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Curves of sign-changing solutions for semilinear equations

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1. Introduction

We study positive and sign-changing solution curves, both radially symmetric and non-symmetric, for the semilinear Dirichlet problem

$$\Delta u + \lambda f(u) = 0 \ \text{ for } |x| < R, \quad u = 0 \ \text{ for } |x| = R, \tag{1.1}$$

on a ball in R^n, depending on a positive parameter λ. Here the space dimension $n \geqslant 1$ is arbitrary, although for the case $n = 1$ we present more detailed results. Recall that in view of the classical results of Gidas et al. [9] any positive solution of (1.1) is radially symmetric, i.e. $u = u(r)$, where $r = |x|$, and hence our problem (1.1) takes the ODE form (2.2).

Our first result in Section 2 deals with the values of any radial solution of (1.1) (positive or not) for large values of the parameter $\lambda > 0$. We show that for large λ the values of any solution must accumulate at stable roots of $f(u)$. This result can be used in particular to prove non-existence of positive (and sign-changing) solutions for large λ, in case nonlinearity $f(u)$ has no stable roots. Our result, Theorem 2.1, gives a considerable extension of the main result of Brown et al. [3], as well as a generalization of Theorem 1 in Ramaswamy and Srikanth [23].

In Section 3 we consider a one-dimensional version of problem (1.1)

$$u'' + \lambda f(u) = 0 \quad \text{for } x \in (0,1), \ \ u(0) = u(1) = 0. \tag{1.2}$$

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We assume that $f(u)$ is a generalization of $f(u)=u|u|^p-1$, with real $p>0$, see conditions (3.2)–(3.4). We show that problem (1.2) has infinitely many points of pitchfork bifurcation, occurring at the following sequence of parameter values $\lambda_k = 2k^2(\int_0^{\theta} \mathrm{d}u/\sqrt{-F(u)})^2$, for $k=1,2\ldots$, which correspond to non-negative solutions u_k with k interior zeros. The pitchfork consists of a curve of symmetric solutions passing through (λ_k,u_k), and a curve of symmetry-breaking solutions, which "opens up" either forward or backward in λ. Away from the pitchfork point (λ_k,u_k) all four branches continue globally, admitting only simple turns. Since our $f(u)$ has no stable roots, it follows by the results of Section 2 that all four solution branches will have to turn around eventually, and tend to infinity as $\lambda\to 0$. (I.e. we can find $\mu_k>\lambda_k$, so that all four branches do not continue in the region $\lambda>\mu_k$.) By contrast, for arbitrary large λ we can still find solutions of (1.2), but with more and more nodes. We thus obtain a kind of nonlinear spectral theory: for large λ only solutions with a large number of roots are possible.

In Section 4 we study symmetry-breaking off a branch of positive solutions of the Dirichlet problem (1.1). As we mentioned above, positive solutions of (1.1) are necessarily radially symmetric. Assume we have a branch of positive solutions of (1.1), which continues for all λ, while it is possible to prove (using for example the result from Section 2) that for large λ problem (1.1) has no positive solutions. This means that our solution branch must cease being positive at some $\bar{\lambda}$. Since the results of Gidas et al. [9] imply also that any positive solution satisfies $u'(r)<0$, there is only one way a branch may loose positivity: to develop a zero slope at $r=1$, i.e. $u'(1,\bar{\lambda})=0$. In view of strong maximum principle this may only happen under the assumption

$$f(0)<0. \tag{1.3}$$

By analogy with the one-dimensional problem from Section 3, it is natural to expect a pitchfork-like bifurcation and symmetry breaking at $\bar{\lambda}$. To prove that bifurcation actually happens, one needs to verify a certain "transversality" condition. We prove this condition, assuming that the branch "loses its positivity forward", i.e. it arrives at the bifurcation point with λ increasing. Symmetry-breaking off a positive branch was studied previously by Smoller and Wasserman, who in [25] present a verification of transversality condition, which they attribute to C. Pospeich. It appears that following [25] symmetry-breaking off a positive branch might have been perceived as well-understood. However the proof of transversality condition in [25] uses implicitly the same assumption that positivity is lost forward. It can be found in condition (b) of [25, p. 282]. That positivity is lost forward is a natural thing to expect, but proving it turns out to be not easy. We show that this will follow from the following disconjugacy condition: any non-trivial solution of the linearized problem

$$\Delta w+\lambda f'(u)w=0 \text{ for } |x|<R, \quad w=0 \text{ for } |x|=R \tag{1.4}$$

can vanish at most once on $(0,R)$. This is the same kind of condition which was recently seen to be important for questions of uniqueness and exact multiplicity for (1.1). This condition will hold if say $f''(u)<0$. In Section 5 we give such a disconjugacy result for a class of superlinear nonlinearities $f(u)$.

2. Shape of solution for large λ

We consider positive solutions of

$$\Delta u + \lambda f(u) = 0 \ \text{ for } |x| < R, \quad u = 0 \ \text{ for } |x| = R, \tag{2.1}$$

i.e. a Dirichlet problem on a ball in R^n, depending on a positive parameter λ. In view of the classical results of Gidas et al. [9] any positive solution of (2.1) is radial, i.e. $u = u(r)$, where $r = |x|$, and problem (2.1) takes the form

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u) = 0 \ \ r \in (0,R), \ \ u'(0) = u(R) = 0. \tag{2.2}$$

Moreover, it was proved in [9] that $u'(r) < 0$ for $r \in (0,R)$.

We assume that the nonlinearity $f(u) \in C^2(R_+)$ has no degenerate roots, i.e. $f'(\bar{u}) \neq 0$, provided $f(\bar{u}) = 0$. We then refer to a root $\bar{u}$ of $f(u)$ as stable if $f'(\bar{u}) < 0$, and unstable if $f'(\bar{u}) > 0$. Our next result shows that for large λ any positive solution has to accumulate at stable roots of $f(u)$.

Theorem 2.1. *Assume that*

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty \tag{2.3}$$

and assume that a positive solution of (2.2) *exists for all* $\lambda > \lambda_0$. *Then the interval* $(0,R)$ *can be decomposed into a union of open intervals, whose total length is* $=R$, *so that on each such subinterval* $u(r,\lambda)$ *tends to a stable root of* $f(u)$ *as* $\lambda \to \infty$.

Proof. We claim that solution cannot tend to infinity over a fixed subinterval as $\lambda \to \infty$. Indeed, assume on the contrary that $u \to \infty$ for all $r \in [0,r_1)$, for some fixed $r_1 > 0$. Then for any constant $M > 0$ we can find a λ_0, so that for all $\lambda \geqslant \lambda_0$

$$\lambda \frac{f(u)}{u} \geqslant M \quad \text{for all } r \in [0, r_1). \tag{2.4}$$

We now write our Eq. (2.2) in the form

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda\frac{f(u)}{u}u = 0$$

and compare it with

$$v''(r) + \frac{n-1}{r}v'(r) + Mu = 0. \tag{2.5}$$

Changing variables $v = zr^{(2-n)/2}$, we reduce (2.5) to a Bessel's equation, and so it has a solution $v = r^{2-n/2}J_{(n-2)/2}(\sqrt{M}r)$, which has a zero on $(0,r_1)$, provided M is large. By Sturm's comparison theorem the same is true for $u(r)$, a contradiction, proving that the length of the interval on which solution gets large is decreasing as $\lambda \to \infty$.

