions $u(x)$ and $u \equiv a$ would share the same initial conditions at $x = \eta$, which is impossible).

Let us now return to the curve of solutions starting at $\lambda = 0$, $u = 0$. Since $f_u < 0$ for $u < a$, it follows by the implicit function theorem that this curve can be continued for all $0 < \lambda < \infty$. We claim that (3.1) has no other solutions (not on the above curve). Indeed, if there were another (positive) solution, we could continue it by the implicit function theorem for decreasing λ, and this new curve of solutions would have nowhere to go (we have local uniqueness at $\lambda = 0$, $u = 0$). Finally, since the nonlinearity in (3.1) is positive for $u < a$, $u(x, \lambda)$ is a strict subsolution of the equation (3.1) at any $\mu > \lambda$, while $u = a$ is a supersolution at μ. It follows that at μ there is a solution of (3.1), which is strictly larger than $u(x, \lambda)$. By uniqueness it must be $u(x, \mu)$, which finishes the proof of the theorem.

We denote the nonlinearity in (3.1) by $f(x, u)$, and define $F(x, u) = \int_0^u f(x, z)\, dz$.

THEOREM 3.2. Assume that a, $b(x)$ and $c(x)$ satisfy (2.10)–(2.13). Assume in addition that

$$\int_{-1}^{1} F(x, a)\, dx < \int_{-1}^{1} F(x, c(x))\, dx. \tag{3.4}$$

All solutions of (3.1) lie on at most countably many unbounded smooth solution curves. One of the curves, referred to as the lower curve, starts at $\lambda = 0$, $u = 0$, it is strictly increasing in λ, and $\lim_{\lambda \to \infty} u(x, \lambda) = a$ for all $x \in (-1, 1)$. Each upper curve has two branches $u^-(x, \lambda) < u^+(x, \lambda)$, and as $\lambda \to \infty$, $u^-(x, \lambda)$ tends to a for all $x \in (-1, 1) \backslash \{0\}$. For $u^+(x, \lambda)$ there is a $p \in (0, 1)$, such that as $\lambda \to \infty$, $u^+(x, \lambda)$ tends to $c(x)$ for $x \in (-p, p)$ and to a for $x \in (-1, 1) \backslash (-p, p)$. The number p is the same for all upper curves. Each upper curve has at most finitely many turns for λ belonging to any bounded interval.

Proof. All the assertions about the lower curve, as well as uniqueness of solutions for sufficiently small $\lambda > 0$, follow exactly as in the previous theorem. We show next that for sufficiently large λ there is a solution, not lying on the lower curve. Solutions of (3.1) are critical points in $H_0^1(-1, 1)$ of the functional

$$J(u) = \int_{-1}^{1} [\tfrac{1}{2} u'^2 - \lambda F(x, u)]\, dx. \tag{3.5}$$

On the lower curve, which we denote by $\bar{u} = \bar{u}(x, \lambda)$,

$$J(\bar{u}) \geq -\lambda \int_{-1}^{1} F(x, \bar{u})\, dx, \tag{3.6}$$

$$-\lambda \int_{-1}^{1} F(x, \bar{u})\, dx \simeq -\lambda \int_{-1}^{1} F(x, a)\, dx \qquad \text{for } \lambda \text{ large.} \tag{3.7}$$

(Notice that $F(x, u)$ is a bounded and continuous function.) We now construct a function for which the value of the functional is less that $J(\bar{u})$. Since the functional $J(u)$ is bounded from below, it will have a point of minimum, different from $\bar{u}$. Define $u_\varepsilon(x)$ to be an even function of class $C^1[-1, 1]$, decreasing on $[0, 1]$, such that $u_\varepsilon(x) = c(x)$ on $[0, 1 - \varepsilon]$, and $u_\varepsilon(1) = 0$. Clearly, $u_\varepsilon \in H_0^1(-1, 1)$ and in view of (3.4),

$$\int_{-1}^{1} F(x, a)\, dx < \int_{-1}^{1} F(x, u_\varepsilon(x))\, dx,$$

for ε sufficiently small. Then $J(u_\varepsilon) < J(\bar{u}(x, \lambda))$ for λ large enough, and so we have a second solution of (3.1) for $\lambda \geq \tilde{\lambda}$, with some $\tilde{\lambda}$ large.

