

Direction of bifurcation for some non-autonomous problems

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Abstract

We study the exact multiplicity of positive solutions, and the global solution structure for several classes of non-autonomous two-point problems. We present two situations where the direction of turn can be computed rather directly. As an application, we consider a problem from combustion theory with a sign-changing potential. We illustrate our results by numerical computations, using a novel method.

Key words: Global solution curves, direction of bifurcation, continuation in a global parameter.

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1 Introduction

In recent years bifurcation theory methods were applied to study the exact multiplicity of positive solutions, and the global solution structure of non-autonomous two-point problems

$$(1.1) \quad u'' + \lambda f(x, u) = 0, \quad a < x < b, \quad u(a) = u(b) = 0,$$

depending on a positive parameter λ . Let us briefly review the bifurcation theory approach, and more details can be found in the author's book [6]. If at some solution (λ_0, u_0) the corresponding linearized problem

$$(1.2) \quad w'' + \lambda_0 f_u(x, u_0)w = 0, \quad a < x < b, \quad w(a) = w(b) = 0$$

admits only the trivial solution, then we can continue the solutions of (1.1) in λ , by using the Implicit Function Theorem, see e.g., L. Nirenberg [12].

If, on the other hand, the problem (1.2) has non-trivial solutions then the Implicit Function Theorem cannot be used, instead one tries to show that the Crandall-Rabinowitz [3] bifurcation theorem applies. The crucial condition one needs to verify is

$$(1.3) \quad \int_a^b f(x, u_0)w \, dx \neq 0.$$

The Crandall-Rabinowitz theorem guarantees existence of a solution curve through the critical point (λ_0, u_0) , and if a turn occurs at (λ_0, u_0) , its direction is governed by (see e.g., the exposition in P. Korman [6])

$$(1.4) \quad I = \frac{\int_a^b f_{uu}(x, u_0)w^3 \, dx}{\int_a^b f(x, u_0)w \, dx}.$$

If $I > 0$ ($I < 0$) the direction of the turn is to the left (right) in the (λ, u) “plane”. If one can show that a turn to the left occurs at any critical point, then there is at most one critical point. Usually, there is exactly one critical point, which provides us with the exact shape of solution curve, and the exact multiplicity count for solutions.

In the present paper we present two situations in which the sign of I can be computed in a rather direct way, differently from the previous works. As an application, we obtain three new exact multiplicity results for non-autonomous equations, including one for sign-changing equations of combustion theory. Sign-changing equations present several new challenges, which we overcome in case of symmetric potentials. We also present some improvements of earlier results. In the last section we develop an algorithm for the numerical computation of global solution curves for non-autonomous equations. This is accomplished by continuation in a global parameter.

2 The direction of bifurcation

We consider positive solutions of a two point non-autonomous boundary value problem

$$(2.1) \quad u'' + \lambda f(x, u) = 0, \quad a < x < b, \quad u(a) = u(b) = 0,$$

depending on a positive parameter λ . Here $f(x, u) \in C^2([a, b] \times \bar{R}_+)$, and we assume that $f(a, 0) \geq 0$ and $f(b, 0) \geq 0$ (which implies that Hopf’s boundary lemma holds). The linearized problem corresponding to (2.1) is

$$(2.2) \quad w'' + \lambda f_u(x, u)w = 0, \quad a < x < b, \quad w(a) = w(b) = 0.$$

When one studies how solutions of (2.1) change in λ , i.e., the *solution curves*, the direction of the turn (or bifurcation) depends on the sign of the integral $\int_a^b f_{uu}(x, u)w^3 dx$, which is a part of (1.4). We have the following crucial lemma.

Lemma 2.1 *Let $u(x)$ be a positive solution of (2.1), and assume that the linearized problem (2.2) has a non-trivial solution $w(x)$, and moreover $w(x) > 0$ on (a, b) . If we have, for some $c > 0$,*

$$(2.3) \quad u^2 f_{uu}(x, u) \geq c(u f_u(x, u) - f(x, u)), \quad \text{for all } u > 0, \text{ and } x \in (a, b)$$

then

$$(2.4) \quad \int_a^b f_{uu}(x, u)w^3 dx > 0.$$

If, on the other hand, for some $c > 0$,

$$(2.5) \quad u^2 f_{uu}(x, u) \leq -c(u f_u(x, u) - f(x, u)), \quad \text{for all } u > 0, \text{ and } x \in (a, b)$$

then

$$(2.6) \quad \int_a^b f_{uu}(x, u)w^3 dx < 0.$$

Proof: We multiply the equation (2.2) by $\frac{w^2}{u}$, and subtract from that the equation (2.1) multiplied by $\frac{w^3}{u^2}$, then integrate

$$\lambda \int_a^b \left[\frac{f_u(x, u)}{u} - \frac{f(x, u)}{u^2} \right] w^3 dx = \int_a^b \left(\frac{w^3}{u^2} u'' - \frac{w^2}{u} w'' \right) dx.$$