It follows that for any $\delta > 0$ we can find a constant $M_0 > 0$ so that

$$u(r) < M_0 \text{ on } (\delta, 1) \quad \text{for all } \lambda \geqslant \lambda_0. \tag{2.6}$$

We claim that on any subinterval of $(\delta, 1)$ solution $u(r,\lambda)$ must tend to a root of $f(u)$. Indeed, if $u(r,\lambda)$ failed to tend to a root of $f(u)$ on some subinterval $I \in (\delta, 1)$, then from Eq. (2.2) we see that $r^{n-1}u'$ would have to become large over I. This would imply that $u(r)$ has a large variation on $I \in (\delta, 1)$, contradicting the fact that the variation of $u(r)$ on $(\delta, 1)$ is less than M_0 by (2.6). Hence the values of $u(r, \lambda)$ have to accumulate at roots of $f(u)$ as $\lambda \to \infty$.

Finally, we show that solution has to accumulate at stable roots of $f(u)$. Assume on the contrary that there is an interval $J \equiv (r_1, r_2) \in (\delta, 1)$, so that

$$u(r, \lambda) \to \bar{u} \text{ over } J \quad \text{as } \lambda \to \infty \tag{2.7}$$

and $\bar{u}$ is an unstable root of $f(u)$, i.e. $f(u)$ is negative to the left of $\bar{u}$, and positive to the right. Denote by $\xi = \xi(\lambda)$ the point where $u(\xi) = \bar{u}$. We claim that there is a $\bar{r} \in (r_1, r_2)$, so that ξ will be excluded from $(r_1, \bar{r})$ as λ gets large. Indeed, we see from Eq. (2.1) that $u(r)$ is convex just below the value of $\bar{u}$, and it remains convex for increasing r at least until $u(r)$ will drop below the root of $f(u)$ which precedes $\bar{u}$ (so that $f(u)$ can change to being positive). But (2.7) implies that $u(r)$ would have to change to being concave at values just a little below $\bar{u}$, a contradiction. (A convex and decreasing function cannot tend to a constant over a subinterval.)

Let again ξ be such that $u(\xi) = \bar{u}$. By above $\xi \in (\bar{r}, r_2)$ for large λ. We claim that $|u'(\xi)|$ is large for large λ. Indeed, denote by $\underline{u}$ the root of $f(u)$ preceding $\bar{u}$, define η by $u(\eta) = \underline{u}$, and $0 < c_0 \equiv -\int_{\underline{u}}^{\bar{u}} f(u)\,\mathrm{d}u$. Multiplying Eq. (2.2) by $r^{n-1}u'$, and integrating over (ξ, η), we get

$$\begin{aligned}\tfrac{1}{2}\xi^2 u'^2(\xi) &= \tfrac{1}{2}\eta^2 u'^2(\eta) + \lambda \int_\xi^\eta r^{2(n-1)} f(u) u' \,\mathrm{d}r \\ &\geqslant \lambda \xi^{2(n-1)} \int_\xi^\eta f(u) u' \,\mathrm{d}r = \lambda \xi^{2(n-1)} c_0,\end{aligned}$$

which implies that $u(\xi) \geqslant c_1\sqrt{\lambda}$ for large λ, with some $c_1 > 0$, proving the claim.

When we decrease r from ξ toward the interval $(r_1, \bar{r})$, we need $u''(r)$ to be large in absolute value and negative in order to decrease $|u'(r)|$. ($|u'(r)|$ must be small over the interval $(r_1, \bar{r})$ for large λ, since $u(r)$ tends to a constant over that interval.) From Eq. (2.2), $u'' \geqslant -\lambda f(u)$, i.e. $|u''| = -u'' \leqslant \lambda f(u)$, so that $\lambda f(u)$ must get large somewhere on $(\bar{r}, \xi)$. But as we decrease r we increase $u(r)$, so the quantity $\lambda f(u)$ once it gets large can only further increase. Also, the assumption (2.3) may be dropped, provided there is some $u - 0 > 0$ so that $f(u) < 0$ for $u > u - 0$. It follows from Eq. (2.2) that $(r^{n-1}u')'$ is large over $(r_1, \bar{r})$. This implies a large variation of $u(r)$ over this interval, a contradiction, finishing the proof. $\square$

Example. Assume that $f(0) < 0$, condition (2.3) holds, and the function $f(u)$ has exactly one positive root. Then this root is unstable, and so there exists λ_1 so that for $\lambda > \lambda_1$ problem (2.2) has no positive solutions. This extends considerably the main result in [3].

Remark. Examining the proof, we see that a similar result holds for sign-changing radial solutions of (2.1). Indeed, replacing condition (2.3) by $\lim_{u \to \pm\infty} f(u)/u = \infty$, we conclude that for large λ any radial solution of (2.1) has to accumulate near the stable roots of $f(u)$.

3. Infinitely many solution curves

We study positive, negative and sign-changing solutions of a family of two-point problems

$$u'' + \lambda f(u) = 0 \quad \text{for } x \in (0,1), \; u(0) = u(1) = 0, \tag{3.1}$$

depending on a positive parameter λ. We assume that the nonlinearity $f(u) \in C^2(R)$ satisfies the following conditions:

$$f(0) < 0, \tag{3.2}$$

$$\lim_{u \to \pm\infty} \frac{f(u)}{u} = \infty. \tag{3.3}$$

We assume also that $f(u)$ changes sign exactly once, i.e. there exists an $\alpha > 0$ such that

$$f(u) > 0 \; \text{ for } u > \alpha, \quad f(u) < 0 \; \text{ for } -\infty < u < \alpha. \tag{3.4}$$

We shall show that infinitely many pitchfork bifurcations occur for problem (3.1), and that for small $\lambda > 0$ infinitely many solution are present. But first we fix the notation, and present some background results.

In the following we shall use alternatively the notation $u(x,\lambda) \equiv u(x)$, and $u_x(x,\lambda) = u'(x,\lambda)$ to denote solution branches and derivatives.

Notice that any solution of (3.1) is symmetric between any two of its roots, i.e. if $u(a) = u(b) = 0$ and $u(x)$ is positive (negative) over the subinterval $(a,b) \in (0,1)$ then $u'(a+b)/2 = 0$, and $u(x) = u(a+b-x)$, $u'(x) < 0$ for all $x \in (a+b)/2, b$. This is because any solution is symmetric with respect to any stationary point, by uniqueness for initial value problems. Notice that in particular no solution can have local minimums (maximums) in the region where it is positive (negative).

Lemma 3.1. *The nodal structure of any solution branch of* (3.1) *is preserved for all* λ, *unless* $u_x(0,\lambda) = 0$ *for some* λ. *More precisely, if* $u_x(0,\lambda) \neq 0$ *for all* $\lambda \in [\lambda_1, \lambda_2]$, *then* $u(x,\lambda_1)$ *and* $u(x,\lambda_2)$ *have the same number of zeros, and the same order of regions where solution is positive or negative.*

Proof. By the above remarks the only way a new zero point can be added, is for solution curve to develop a zero slope at an existing zero, and then a new zero appearing nearby when the parameter λ is varied. However, by symmetry having zero slope at any root would imply $u'(0,\lambda) = 0$, a contradiction. $\square$

We recall several lemmas. The following lemma was proved in Korman [13].

Lemma 3.2. *Consider problem* (3.1) *with any* $f(u) \in C^2(R)$. *A positive solution branch* $u(x,\lambda) > 0$ *can cease being positive only for increasing* λ.