Condition (2.11) implies that $c(x)$ is a supersolution of (3.1). This means that (3.1) has a maximal solution, which for $\lambda \geq \tilde{\lambda}$ is different from the lower curve (if there were a solution below the lower branch, then we could continue it for decreasing λ, using the implicit function theorem, obtaining the same contradiction as in the Theorem 3.1). Let $\tilde{u}(x)$ be the maximal solution at $\tilde{\lambda}$. Define $\mathfrak{F}(\lambda, u)$ to be the left-hand side of (3.1), where $\mathfrak{F}: R_+ \times C_0^2[-1, 1] \to C[-1, 1]$. Notice that $\mathfrak{F}_u(\tilde{\lambda}, \tilde{u})$ is given by the left-hand side of (2.45). If $\mathfrak{F}_u(\tilde{\lambda}, \tilde{u})$ is singular, then the Crandall-Rabinowitz theorem applies, as will be explained below. If $\mathfrak{F}_u(\tilde{\lambda}, \tilde{u})$ is nonsingular, then using the implicit function theorem we can continue the curve of solutions for decreasing λ. This curve of solutions cannot be continued left for all λ, since for $\lambda > 0$ small the problem (3.1) has only one solution (corresponding to the lower curve). Let λ_0 be the infimum of the λ, for which the upper curve can be continued to the left. A standard argument, see [1, 2], shows that there is a solution $u(x, \lambda_0)$ of (3.1) on that curve.

By the definition of λ_0, it follows that $\mathfrak{F}_u(\lambda_0, u(x, \lambda_0))$ is singular, i.e. (2.45) has a nontrivial solution $w(x) > 0$. As in [1, 2] we see that the Crandall-Rabinowitz theorem applies at $(\lambda_0, u(x, \lambda_0))$ if we can show that $\mathfrak{F}_\lambda \notin R(\mathfrak{F}_u)$ at this solution ($\mathfrak{F}_\lambda(\lambda_0, u(x, \lambda_0)) = f(x, u(x, \lambda_0))$, the nonlinearity in (3.1)). Notice first that $w(x)$ is an even function (otherwise the problem (2.45) would have another positive solution $w(-x)$, which is impossible), and then we conclude using the Fredholm alternative (from (3.1), $-u'' = f(x, u)$)

$$\int_0^1 u''w \, dx = \int_0^1 u'w' \, dx = 0. \tag{3.8}$$

Differentiate the equation (3.1)

$$u_x'' + \lambda f_u u_x + \lambda f_x(x, u) = 0. \tag{3.9}$$

Let $0 < \eta < 1$ be the point where $u(\eta) = a$. Multiply the equation (2.45) by xu_x, the equation (3.9) by xw, integrate from 0 to η, and subtract. Obtain

$$xu'w'|_0^\eta - xwu''|_0^\eta - \int_0^\eta u'w' \, dx + \int_0^\eta wu'' \, dx - \lambda \int_0^\eta xf_x w \, dx = 0. \tag{3.10}$$

From the equation (3.1), $u''(\eta) = 0$. So the second term on the left in (3.10) is zero. From the proof of Lemma 2.3 we know that $f_x < 0$ for $x \in [0, \eta)$, since then $u > a$. This implies that the last term in (3.10) is positive. On the interval $(\eta, 1)$ we know from the equation (3.1) that $u''(x) < 0$. Also $f_u < 0$ on $[\eta, 1]$, so that by (2.45) $w'' > 0$ on that interval. This together with $w(1) = 0$ and $w > 0$ implies that

$$w'(x) < 0, \qquad \text{when } x \in [\eta, 1]. \tag{3.11}$$

However, using (3.8),

$$-\int_0^\eta u'w' \, dx = \int_\eta^1 u'w' \, dx > 0;$$

$$\int_0^\eta wu'' \, dx = -\int_\eta^1 wu'' \, dx > 0,$$

i.e. the third and fourth terms in (3.10) are positive. The first term in (3.10) is equal to $\eta u'(\eta)w'(\eta)$, and it is also positive in view of (3.11). We conclude that the left-hand side of (3.10) is positive. The resulting contradiction proves that $\mathfrak{F}_\lambda \notin R(\mathfrak{F}_u)$.