In the last integral we integrate by parts. The boundary terms vanish, since $u'(a)$ and $u'(b)$ are not zero by Hopf's boundary lemma, and hence $u(x)$ is asymptotically linear near the end points. We have

$$\begin{aligned} \int_a^b \left(\frac{w^3}{u^2} u'' - \frac{w^2}{u} w'' \right) dx &= \int_a^b \frac{2w^3 u u'^2 - 4w^2 w' u^2 u' + 2w w'^2 u^2}{u^4} dx \\ &= \int_a^b \frac{2w(wu' - uw')^2}{u^3} dx > 0. \end{aligned}$$

If the condition (2.3) holds, then

$$\int_a^b f_{uu}(x, u)w^3 dx \geq c \int_a^b \left[\frac{f_u(x, u)}{u} - \frac{f(x, u)}{u^2} \right] w^3 dx > 0.$$

Similarly, the condition (2.5) implies

$$-\int_a^b f_{uu}(x, u)w^3 dx \geq c \int_a^b \left[\frac{f_u(x, u)}{u} - \frac{f(x, u)}{u^2} \right] w^3 dx > 0. \quad \diamond$$

Remarks

1. After this paper was written, we became aware that a similar result was proved in J. Shi [14].
2. The condition (2.3), when $c = 1$, is equivalent to

$$\left[u \left(\frac{f(u)}{u} \right)' \right]' = \left(f'(u) - \frac{f(u)}{u} \right)' > 0.$$

In T. Ouyang and J. Shi [13] it has been pointed out that the turning direction is sometimes related to monotonicity of the function $f(u)/u$. This form of (2.3) again shows a connection to the function $f(u)/u$.

Example $f(u) = a + u^p + u^q$, with a constant $a \geq 0$. One computes, with $c = 1$,

$$u^2 f''(u) - u f'(u) + f(u) = (p-1)^2 u^p + (q-1)^2 u^q + a > 0 \text{ for all } u > 0.$$

The case when $0 < p < 1 < q$ is of particular interest. Then $f(u)$ is *concave-convex*, i.e., concave on $(0, u_0)$ and convex on (u_0, ∞) , for some $u_0 > 0$. Similarly, the condition (2.3) holds for $f(x, u) = \sum_{i=1}^m a_i(x) u^{p_i}$, with $a_i(x) > 0$ for all x , and any positive p_i . While in the case of constant $a_i(x)$, the direction of bifurcation was known before (see the Theorem 6.1 below), the non-autonomous case is new.

3 A class of symmetric nonlinearities

We study positive solutions of a class of symmetric problems of the type

$$(3.1) \quad u'' + \lambda f(x, u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0.$$

In several papers of P. Korman and T. Ouyang, a class of symmetric $f(x, u)$ has been identified, for which the theory of positive solutions is very similar to that for the autonomous case, see e.g., [8] and [9]. Further results in this direction have been given in P. Korman, Y. Li and T. Ouyang [7], and P.

Korman and J. Shi [11]. Namely, we assume that $f(x, u) \in C^1([-1, 1] \times \bar{R}_+)$ satisfies

$$(3.2) \quad f(-x, u) = f(x, u) \quad \text{for all } -1 < x < 1, \text{ and } u > 0,$$

$$(3.3) \quad f_x(x, u) \leq 0 \quad \text{for all } 0 < x < 1, \text{ and } u > 0.$$

Under the above conditions the following facts, similar to those for autonomous problems, have been established.

1. Any positive solution of (3.1) is an even function, with $u'(x) < 0$ for all $x \in (0, 1]$, so that $x = 0$ is a point of global maximum. This follows from B. Gidas, W.-M. Ni and L. Nirenberg [4].

2. Assume, additionally, that $f(x, u) > 0$. Then the maximum value of solution, $\alpha = u(0)$, uniquely identifies the solution pair $(\lambda, u(x))$, as proved in P. Korman and J. Shi [11]. i.e., $\alpha = u(0)$ gives a global parameter on any solution curve. We shall generalize this result below, dropping the condition that $f(x, u) > 0$.