The following lemma is a variation of the corresponding result in Korman et al. [15].

Lemma 3.3. *Let* $u(x,\lambda_0)\in C^3(0,1)\cap C[0,1]$ *be a singular solution of* (3.1), *i.e. the problem*

$$w''+\lambda_0 f'(u)w=0 \quad \text{for } x\in(0,1),\ \ w(0)=w(1)=0 \tag{3.5}$$

has a non-trivial solution. Assume that $u'(1,\lambda_0)\neq 0$ (*or* $u'(0,\lambda_0)\neq 0$). *Then*

$$\int_0^1 f(u)w\,\mathrm{d}x\neq 0. \tag{3.6}$$

Proof. Differentiating (3.1), we get

$$u_x''+\lambda f'(u)u_x=0. \tag{3.7}$$

From Eqs. (3.4) and (3.7) at $\lambda=\lambda_0$, we obtain

$$u'w'-u''w=\text{constant}=u'(1)w'(1). \tag{3.8}$$

Since we may assume that $w'(1)\neq 0$ (otherwise $w(x)$ is trivial), we see that the constant in (3.8) is non-zero. Integrating (3.8), we then conclude (3.6).

The following lemma addresses the complementary situation when $u'(1,\lambda_0)=0$. It gives explicitly the solution of the equation for u_λ

$$u_\lambda''+\lambda f'(u)u_\lambda+f(u)=0 \quad \text{for } x\in(0,1),\ \ u_\lambda(0)=u_\lambda(1)=0 \tag{3.9}$$

at $\lambda=\lambda_0$.

Lemma 3.4. *Assume that* $u(x,\lambda)\in C^3(0,1)\cap C[0,1]$ *is a curve of solutions of* (3.1), *which are symmetric with respect to* $x=\frac{1}{2}$ *near* $\lambda=\lambda_0$, *and such that* $u_x(1,\lambda_0)=0$. *Then problem* (3.9) *is solvable at* $\lambda=\lambda_0$, *and in fact*

$$u_\lambda(x,\lambda_0)=\frac{1}{2\lambda_0}\left(x-\frac{1}{2}\right)u_x(x,\lambda_0) \quad \text{for all } x\in(0,1). \tag{3.10}$$

Proof. One easily checks that the function $w(x)\equiv 2\lambda_0 u_\lambda-(x-\frac{1}{2})u_x$ satisfies

$$w''+\lambda_0 f'(u)w=0 \text{ on } (0,1), \quad w(0)=w(1)=0.$$

This function is even with respect to $x=\frac{1}{2}$, while the null-space of the above equation is spanned by the odd with respect to $x=\frac{1}{2}$ function u_x. This implies that $w\equiv 0$, and the proof follows. □

We define the point $\theta>0$ by the relation $\int_0^\theta f(u)\,\mathrm{d}u=0$. We also use a standard notation $F(u)=\int_0^u f(u)\,\mathrm{d}u$. It turns out that pitchfork bifurcation occurs at the following sequence of parameter values:

$$\lambda_k=2k^2\left(\int_0^\theta \frac{\mathrm{d}u}{\sqrt{-F(u)}}\right)^2 \quad \text{for } k=1,2\ldots. \tag{3.11}$$

Lemma 3.5. *Let λ_k be defined by* (3.11). *At $\lambda = \lambda_1$ problem* (3.1) *has a positive solution $u_1(x)$, with $u_1'(0) = 0$ and $u_1(\frac{1}{2}) = \theta$. At $\lambda = \lambda_k$, $k \geqslant 2$ problem* (3.1) *has a non-negative solution $u_k(x)$ with $k-1$ interior zeros, and $u_k'(0) = 0$, $u(1/2k) = \theta$.*

Proof. We begin by establishing existence of solutions $u_k(x)$. Consider the initial value problem (for $x \geqslant 0$)

$$u''(x) + f(u(x)) = 0, \quad u(0) = \theta, \; u'(0) = 0. \tag{3.12}$$

Since $f(\theta) > 0$, it follows that $u(x)$ is concave, and hence decreasing for small $x > 0$. Multiplying Eq. (3.12) by u' and integrating, we conclude in view of our initial conditions that

$$\tfrac{1}{2}u'^2 + F(u) = 0. \tag{3.13}$$

Solution of (3.12) cannot decrease and be positive indefinitely. Indeed, if that was the case, $u(x)$ would have to converge to a constant $0 < c_0 < \theta$ as $x \to \infty$. We obtain a contradiction from (3.13), since for large x $u'(x)$ must be small, while $F(u)$ term is negative and bounded away from zero.

It follows that either solution becomes zero at some ξ, or it stops decreasing at some η, i.e $u'(\eta) = 0$. In the first case it follows from (3.13) that $u'(\xi) = 0$, and in the second case it follows from the same equation that $u(\eta) = 0$. In either case we obtain a positive solution, with zero slope at its first root. By scaling the equation we can make sure that this first root happens at $x = \frac{1}{2}$. This will introduce a factor into the equation, and hence for a proper λ we obtain solution $u_1(x)$. Similarly, by scaling the first root to appear at $x = \frac{1}{4}$, we obtain the solution $u_2(x)$ at some other λ. Finally, by a direct integration of (3.1) we get formula (3.11) for λ_k. □

Our next lemma shows that solution curves stay bounded when λ ranges over a bounded interval, not including $\lambda = 0$.

Lemma 3.6. *Assume that $f(u)$ satisfies conditions* (3.1)–(3.3), *and $0 < \lambda_1 \leqslant \lambda \leqslant \lambda_2 < \infty$, with some constants λ_1, λ_2. Then for any solution branch $u(x, \lambda)$ of* (3.1) *there exists a constant c, such that*

$$|u(x, \lambda)| < c \quad \textit{for all } \lambda \in [\lambda_1, \lambda_2]. \tag{3.14}$$

Proof. Assume that on the contrary solution $u(x, \lambda)$ becomes unbounded when $\lambda \to \lambda_0 \in [\lambda_1, \lambda_2]$. Assume that $u(x)$ is positive on some subinterval $I \equiv (a, b) \in (0, 1)$, and that $u(a) = u(b) = 0$. As we change λ the roots change, so we can think of roots as continuous functions $a = a(\lambda)$ and $b = b(\lambda)$, and consider a variable interval $I = I_\lambda = (a(\lambda), b(\lambda))$. Writing our equation in the form

$$u'' + \lambda \frac{f(u)}{u} u = 0,$$

then using Sturm's comparison theorem and our condition (3.3), we see that the length of the interval on which solution becomes large must be decreasing. i.e. given any

$\varepsilon > 0$ small we can find a constant $M > 0$ and an interval $J \in I$ of length less than ε, and centered around the midpoint of I, so that

$$u(x,\lambda) < M \quad \text{for } x \in I \setminus J \text{ and all } \lambda \in [\lambda_1, \lambda_2]. \tag{3.15}$$

Multiplying Eq. (3.1) by u' and integrating from a to $(a+b)/2$, and denoting $u_0 = u(a+b)/2$, we have

$$u'^2(a) = \lambda \int_0^{u_0} f(u)\,\mathrm{d}u. \tag{3.16}$$

Since $u_0 \to \infty$ as $\lambda \to \lambda_0$, it follows that $u'(a)$ must get large. Similarly, $u'(x)$ must get large for all $x \in I \setminus J$. We conclude that the interval I must be shrinking, since otherwise we get a contradiction with (3.15) (a bounded increasing function cannot have unbounded derivatives on an interval, whose length is bounded from below).