Applying the Crandall-Rabinowitz theorem, we conclude that $(\lambda_0, u(x, \lambda_0))$ is a bifurcation point, near which solutions of (3.1) form a curve $(\lambda_0 + \tau(s), u(x, \lambda_0) + sw + z(s))$, and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$. It follows that for (λ, u) close to $(\lambda_0, u(x, \lambda_0))$ and $\lambda > \lambda_0$, we have two solutions $u^-(x, \lambda) < u^+(x, \lambda)$, and that $u^+(x, \lambda)$ is strictly increasing in λ, while $u^-(x, \lambda)$ *is strictly decreasing in* λ. *This implies that* $u^+(x, \lambda) > u^-(x, \lambda)$ *for* λ *close to* λ_0, *and* by Lemma 1.1 the same inequality holds for all λ (since solutions cannot touch; they also cannot coincide and then go through each other, since this type of bifurcation is ruled out by the Crandall-Rabinowitz theorem).

We show next that

$$u_\lambda(0, \lambda) \neq 0 \qquad \text{for all } \lambda > \lambda_0, \tag{3.12}$$

which will imply that $u^+(0, \lambda)$ is increasing and $u^-(0, \lambda)$ is decreasing for all $\lambda > \lambda_0$ (since $u_\lambda^+(0, \lambda) > 0$ and $u_\lambda^-(0, \lambda) < 0$ for λ near λ_0). Assume that for some λ_1 we have on the contrary $u_\lambda(0, \lambda_1) = 0$. Differentiate (3.1) in λ

$$u_\lambda'' + \lambda f_u u_\lambda + f = 0 \tag{3.13}$$

(where $f = (u - a)(u - b(x))(c(x) - u)$). We see from (3.13) that $u_\lambda''(0, \lambda_1) < 0$ and, hence, $u_\lambda(x) \equiv u_\lambda(x, \lambda_1)$ is negative in some neighborhood of $x = 0$. Let $0 < \xi \le 1$ be the smallest positive root of $u_\lambda(x)$. Let $u(\eta, \lambda_1) = a$. We distinguish two cases.

Case (i) $\xi \le \eta$. From the proof of Lemma 2.3 we know that $f_x(x, u) < 0$ on $(0, \xi)$. Multiply the equation (3.13) by u', and subtract the equation (3.9) multiplied by u_λ, then integrate over $(0, \xi)$. Obtain

$$(u'u_\lambda' - u_\lambda u'')|_0^\xi + \int_0^\xi (u'f - \lambda u_\lambda f_x)\, dx = 0. \tag{3.14}$$

The integral term in (3.14) is negative (to see that $\int_0^\xi u'f\, dx < 0$, one multiplies (3.1) by u', then integrates over $(0, \xi)$). The first term in (3.14) is equal to $u'(\xi)u_\lambda'(\xi) \le 0$, a contradiction.

Case (ii) $\eta < \xi$. Subcase (a) $u_\lambda'(\eta) \ge 0$. Notice that we have $u''(\eta) = 0$. We obtain the same contradiction as above by integrating over $(0, \eta)$ instead of $(0, \xi)$.

Subcase (b) $u_\lambda'(\eta) < 0$. We know from the proof of Lemma 2.4 that $f_u(x, u) < 0$ for $x \in (\eta, 1)$. On (η, ξ) the function $u_\lambda(x)$ must have a point of minimum, say at x_0. However, at x_0 all terms in (3.13) will be positive, a contradiction.

If there are further bifurcations on the upper curve, the Crandall-Rabinowitz theorem applies as before. Since solutions of (3.1) are bounded it easily follows (see [1, 2]) that over any finite interval of λs there is only a finite number of turns.