3. Any non-trivial solution of the corresponding linearized problem

$$(3.4) \quad w'' + \lambda f_u(x, u)w = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0$$

is of one sign on $(-1, 1)$.

We have the following exact multiplicity result.

Theorem 3.1 *Consider the problem*

$$(3.5) \quad u'' + \lambda(a_1(x)u^p + a_2(x)u^q) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0.$$

Assume that $0 < p < 1 < q$, while the given functions $a_1(x)$ and $a_2(x)$ satisfy

$$(3.6) \quad a_i(x) > 0, \quad a_i(-x) = a_i(x) \quad \text{for } x \in (-1, 1), \quad i = 1, 2,$$

$$(3.7) \quad a'_i(x) < 0 \quad \text{for } x \in (0, 1), \quad i = 1, 2.$$

Then there is a critical $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ the problem (3.5) has no positive solutions, it has exactly one positive solution for $\lambda = \lambda_0$, and exactly two positive solutions for $0 < \lambda < \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve $u(x, \lambda)$, which for $0 < \lambda < \lambda_0$ has two branches denoted by $0 < u^-(x, \lambda) < u^+(x, \lambda)$, with $u^-(x, 0) = 0$, $u^-(x, \lambda)$ strictly monotone increasing in λ for all $x \in (-1, 1)$, and $\lim_{\lambda \rightarrow 0} u^+(0, \lambda) = \infty$. The maximal value of solution, $u(0, \lambda)$, serves as a global parameter on this solution curve.

Proof: Conditions (3.6) and (3.7) imply that the above mentioned results on symmetric problems apply, and in particular any non-trivial solution of the linearized problem, corresponding to (3.5) is positive on $(-1, 1)$. Then by Lemma 2.1 only turns to the left are possible on the solution curve. The rest of the proof is similar to that for similar results in P. Korman and T. Ouyang [8], or P. Korman and J. Shi [11], so we just sketch it. By the Implicit Function Theorem, there is curve of positive solutions of (3.5) starting at $(\lambda = 0, u = 0)$. By Sturm's comparison theorem, this curve cannot be continued indefinitely in λ , so that it will have to reach a critical point (λ_0, u_0) at which the Crandall-Rabinowitz Theorem [3] applies. By Lemma 2.1, a turn to the left occurs at (λ_0, u_0) , and at any other critical point. Hence, there are no other turning points, and the solution curve continues for all decreasing $\lambda > 0$, tending to infinity as $\lambda \rightarrow 0+$. \diamond

Remark This theorem also holds for more general $f(u) = \sum_{i=1}^m a_i(x)u^{p_i}$, with $a_i(x)$ satisfying (3.6) and (3.7), and $p_i \geq 0$, with at least one of p_i less than one, and at least one of p_i greater than one.

4 Non-symmetric nonlinearities

Without the symmetry assumptions on $f(x, u)$, the problem is much harder. We restrict to a subclass of such problems, i.e., we now consider positive solutions of the boundary value problem

$$(4.1) \quad u'' + \lambda \alpha(x)f(u) = 0 \quad \text{for } a < x < b, \quad u(a) = u(b) = 0,$$

on an arbitrary interval (a, b) . We assume that $f(u)$ and $\alpha(x)$ are positive functions of class C^2 , i.e.,

$$(4.2) \quad f(u) > 0 \text{ for } u > 0, \quad \alpha(x) > 0 \text{ for } x \in [a, b].$$

As before, it is crucial for bifurcation analysis to prove positivity for the corresponding linearized problem

$$(4.3) \quad w'' + \lambda \alpha(x)f'(u)w = 0 \quad \text{for } a < x < b, \quad w(a) = w(b) = 0.$$

The following lemma was proved in P. Korman and T. Ouyang [10].

Lemma 4.1 *In addition to the conditions (4.2), assume that*

$$(4.4) \quad \frac{3}{2} \frac{\alpha'^2}{\alpha} - \alpha'' \leq 0 \quad \text{for all } x \in (a, b).$$

If the linearized problem (4.3) admits a non-trivial solution, then we may assume that $w(x) > 0$ on (a, b) .

Using Lemma 2.1, we have the following exact multiplicity result, whose proof is similar to that of Theorem 3.1.