If $u(x,\lambda)$ is a positive solution, we are done, since we may take $I = (0,1)$, and this interval is not shrinking. Otherwise, we turn our attention to an adjacent to I interval, where the solution is negative, call it I_1. Denoting by $u_1 < 0$ the minimum value of $u(x,\lambda)$ on this interval, we see using (3.16) and a formula similar to (3.16) and involving u_1, that $u_1 \to -\infty$ as $\lambda \to \lambda_0$. Using Sturm's comparison theorem we see that I_1 is also shrinking. Since by Lemma 3.1 the nodal structure of $u(x,\lambda)$ is preserved, and the length of all intervals is decreasing, we obtain a contradiction. $\square$

We are ready to state the main result of this section.

Theorem 3.1. *Consider problem* (3.1) *and assume that conditions* (3.2)–(3.4) *are satisfied. Then in addition to the numbers* λ_k *defined by* (3.11) *there exist a sequence of positive numbers* $\{\mu_k\}$, *with* $\lambda_k < \mu_k$ *for all* k, *so that for all* $0 < \lambda < \lambda_k$ *problem* (3.1) *has at least two symmetric solutions with* $2k$ *interior zeros, and at least two asymmetric solutions with* $2k-1$ *interior zeros, while for* $\lambda > \mu_k$ *only solutions with at least* $2k+1$ *interior zeros may exist.*

Proof. We show similarly to [14] that pitchfork bifurcation occurs at each of the points (λ_k, u_k). Differentiating Eq. (3.1), we see that $u_k'(x)$ is solution of the linearized problem (3.5), and that any solution of (3.5) is a multiple of $u_k'(x)$.

We recast our Eq. (3.1) in the operator form $F(\lambda,u)=0$, where $F: R \times C_0^2(0,1) \to C(0,1)$ is defined as follows:

$$F(\lambda,u) = u'' + \lambda f(u) = 0. \tag{3.17}$$

Define by X the subspace of $C_0^2(0,1)$ which consists of functions that are even with respect to $x = \frac{1}{2}$, and by Y the subspace of functions in $C_0^2(0,1)$ that are odd with respect to $x = \frac{1}{2}$. We now restrict our equation (3.17) to X, i.e., consider F as a map $R \times X \to X$. In fact, we shall show that (3.17) defines a curve in $R \times X$. The point $\lambda = \lambda_k$, $u = u_k$ lies on this curve. The linearized equation at this point

$$F_u(\lambda_k, u_k)w = w'' + \lambda_k f'(u_k)w = 0, \quad w(0) = w(1) = 0$$

has no non-trivial solutions in X, since its solution set is spanned by $u_k'(x) \notin X$. This implies that $F_u(\lambda_k, u_k)$ is an injection of X into X. To see that it is also onto, take an arbitrary $g(x) \in X$, and consider the problem

$$w'' + \lambda_k f'(u_k)w = g(x), \quad w(0) = w(1) = 0. \tag{3.18}$$

Since $\int_0^1 g u_k' \,\mathrm{d}x = 0$, we see that problem (3.18) is solvable. Writing its solution in the form $w = w_e + w_o$ with $w_e \in X$ and $w_o \in Y$, we see that w_e is also a solution of (3.18), and hence $F_u(\lambda_k, u_k)$ is onto (w_o is a constant multiple of u_k'). By the implicit function theorem we have a curve of symmetric solutions passing through (λ_k, u_k).

We claim next that solutions on this curve become negative near $x = 0, 1$ for $\lambda > \lambda_k$. This will follow from the fact that $u_\lambda(x, \lambda_k)$ is negative near $x = 0, 1$. Indeed, by Lemma 3.4

$$u_\lambda'(0, \lambda_k) = \tfrac{1}{4} f(0) < 0 \tag{3.19}$$

and hence the symmetric solution becomes negative near the end-points for increasing λ.

We denote by $U = U(x, \lambda) = U(x)$ the curve of symmetric solutions passing through (λ_k, u_k). Set $u(x) = U(x) + v(x)$. From the previous discussion it follows that

$$U(x, \lambda_k) = u_k(x), \quad U_\lambda(x, \lambda_k) = u_\lambda(x, \lambda_k -). \tag{3.20}$$

We rewrite our Eq. (3.1)

$$G(\lambda, v) \equiv v'' + \lambda[f(U + v) - f(U)] = 0, \quad v(0) = v(1) = 0. \tag{3.21}$$

Problem (3.21) has a trivial solution for all $\lambda > \lambda_k$. We are looking now for non-trivial solutions of (3.21), bifurcating off the point $\lambda = \lambda_k$, $v = 0$. The linearized equation at this point

$$w'' + \lambda_k f'(u_k)w = 0, \quad w(0) = w(1) = 0 \tag{3.22}$$

has a one-dimensional null-space, spanned by u_k'. It is well known by the elliptic theory that the range of $G_v(\lambda_k, 0)$ has codimension one. We shall show that the Crandall–Rabinowitz theorem on bifurcation from simple eigenvalues applies at this point ([6, Theorem 1.7]). In view of the above remarks we only need to verify the crucial "transversality" condition: $G_{\lambda v}(\lambda_k, 0)u_k' \notin R(G_v(\lambda_k, 0))$. Since by (3.20) $G_{\lambda v}(\lambda_k, 0)u_k' = \lambda_k f''(u_k)u_\lambda u_k' + f'(u_k)u_k'$, we need to show, in view of Fredholm alternative, that $\lambda_k \int_0^1 f''(u_k)u_\lambda u_k'^2 \,\mathrm{d}x + \int_0^1 f'(u_k)u_k'^2 \,\mathrm{d}x \neq 0$. This will follow from a more precise inequality,

$$\lambda_k \int_0^1 f''(u_k)u_\lambda u_k'^2 \,\mathrm{d}x + \int_0^1 f'(u_k)u_k'^2 \,\mathrm{d}x > 0. \tag{3.23}$$

We proceed similarly to Ramaswamy [22]. To simplify the notation, we drop the subscript k in the intermediate steps. Integrating by parts,

$$\begin{aligned}\int_0^1 f''(u)u_\lambda u'^2 \,\mathrm{d}x &= \int_0^1 (f'(u))' u_\lambda u' \,\mathrm{d}x \\ &= -\int_0^1 f'(u)u_\lambda' u' \,\mathrm{d}x - \int_0^1 f'(u)u_\lambda u'' \,\mathrm{d}x.\end{aligned} \tag{3.24}$$

We rewrite similarly the first term on the right-hand side in (3.24)

$$-\int_0^1 f'(u)u'_\lambda u'\,\mathrm{d}x = \int_0^1 f(u)u''_\lambda\,\mathrm{d}x + 2f(0)u'_\lambda(0)$$

and the second term on the left-hand side in (3.23)

$$\int_0^1 f'(u)u'^2\,\mathrm{d}x \equiv \int_0^1 (f(u))'u'\,\mathrm{d}x = -\int_0^1 fu''\,\mathrm{d}x = \lambda\int_0^1 f^2\,\mathrm{d}x.$$

Plugging these formulas into (3.23), and using the equations satisfied by u and u_λ, we get

$$\lambda_k\int_0^1 f''(u_k)u_\lambda u_k'^2\,\mathrm{d}x + \int_0^1 f'(u_k)u_k'^2\,\mathrm{d}x = 2\lambda_k f(0)u'_\lambda(0).$$

In view of (3.19) and of above identity we conclude (3.23).