Recall that $u^+(0, \lambda)$ *is increasing in* λ. *By Lemma 3.1, it follows that on some set containing* the origin, $u(x, \lambda)$ tends to $c(x)$ as $\lambda \to \infty$.

Next we show that the upper branches of all upper curves tend to the same limit. Assume that $u(x, \lambda) \to c(x)$ on $(-p, p)$, $v(x, \lambda) \to c(x)$ on $(-q, q)$ as $\lambda \to \infty$, and $q < p$. We shall show that this leads to a contradiction, implying that $p = q$.

Let α, β be such that $v(\alpha) = u(\beta) = a$, and let $u_0 = u(0, \lambda)$, $v_0 = v(0, \lambda)$, with $u_0 > v_0$. Multiply the equation (3.1) (with $f(x, u) \equiv (u - a)(u - b(x))(c(x) - u))$ by u', and integrate from 0 to β,

$$\tfrac{1}{2}u'^2(\beta) + \lambda \int_{u_0}^{a} f(x_2(u), u)\,du = 0,$$

with $x = x_2(u)$ the inverse function of $u(x)$. Multiplying the same equation for $v(x)$ by v', and integrating from 0 to α,

$$\tfrac{1}{2}v'^2(\alpha) + \lambda \int_{v_0}^{a} f(x_1(v), v)\,dv = 0,$$

with $x_1(v)$ the inverse function of $v(x)$, and $x_2(u) > x_1(u)$ for $u \in (0, v_0)$. Subtracting,

$$\tfrac{1}{2}(u'^2(\beta) - v'^2(\alpha)) + \lambda \int_{a}^{v_0} [f(x_1(u), u) - f(x_2(u), u)]\,du - \lambda \int_{v_0}^{u_0} f(x_2(u), u)\,du = 0.$$

The first term in the above formula is positive by Lemma 2.4. The second integral is also positive, and in fact is bounded below by a positive constant, since for λ large we have $x_2(u) \simeq p$ and $x_1(u) \simeq q$ over some u-interval. The third integral goes to 0, since $u_0, v_0 \to c(0)$ as $\lambda \to \infty$, and we have a contradiction.

Finally, we rule out the case of isolated solution curves. Assume that such a curve exists and has two turning points, the case of more than two turning points being similar. Near the left turning point there is an upper branch that is increasing in λ at $x = 0$ for all λ. The same branch would have to be decreasing in λ at $x = 0$ when approaching the right turning point, which is impossible.

More detailed information on the asymptotic behavior of solutions on the upper curve can be obtained if one replaces the condition (3.4) by a more stringent one:

$$F(x, a) < F(x, c(x)) \qquad \text{for all } x \in (-1, 1). \tag{3.15}$$

Let $r_1(x) < r_2(x)$ denote the roots of $f_u(x, u)$. We assume that

$$r_2(x) < c(1) \qquad \text{for all } x \in (-1, 1). \tag{3.16}$$

(This condition is trivially satisfied for constant b and c.)

THEOREM 3.3. Assume all conditions of the Theorem 3.2 hold with the condition (3.4) replaced by (3.15) and assume additionally (3.16). Then all of the conclusions of Theorem 3.2 hold and, in addition, the upper curve is unique and it consists for λ sufficiently large of two branches, referred to as an upper and lower branch, $u^+(x, \lambda) > u^-(x, \lambda)$ for all x, and $\lim_{\lambda \to +\infty} u^+(x, \lambda) = c(x)$ for all $x \in (-1, 1)$, $\lim_{\lambda \to +\infty} u^-(x, \lambda) = a$ for all $x \in (-1, 1)\backslash\{0\}$, and $u^-(0, \lambda) > b(0)$ for all λ (i.e. the lower branch approaches a spike-layer). In particular, for sufficiently large λ the problem (3.1) has exactly three solutions.