Applying Theorem 1.7 of [6] we conclude that in addition to $U(\lambda,x)$ there is another curve of solutions passing through (λ_k,u_k)

$$\lambda = \lambda_k + \tau(s), \quad u = u_k + su'_k + z(s), \tag{3.25}$$

with the parameter s defined on some interval around $s=0$, and $\tau(0)=\tau'(0)=0$, $z(0)=z'(0)=0$. This provides us with infinitely many points of pitchfork bifurcation. The solution $u(x)$ in (3.25) has $2k-1$ interior zeros: two zeros near each of $x_k=1/2k$, and one near $x=1$.

We claim that no solution curve can pass through two points of pitchfork bifurcation. Indeed, non-symmetric solution curves passing through different u_k have different number of zeros. Since the number of zeros is preserved on each solution curve, no such curve can pass through two different u_k. For the symmetric curves, the front curve at u_k and the tail end at u_{k+1} do have the same number of zeros, but they have the opposite order for regions of positivity and negativity, and hence cannot link up. Hence we can assume that the condition

$$u'(1,\lambda)\neq 0$$

is satisfied everywhere on each solution curve, except for a single point of pitchfork bifurcation. If the operator $F_u(\lambda,u)w$, defined previously, is invertible then solution curve can be continued by the implicit function theorem. Otherwise, in view of Lemma 3.3, the Crandall–Rabinowitz bifurcation theorem [5] applies, and hence all solution curves can be continued globally (see [15] for a similar argument; inequality (3.6) verifies the crucial “transversality” condition of that theorem).

Next, we show that no solution curve can continue for all $\lambda>0$, and hence all the curves will have to turn around eventually, and tend to infinity as $\lambda\to 0$. Assume on the contrary that some solution branch continues for all $\lambda>0$. By Lemma 3.1 the nodal structure of each solution is preserved. By Theorem 2.1 it follows that all intervals where solution is positive would have to shrink as $\lambda\to\infty$. It follows that there is at least one subinterval $I\in(0,1)$ on which solution is negative, and the length of I does not go to zero as $\lambda\to\infty$. On the interval I $u''(x)$ gets uniformly large and positive, as is clear from Eq. (3.1). It follows that as $\lambda\to\infty$, $u(x)$ will have to get uniformly large and negative on I. But by the Sturm's comparison theorem the interval on which $|u(x)|$ is large must shrink, a contradiction. Hence all solution curves will have to turn around. Since by Lemma 3.6 solution curves cannot go to infinity at a positive λ, and they all disappear at $\lambda = 0$, it follows that all solution curves tend to infinity as $\lambda \to 0$. $\square$

Remark. An example of nonlinearity for which the theorem applies is furnished by $f(u) = u|u|^p - 1$, with any real $p > 0$. (For $0 < p < 1$ the function $f(u)$ is not of class C^2 at $u = 0$. However, examining the proof, we see that all of the integrals are still defined, and the proof does not need any changes.) On the other hand the result fails for e.g. $f(u) = u^2 - 1$. This time $u = -1$ is a stable root, at which negative parts of solutions can accumulate, and in fact all solution curves continue as $\lambda \to \infty$ (see McKean and Scovel [18], where a detailed study is given for this nonlinearity).

4. Symmetry breaking for a ball

We study symmetry-breaking off the branch of positive solutions for the semilinear Dirichlet problem (2.1) on a ball in R^n. As we mentioned previously, in view of the results of [9], any positive solution of (2.1) is radially symmetric, and hence our problem reduces to the radial problem (2.2).

We shall also need the corresponding linearized equation

$$w'' + \frac{n-1}{r}w' + \lambda f'(u)w = 0 \quad \text{for } 0 < r < 1, \;\; w'(0) = w(1) = 0. \tag{4.1}$$

The following lemma was proved in [12] (see also [13] for a simpler proof). In both of these papers we worked in the context of positive solutions. Examining the proof, one sees that the lemma holds for sign-changing radial solutions $u(r)$ as well.

Lemma 4.1. *Assume that the function $f(u) \in C^2(\bar{R})$, and problem* (4.1) *has a non-trivial solution w at some λ and $u(r,\lambda)$. ($u(r,\lambda)$ is a radial solution of* (2.1).) *Then*

$$\int_0^1 f(u)wr^{n-1}\,\mathrm{d}r = \frac{1}{2\lambda}u'(1)w'(1). \tag{4.2}$$

We see that the integral in (4.2) is non-zero, unless

$$u'(1) = 0. \tag{4.3}$$

Recall that we say that symmetry breaks at a radial solution $u(r)$, provided that $u(r)$ bifurcates into non-radial solution.

Lemma 4.2. *Assume that $f(u) \in C^2(\bar{R})$, and $f(0) < 0$. Then condition* (4.3) *is necessary for symmetry breaking on a branch of radial solutions $u(r,\lambda)$ of* (2.1) (*i.e. solutions on the branch stay radial for all λ, provided $u'(1) \neq 0$*).

Proof. According to Dancer [8], symmetry of the solution of (2.1) can be broken only if the corresponding linearized equation has a non-radial solution. This immediately implies that the symmetric solution branch stays symmetric through all the regular points of (2.1). If condition $u'(1) \neq 0$ holds at singular points, then at any singular point $(\bar{\lambda}, \bar{u})$ the Crandall–Rabinowitz Theorem 1.1 applies, see e.g. [15] for the proof. Since this theorem implies that near the singular point $u(s) = \bar{u} + sw + \cdots$, we conclude that w is symmetric (since solutions are symmetric before entering the turning point), and hence symmetry is preserved at any possible turn. It follows that symmetry can never be broken if condition (4.3) is violated. □

The following lemma provides a condition for the crucial "transversality" condition to hold at the points where symmetry breaking is possible.

Lemma 4.3. *Assume that a branch of radial solutions $u(r,\lambda)$ of (2.1) is defined for $\lambda < \bar{\lambda}$ or for $\lambda > \bar{\lambda}$. Assume that $u'(1,\bar{\lambda}) = 0$. Then*

$$\bar{\lambda} \int_0^1 f''(u) u_\lambda u'^2 r^{n-1}\,\mathrm{d}r + \int_0^1 f'(u) u'^2 r^{n-1}\,\mathrm{d}r = -\bar{\lambda} f(0) u'_\lambda(1,\bar{\lambda}). \tag{4.4}$$

Proof. Let us assume first that $n > 2$, as there is a slight complication in case $n = 2$. The proof is similar to that for one-dimensional case. Denoting the left-hand side of (4.4) by $\bar{\lambda} I + J$, we express

$$\begin{aligned} I &\equiv \int_0^1 f''(u) u_\lambda u'^2 r^{n-1}\,\mathrm{d}r = \int_0^1 f'(u)' u_\lambda u' r^{n-1}\,\mathrm{d}r \\ &= -\int_0^1 [f'(u) u'' u_\lambda r^{n-1} + f'(u) u' u'_\lambda r^{n-1} + f'(u) u' u_\lambda (n-1) r^{n-2}]\,\mathrm{d}r \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{4.5}$$

Proceeding similarly, we express

$$\begin{aligned} I_2 &\equiv -\int_0^1 f(u)' u'_\lambda r^{n-1}\,\mathrm{d}r = \int_0^1 f(u) u''_\lambda r^{n-1}\,\mathrm{d}r + (n-1) \\ &\quad \times \int_0^1 f(u) u'_\lambda r^{n-2}\,\mathrm{d}r - f(0) u'_\lambda(1), \end{aligned}$$

$$I_3 = \int_0^1 [(n-1) f(u) u'_\lambda r^{n-1}\,\mathrm{d}r + (n-1)(n-2) f(u) u_\lambda r^{n-2}]\,\mathrm{d}r.$$

Using these expressions in (4.5), we have

$$\begin{aligned} I = \int_0^1 &[-f'(u) u'' u_\lambda r^{n-1} + f(u) u''_\lambda r^{n-1} + 2 f(u) u'_\lambda (n-1) r^{n-2} \\ &+ f(u) u_\lambda (n-1)(n-2) r^{n-3}]\,\mathrm{d}r - f(0) u'_\lambda(1). \end{aligned} \tag{4.6}$$

From the corresponding equations we express

$$r^{n-1}u'' = -(n-1)r^{n-2}u' - \bar{\lambda} f r^{n-1},$$

$$r^{n-1}u_\lambda'' = -(n-1)r^{n-2}u_\lambda' - \bar{\lambda} f' u_\lambda r^{n-1} - f r^{n-1}.$$

Using these relations we rewrite the first two terms in (4.6) as follows:

$$\int_0^1 [-f'(u)u''u_\lambda r^{n-1} + f(u)u_\lambda'' r^{n-1}]\,\mathrm{d}r$$

$$= \int_0^1 [f'(u)u'u_\lambda(n-1)r^{n-2} - f(u)u_\lambda'(n-1)r^{n-2} - f^2(u)r^{n-1}]\,\mathrm{d}r$$

$$= \int_0^1 [-2f(u)u_\lambda'(n-1)r^{n-2} - f(u)u_\lambda(n-1)(n-2)r^{n-3} - f^2(u)r^{n-1}]\,\mathrm{d}r. \tag{4.7}$$

Using this in (4.6), we have

$$I = -\int_0^1 f^2(u)r^{n-1}\,\mathrm{d}r - f(0)u_\lambda'(1). \tag{4.8}$$

Turning to the second integral in (4.4), we express (using that $u'(1)=0$ on the first step, and Eq. (2.2) on the last step)

$$J \equiv \int_0^1 f'(u)u'^2 r^{n-1}\,\mathrm{d}r = \int_0^1 f'(u)'u'r^{n-1}\,\mathrm{d}r$$

$$= -\int_0^1 [f(u)u''r^{n-1} + (n-1)f(u)u'r^{n-2}]\,\mathrm{d}r = \bar{\lambda}\int_0^1 f^2(u)r^{n-1}\,\mathrm{d}r. \tag{4.9}$$

Putting together (4.8) and (4.9), we conclude the lemma in case $n > 2$.

Turning to the case $n = 2$, we proceed similarly. There is now an extra term $-f(u(0))u_\lambda(0)$ in the expression for I_3, and consequently in (4.6). However in (4.7) we pick up an extra term, which is negative of the extra term above. Hence these extra terms cancel in (4.8), and the proof proceeds exactly as before. □

Remark. Notice that in both lemmas above we do not require $u(r)$ to be positive.

We say that a positive solution branch $u(r,\lambda)$ *loses positivity forward* at some $\bar{\lambda}$, if $u'(1,\bar{\lambda}) = 0$ and the solution branch arrives at the point $(\bar{\lambda}, u(r,\bar{\lambda}))$ with λ increasing (i.e. for $\lambda < \bar{\lambda}$ solutions are positive with $u'(1,\lambda) < 0$).

Lemma 4.4. *Assume that a positive solution branch $u(r,\lambda)$ loses positivity forward at some $\bar{\lambda}$. Then the transversality condition holds, i.e. the sum of the integrals in* (4.4) *is not zero.*

Proof. Assume on the contrary that $u_\lambda'(1,\bar{\lambda}) = 0$. From the equation for u_λ

$$u_\lambda'' + \frac{n-1}{r}u_\lambda' + \lambda f_u(u)u_\lambda + f(u) = 0, \tag{4.10}$$

we conclude that $u''_\lambda(1,\bar{\lambda}) > 0$. This implies that $u_\lambda(r,\bar{\lambda}) > 0$ for r near 1, which is inconsistent with the definition of $\bar{\lambda}$, since solution must be decreasing near the end-point $r=1$ before losing positivity. □

It is not easy to verify that a solution branch loses positivity forward. One case when we can prove this is when we can show that any non-trivial solution of the linearized problem (4.1) is of one sign, or without restricting generality that it is positive, i.e. $w(r) > 0$ for $r \in (0,1)$.

Lemma 4.5. *Assume that* $u'(1,\bar{\lambda})=0$. *Assume that any non-trivial solution* $w(r)$ *of the linearized problem* (4.1) *at* $\lambda=\bar{\lambda}$ *has to be positive. Then solution* $u(r,\bar{\lambda})$ *is radially non-degenerate*, *i.e. the linearized problem* (4.1) *has no non-trivial solutions at* $(\bar{\lambda}, u(r,\bar{\lambda}))$. (*i.e.* $w(r) \equiv 0$.)

Proof. Assume on the contrary that (4.1) has a non-trivial solution, $w(r) > 0$. Differentiating Eq. (2.2), we get

$$u''_r + \frac{n-1}{r}u'_r + \lambda f'(u)u_r = \frac{n-1}{r^2}u'. \tag{4.11}$$

From Eqs. (4.1) and (4.11) we obtain

$$[r^{n-1}(u''w - u'w')]|_0^1 = \int_0^1 (n-1)r^{n-3}u'w\,\mathrm{d}r.$$

The integral on the right is negative, while the quantity on the left is equal to zero, a contradiction. □

Lemma 4.6. *Assume that any non-trivial solution of the linearized problem* (4.1) *is either positive or changes sign exactly once on* $(0,1)$. *Assume that* $u(0,\lambda)$ *is decreasing on a branch as we arrive at the point where* $u'(1)=0$. *Then positivity can be lost only forward*, *and moreover the transversality condition holds*, *i.e. the integral in* (4.4) *is positive*.

Proof. Define $\bar{\lambda}$ as before by $u'(1,\bar{\lambda})=0$. One easily checks that $w=ru_r-2\bar{\lambda}u_\lambda$ satisfies the linearized problem (4.1). Assume contrary to what we want to prove that positivity is lost backward, i.e. $u(r,\lambda)$ arrives at $\bar{\lambda}$ when λ is decreasing. If positivity is lost backward, we know from [12] that $u_\lambda(r,\bar{\lambda})$ is positive near $r=1$, and hence

$$w(r) \text{ is negative near } r=1. \tag{4.12}$$

We have

$$w(0) = -2\bar{\lambda}u_\lambda(0,\bar{\lambda}) < 0, \tag{4.13}$$

since $u(0,\lambda)$ is increasing in λ near $\lambda=\bar{\lambda}$, and the possibility of $w(0)=0$ is excluded by the uniqueness result of [21]. (By uniqueness result we would have $w \equiv 0$, contradicting (4.12).) If $w(r) < 0$ everywhere on $(0,1)$, we conclude by the previous lemma that w is identically zero, again contradicting (4.12). If on the other hand, $w(r)$ changes sign on $(0,1)$, it would have to have at least two roots, since it is negative near $r=0$ and $r=1$, contradicting the assumptions of the lemma. It follows that positivity is lost forward, and then u_λ is negative near $r=1$, and hence $u_\lambda'(1) \geqslant 0$. If one assumes that $u_\lambda'(1)=0$ then from (4.10) we get $u_\lambda''(1)>0$, which is not possible for a function negative near $r=1$. Hence, $u_\lambda'(1)>0$, which together with (4.4) proves the last claim of the lemma. □