Proof. We begin by showing that for each upper branch $\lim_{x \to \infty} u^+(x, \lambda) = c(x)$. From the proof of Theorem 3.2, we know that on the lower curve, denoted by $u_l(x, \lambda)$, we have $u_l(x, \lambda) \to a$ as $\lambda \to \infty$ and, hence, for large λ,

$$J(u_l(x, \lambda)) \geq -\lambda \int_{-1}^{1} F(x, u_l)\,dx \simeq -\lambda \int_{-1}^{1} F(x, a)\,dx, \tag{3.17}$$

where $J(u)$ is the energy functional defined in (3.5). We know that for large λ each upper curve consists of two branches $u^+(x, \lambda) > u^-(x, \lambda)$. By the previous theorem, $u^+(x, \lambda)$ approaches $c(x)$ on $(-p, p)$. If $p < 1$, then the case (iii) of Lemma 3.1 holds and

$$J(u) \simeq \tfrac{1}{2} \int_{-1}^{1} u'^2(x, \lambda)\,\mathrm{d}x - \lambda \int_{-p}^{p} F(x, c(x))\,\mathrm{d}x - \lambda \int_{(-1,1)\setminus(-p,p)} F(x, a)\,\mathrm{d}x. \tag{3.18}$$

From the equation (3.1),

$$\int_{-1}^{1} u'^2\,\mathrm{d}x = \lambda \int_{-1}^{1} uf(x, u)\,\mathrm{d}x. \tag{3.19}$$

Using (3.19) and Lemma 3.1 it is easy to see that

$$\int_{-1}^{1} u'^2\,\mathrm{d}x = \lambda\delta(\lambda), \qquad \text{with } \delta(\lambda) \to 0 \quad \text{as } \lambda \to \infty.$$

If now $u_\varepsilon(x)$ is the function defined in the proof of the Theorem 3.2, then one sees from (3.17)–(3.19) and (3.15) that for ε small and λ large $J(u_\varepsilon)$ is smaller than each of $J(u_l)$, $J(u^-)$ and $J(u^+)$. Since $J(u)$ is bounded from below, its minimum would be a solution of (3.1) which is qualitatively different from the three possible types of solutions for λ large, that are described by Theorem 3.2. The resulting contradiction proves that $p = 1$.

We show next that for λ large each branch $u^+(x, \lambda)$, approaching $c(x)$, must become stable. Indeed, if $u = u^+(x, \lambda)$ were not stable, we could find a constant $\mu \geq 0$ and $w(x) > 0$, so that

$$w'' + f_u(x, u)w = \mu w \quad \text{for } x \in (-1, 1), \qquad w(-1) = w(1) = 0. \tag{3.20}$$

We may assume that $\int_0^1 w^2\,\mathrm{d}x = 1$. Define η to be the smallest number in $(0, 1)$, where $f_u(x, u(x)) = 0$. Clearly, $f_u < 0$ on $(0, \eta)$.

We claim that

$$|u'(x)| \geq c_0\sqrt{\lambda}, \tag{3.21}$$

for some c_0 when λ is large, for all $x \in (\eta, 1]$. It suffices to prove (3.21) for $x \in (\eta, \alpha]$ (i.e. when $u > a$), since $u'' < 0$ on $(\alpha, 1)$. Define $\theta \in (0, 1)$ by $u(\theta) = c(1)$. By (3.16) we know that $\theta < \eta$. From the equation (3.1) we obtain, using (3.15)

$$\frac{u'^2(x)}{2} = \frac{u'^2(\theta)}{2} - \int_\theta^x \lambda f(x, u)u'\,\mathrm{d}x \geq -\lambda \int_\theta^x f(1, u)\,u'\,\mathrm{d}x = \lambda \int_{u(x)}^{c(1)} f(1, u)\,\mathrm{d}u \geq c_0\lambda,$$

and (3.21) follows.