Theorem 4.1. *Assume that any non-trivial solution of the linearized problem* (4.1) *is either positive or changes sign exactly once. Assume that problem* (2.2) *has a curve of positive solutions with* $u(0,\lambda)$ *decreasing, assume that this curve does not continue as a curve of positive solutions for large* λ. *Then this curve loses positivity forward at some* $\lambda=\bar{\lambda}$, *and a family of symmetry-breaking solutions bifurcates off this point. In case any non-trivial solution of the linearized problem is positive, we obtain the following more detailed conclusions. The curve of radial solutions* $u(r,\lambda)$ *continues locally for* $\lambda>\bar{\lambda}$ *as a curve of radial sign changing solutions, with* $u_\lambda=(1/2\bar{\lambda})ru_r$. *Moreover, an n-parameter family of non-radial solutions bifurcates off this curve at* $\lambda=\bar{\lambda}$.

Proof. Since by Lemma 4.5 the point $(\bar{\lambda},u(r,\bar{\lambda}))$ is radially non-degenerate, we see by restricting to the spaces of radial functions and using implicit function theorem, that the curve of radial solutions continues past the point $(\bar{\lambda},u(r,\bar{\lambda}))$. Since u_λ is negative near $r=1$ at this point, it follows that solutions must become negative near the end-point $r=1$ for $\lambda>\bar{\lambda}$.

Differentiating Eq. (2.1) we see that the functions $w_i=(x_i/r)u_r(r,\bar{\lambda})$ are non-trivial solutions of the linearized problem

$$\Delta w+\lambda f'(u)w=0 \text{ for } |x|<R, \quad w=0 \text{ for } |x|=R. \tag{4.14}$$

By the result of Cerami [4], the null-space of (4.14) is either n-dimensional or $n+1$-dimensional with one of its elements being a radial function (a similar result was proved by Smoller and Wasserman [25]). In view of Lemma 4.5 the first alternative holds, and so w_i, $i=1,2,\dots n$, span the null-space of (4.14). It follows that the null-space is invariant under the action of the orthogonal group $O(n)$. We now proceed similarly to the proof of Theorem 3.1, replacing the use of Crandall–Rabinowitz theorem by its generalization to the case of kernels invariant under the action of $O(n)$, due to Vanderbauwhede [26], concluding existence of an n-parameter family of non-radial solutions. (See Smoller and Wasserman [24] for a similar argument.) □

Example. Assume that the function $f(u)\in C^2(R)$ satisfies

$$f(0)<0, \tag{4.15}$$

$$f''(u)<0 \quad \text{for all } u>0. \tag{4.16}$$

Assume also there is a $\theta>0$ such that

$$\int_0^\theta f(u)\,\mathrm{d}u>0. \tag{4.17}$$

We have proved in [11] that under conditions (4.15) and (4.16) any non-trivial solution of the linearized Eq. (4.1) is of one sign, i.e. that we can take $w(r)>0$. Adding condition (4.17) guarantees existence of positive solutions. Using the results in Korman [11], we conclude that under conditions (4.15)–(4.17), problem (2.1) has a curve of positive solutions. This curve makes exactly one turn at some $\lambda = \lambda_0$. For $\lambda > \lambda_0$ the curve has two branches, the upper one which continues without any turns for all $\lambda > \lambda_0$, and the lower one which continues without turns, until at some finite $\bar{\lambda} > \lambda_0$ it develops a zero slope at the end-point (i.e. $u'(1, \bar{\lambda}) = 0$), and loses positivity for $\lambda > \bar{\lambda}$. Moreover, an n-parameter family of symmetry-breaking solutions bifurcates off the lower branch at $\lambda = \bar{\lambda}$.

5. Positivity for the linearized problem

We have seen that when discussing symmetry breaking it is very useful to know that any non-trivial solution of the linearized equation cannot have more than one interior zero. As we mentioned above, such a result holds for concave $f(u)$. In this section we present another such result, which applies to superlinear $f(u)$.

We shall need the following form of Sturm's comparison theorem, taken from [17]. For completeness we present its proof. We consider a differential operator, defined on functions $u = u(r)$ of class C^2

$$L[u] \equiv a(r)u'' + b(r)u' + c(r)u,$$

with continuous coefficients $a(r)$, $b(r)$ and $c(r)$.

Lemma 5.1. *Assume that on some interval $I \subseteq (-\infty, \infty)$ we have $a(r) > 0$ and*

$$L[u] \geqslant 0, \tag{5.1}$$

while

$$L[v] \leqslant 0, \tag{5.2}$$

with at least one of the inequalities being strict on a set of positive measure. Then the function $v(r)$ oscillates faster than $u(r)$, provided that they are both non-negative. More precisely, assume that $u(a) = u(b) = 0$ for some $a, b \in I$, $u(r) > 0$ on (a, b), while $v(a) \geqslant 0$. Then $v(r)$ must vanish on (a, b).

Proof. Assume that on the contrary $v(r) > 0$ for all $r \in (a, b)$. From Eqs. (5.1) and (5.2), we obtain

$$[\mu(r)(u'v - uv')]' \geqslant 0, \tag{5.3}$$

where $\mu(r) = e^{\int b(r)/a(r)\,dr}$ is the integrating factor. Integrating (5.3) over (a, b) we get

$$\mu(b)u'(b)v(b) - \mu(a)u'(a)v(a) > 0,$$

which is a contradiction, since the quantity on the left is non-positive.

We shall denote by $L[w]$ the left-hand side of (4.1), so that (4.1) takes the form

$$L[w] = 0 \quad \text{for } 0 < r < 1, \;\; w'(0) = w(1) = 0. \tag{5.4}$$

The typical way to prove positivity of $w(r)$ is by considering various test functions. The canonical situation is given by the following lemma that was used in [15]. We present a simpler proof for completeness. □

Lemma 5.2. *Assume that we can find a test function $v(r) \in C^2[0,1)$ such that $v'(0)=0$, and for some $r_0 \in (0,1)$ we have*

$$v > 0 \text{ and } L[v] < 0 \quad \text{for } 0 < r < r_0,$$

$$v < 0 \text{ and } L[v] > 0 \quad \text{for } r_0 < r < 1. \tag{5.5}$$

Then any non-trivial solution of (5.4) *does not change sign, i.e. we can choose* $w(r) > 0$.

Proof. We extend both $w(r)$ and $v(r)$ to $(-1,0)$ as even functions. Clearly, $w(r)$ satisfies (5.4) on $(-1,1)$, while $v(r)$ satisfies the conditions in the first line of (5.5) on $I_1 \equiv (-r_0, r_0)$, and the conditions in the second line of (5.5) on $I_2 \equiv (-1,1)\backslash(-r_0,r_0)$. By Lemma 4.1 v oscillates faster than w on both I_1 and I_2. Hence w cannot vanish on either $\bar{I}_1$ or I_2, since v has no roots inside these intervals (if an even function $w(r)$ vanishes on $\bar{I}_1$, it has to have two roots on $\bar{I}_1$). □

Remark. The lemma clearly holds if $r_0 = 0$ or $r_0 = 1$. (One of the conditions in (5.5) is then empty.)

Theorem 5.1. *Let $u = u(r)$ be a positive solution of*

$$\Delta u + f(u) = 0 \text{ for } |x| < 1, \quad u = 0 \text{ for } |x| = 1. \tag{5.6}$$

Assume that $n > 2$, and the function $f(u) \in C^2[0,\infty)$ satisfies the following conditions:

$$f(u) < 0 \text{ for } 0 < u < u_0, \quad f(u) > 0 \text{ for } u > u_0, \text{ for some } u_0 > 0, \tag{5.7}$$

$$f'(u) > 0 \quad \text{for all } u > 0, \tag{5.8}$$

$$uf'(u) - f(u) \geqslant 0 \quad \text{for all } u > 0, \tag{5.9}$$

$$2f'^2(u) - nff'' \geqslant 0 \quad \text{for all } u > 0. \tag{5.10}$$

Then any solution of the linearized equation $w(r)$ can have at most one root on $(0,1)$.

Proof. We consider a test function $v = ru'(r) + (n-2)u + \alpha$ with a constant α to be specified. Recall that we denote the left-hand side of the linearized equation (5.6) by $L[w]$. Compute

$$L[v] = (n-2)uf'(u) - nf(u) + \alpha f'(u) \equiv g_\alpha(u). \tag{5.11}$$

The sign of the test function $v(r)$ is governed by the function $\alpha = h(r) \equiv -ru'(r) - (n-2)u$, which is obtained by setting v equal to zero, and solving for α. Indeed, $v > 0$ (< 0) when $h(r) < \alpha$ $(> \alpha)$. Similarly, the sign of $g_\alpha(u)$ is governed by $\alpha = j(r) \equiv$

$-(n-2)uf'(u)-nf(u)/f'(u)$. This time, in view of (5.8), $g_\alpha(u)>0$ (<0) provided $\alpha>j(r)$ ($\alpha<j(r)$).

We claim that $h(r)$ changes sign exactly once on $(0,1)$. Indeed, $h(0)=-(n-2)u(0)<0$, $h(1)=-u'(1)\geqslant 0$, and

$$h'(r)=-ru''-(n-1)u'=rf(u). \tag{5.12}$$

From (5.12) and our assumption (5.7) we see that $h'<0$ near $r=1$, and so h is positive near $r=1$, and hence h changes sign on $(0,1)$. Since h' changes sign only once, it follows that $h(r)$ has exactly one root inside $(0,1)$, which we denote by r_0.

Using (5.10) we verify that $j'(r)\leqslant 0$ for all r, and so the function $j(r)$ is non-increasing on $(0,1)$. We distinguish two cases.

Case 1: $j(0)>0$. Since $j(1)=nf(0)/f'(0)<0$, it follows that the functions $j(r)$ and $h(r)$ intersect on $(0,1)$. Denote by $r_1\in(0,1)$ the point where $u(r_1)=u_0$, i.e. where $f(u(r_1))=0$. We see that j and h must intersect on $(0,r_1)$, since $j<0$ while by (5.12) $h>0$ on $(0,r_1)$. Since on $(0,r_1)$ h is increasing and j is decreasing, it follows that these function have a unique point of intersection, call it $\bar{r}\in(0,r_1)$. We consider further subcases.

(i) $\bar{r}\in(0,r_0)$. (Recall that $h(r_0)=0$.) We now fix $\alpha=\alpha_0=h(\bar{r})=j(\bar{r})$ in the definition of the test function v. Here $\alpha_0<0$, by the definition of r_0. The test function $v(r)$ satisfies the conditions of Lemma 5.2, and hence $w>0$ on $(0,1)$.

(ii) $\bar{r}\in(r_0,r_1)$. We take $\bar{\alpha}=h(\bar{r})$. Arguing as above, we conclude that $w>0$ on $(0,\bar{r}]$. If one could show that $h(1)>h(\bar{r})$, we would conclude as before that $w>0$ on $(0,1)$ (see a remark after the end of proof). Instead, we will show that on $(\bar{r},1)$ $w(r)$ cannot vanish more than once. Define $r_2\in(\bar{r},1)$ by $j(r_2)=0$. By fixing $\alpha=0$, we obtain $v<0$ and $L[v]>0$ on $(r_2,1)$, and hence w cannot vanish on $[r_2,1)$. So it remains to show that w cannot vanish twice on $(\bar{r},r_2)$. On this interval we shall work with two test functions $v_0=ru'(r)+(n-2)u$, and $v_1=(1/r)v_0$. On $(\bar{r},r_2)$ we have $v_0(r)=-h(r)<0$ and $L[v_0]<0$. Compute

$$L[v_1]=\frac{n-2}{r}(uf'(u)-f(u))-(n-3)\frac{v_0}{r^3}\geqslant 0$$

on $(\bar{r},r_2)$, while $v_1<0$ on this interval. Using Lemma 5.2 with the test function v_1, we conclude that $w(r)$ cannot vanish twice on $(\bar{r},r_2)$. (We remark that the restriction $n>2$ as well as our assumption (5.9) are needed only on this step. The trick of considering the test function v_1 is due to Ni and Nussbaum [19] and to Kwong [16], see also Ouyang and Shi [20].)

Case 2: $j(0)\leqslant 0$. Since $j(r)$ is a decreasing function, it is negative on the entire interval $(0,1)$. If j intersects h, we proceed the same way as in subcase (i) of Case 1. Otherwise, $j(r)<h(r)$ for all $r\in(0,1)$. Selecting $\alpha=h(0)$, we obtain $v<0$ and $L[v]>0$ everywhere on $(0,1)$. By Lemma 5.2 we conclude that $w>0$ on $(0,1)$. $\square$

Remark. In case $j(0)\leqslant 0$ both restriction $n>2$ and condition (5.9) are not necessary. In particular for $n=2$ we have by our conditions $j(0)<0$, and we conclude that $w>0$ under conditions (5.7), (5.8) and (5.10). This covers e.g. $f(u)=e^u-a$, $a>1$.

Example 1. $f(u)=u^p-a$. It is easy to check that the theorem applies for any constants $a>0$ and $1<p<n/(n-2)$, with $n>2$.

Example 2. $f(u)=u^p+u^q-a$. It turns out that for $n>3$ the theorem applies for any constants $a>0$ and

$$1<p,q<\frac{n}{n-2}, \tag{5.13}$$

while for $n=3$ there is a slight additional assumption. Indeed, one easily verifies that (5.13) implies conditions (5.7)–(5.9). A straightforward computation shows that the remaining condition (5.10) will be also satisfied, provided

$$F(p,q)\equiv 4pq-np(p-1)-nq(q-1)>0 \quad \text{over } Q\equiv\left[1,\frac{n}{n-2}\right]^2. \tag{5.14}$$

For $n\neq 4$ the function F has a critical point $p=q=(2n^2+4n)/(4n^2-16)$, while no critical points exist for $n=4$. Using (5.13), one checks that $F(p,p)>0$ for any p, and in particular at the above critical point. Then one checks that $F>0$ at all four sides of the square Q, provided $n\geqslant 4$. In case $n=3$ one has to add (5.14) as an extra condition, which is easily seen to be not very restrictive.

Remark. As mentioned in the proof, we could prove $w>0$ if we had an inequality $h(1)>h(\bar{r})$. In view of (5.12) such an inequality holds true, provided $f(0)\geqslant 0$. We thus conclude that $w(r)>0$, provided $f(0)\geqslant 0$, and conditions (5.8) and (5.10) are satisfied.

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