Since $u(x, \lambda) \to c(x)$, we can find a constant A independent of λ and $\xi = \xi(\lambda)$ near $x = 0$ (say in $(0, \frac{1}{4})$), such that $|u''(\xi)| \leq A$. Define α by $u(\alpha) = a$. From the equations (3.20) and (3.9) we derive

$$-u'(\alpha)w'(\alpha) + u'(\xi)w'(\xi) - w(\xi)u''(\xi) + \lambda \int_\xi^\alpha f_x w\,\mathrm{d}x + \mu \int_\xi^\alpha wu'\,\mathrm{d}x = 0. \tag{3.22}$$

Since $w''(x) > 0$ on $(0, \eta)$, it follows that $w'(\xi) > 0$ and that $w(\xi)$ is bounded (because $\int_0^1 w^2\,\mathrm{d}x = 1$). We see then that the second, fourth and fifth terms in (3.22) are negative, while the third term in (3.22) is bounded. The first term in (3.22) is negative, and we show next that it is large in absolute value (as $\lambda \to \infty$), which leads to a contradiction in (3.22). Indeed, assuming otherwise would imply in view of (3.21) that $|w'(\alpha)| = 0(1/\sqrt{\lambda})$. We claim that $w(\alpha)$ is also small. By (3.21) we conclude that $1 - \alpha = 0(1/\sqrt{\lambda})$ and $\alpha - \eta = 0(1/\sqrt{\lambda})$. By Lemma 2.4, $f_u < 0$ on $(\alpha, 1)$ and, hence, $w'' > 0$ there, which means that $|w'(\alpha)| = \max_{[\alpha,1]}|w'(x)|$ and, therefore, $w(\alpha) = 0(1/\lambda)$. Since $w(x)$ is convex on $(0, \eta)$, it must take its maximum on $(\eta, 1)$. The maximum value must be at least 1, since $\int_0^1 w^2\,dx = 1$. On (η, α) we set $t = \alpha - x$, and estimate from (3.20)

$$w'' \leq c\lambda w, \qquad w(0) = 0\left(\frac{1}{\lambda}\right), \qquad w'(0) = 0\left(\frac{1}{\sqrt{\lambda}}\right), \qquad 0 < t < \frac{c_1}{\sqrt{\lambda}},$$

with positive constants c and c_1. Integrating

$$w(t) \leq c\lambda \int_0^t (t - s)w(s)\,ds + w'(0)t + w(0) \leq c_2\left(\sqrt{\lambda}\int_0^t w(s)\,ds + \frac{1}{\lambda}\right).$$

Applying Gronwall's inequality, we conclude that $w(t) = 0(1/\lambda)$ on $(\eta, 1)$. This is a contradiction, which in turn implies a contradiction in (3.22), proving stability of $u^+(x, \lambda)$ for large λ.

By a similar (easier) argument one shows that any lower branch tending to zero for $x \neq 0$ is unstable. Uniqueness of the upper solution curve then follows by a standard argument, see e.g. [7, p. 68].

Acknowledgements—Part of the work of the first-named author (P.K.) was done while he visited the Department of Mathematics and Computer Science at the University of Miami. He wishes to thank the department, and particularly A. C. Lazer, for their hospitality.

REFERENCES

1. KORMAN P. & OUYANG T., Exact multiplicity results for two classes of boundary value problems, *Diff. Integral Eqns* **6**, 1507–1517 (1993).
2. KORMAN P. & OUYANG T., Multiplicity results for two classes of boundary-value problems, *SIAM J. math. Analysis* **26**, 180–189 (1995).
3. CRANDALL M. G. & RABINOWITZ P. H., Bifurcation, perturbation of simple eigenvalues and linearized stability, *Archs ration. Mech. Analysis* **52**, 161–180 (1973).
4. SMOLLER J. & WASSERMAN A., Global bifurcation of steady-state solutions, *J. diff. Eqns* **39**, 269–290 (1981).
5. WANG S.-H. & KAZARINOFF N. D., Bifurcation and stability of positive solutions of two-point boundary value problem, *J. Aust. math. Soc.* **52**, 334–342 (1992).
6. WANG S.-H. & KAZARINOFF N. D., Bifurcation and steady-state solutions of a scalar reaction-diffusion equation in one space variable, *J. Aust. math. Soc.* **52**, 343–355 (1992).
7. GARDNER R. & PELETIER L. A., The set of positive solutions of semilinear equations in large balls, *Proc. R. Soc. Edinb.* **104A**, 53–72 (1986).