Theorem 4.1 *Consider the problem*

$$(4.5) \quad u'' + \lambda \alpha(x) \sum_{i=1}^m a_i u^{p_i} = 0 \text{ for } a < x < b, \quad u(a) = u(b) = 0,$$

where $\alpha(x)$ satisfies the conditions (4.2) and (4.4), a_i are positive constants, the constants $p_i \geq 0$, with at least one of p_i less than one, and at least one of p_i greater than one. Then there is a critical $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ the problem (4.5) has no positive solutions, it has exactly one positive solution for $\lambda = \lambda_0$, and exactly two positive solutions for $0 < \lambda < \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve $u(x, \lambda)$, which for $0 < \lambda < \lambda_0$ has two branches denoted by $0 < u^-(x, \lambda) < u^+(x, \lambda)$, with $u^-(x, 0) = 0$, $u^-(x, \lambda)$ strictly monotone increasing in λ for all $x \in (a, b)$, and $\lim_{\lambda \rightarrow 0} \max_{x \in (a, b)} u^+(x, \lambda) = \infty$.

5 Sign-changing equation of combustion theory

We begin with the following generalization of the corresponding result in P. Korman and J. Shi [11], which we shall use for an equation in combustion theory. Recall that positive solutions of (3.1) are even functions, with $u'(x) < 0$ for $x > 0$, i.e., $u(0)$ is the global maximum of solution $u(x)$.

Theorem 5.1 *Consider the problem (3.1), with $f(x, u)$ satisfying (3.2) and (3.3), with the inequality (3.3) being strict for almost all $x \in (-1, 1)$ and $u > 0$. Then the set of positive solutions of (3.1) can be globally parameterized by the maximum values $u(0)$. (I.e., the value of $u(0)$ uniquely determines the solution pair $(\lambda, u(x))$.)*

Proof: Assume, on the contrary, that $v(x)$ is another solution of (3.1), with $v(0) = u(0)$, and $v'(0) = u'(0) = 0$. We may assume that $\mu > \lambda$. Setting $x = \frac{1}{\sqrt{\lambda}}t$, we see that $u = u(t)$ satisfies

$$(5.1) \quad u'' + f\left(\frac{1}{\sqrt{\lambda}}t, u\right) = 0, \quad u'(0) = u(\sqrt{\lambda}) = 0.$$

Similarly, letting $x = \frac{1}{\sqrt{\mu}}z$, and then renaming z by t , we see that $v = v(t)$ satisfies

$$v'' + f\left(\frac{1}{\sqrt{\mu}}t, v\right) = 0, \quad v'(0) = v(\sqrt{\mu}) = 0,$$

and in view of (3.3)

$$(5.2) \quad v'' + f\left(\frac{1}{\sqrt{\lambda}}t, v\right) < 0,$$

i.e., $v(t)$ is a supersolution of (5.1). We may assume that

$$(5.3) \quad v(t) < u(t) \text{ for } t > 0 \text{ and small.}$$

Indeed, the opposite inequality is impossible by the strong maximum principle (a supersolution cannot touch a solution from above, and the possibility of infinitely many points of intersection of $u(t)$ and $v(t)$ near $t = 0$ is ruled out by the Sturm comparison theorem, applied to $w = u - v$).

Since $v(t)$ is positive on $(0, \sqrt{\lambda})$, we can find a point $\xi \in (0, \sqrt{\lambda})$ so that $u(\xi) = v(\xi)$, $|u'(\xi)| \geq |v'(\xi)|$, and (5.3) holding on $(0, \xi)$. We now multiply the equation (5.1) by $u'(t)$, and integrate over $(0, \xi)$. Since the function $u(t)$ is decreasing, its inverse function exists. Denoting by $t = t_2(u)$, the inverse function of $u(t)$ on $(0, \xi)$, we have

$$(5.4) \quad \frac{1}{2}u'^2(\xi) + \int_{u(0)}^{u(\xi)} f\left(\frac{1}{\sqrt{\lambda}}t_2(u), u\right) du = 0.$$

Similarly denoting by $t = t_1(u)$ the inverse function of $v(t)$ on $(0, \xi)$, we have multiplying (5.2) by $v'(t) < 0$, and integrating

$$(5.5) \quad \frac{1}{2}v'^2(\xi) + \int_{u(0)}^{u(\xi)} f\left(\frac{1}{\sqrt{\lambda}}t_1(u), u\right) du > 0.$$

Subtracting (5.5) from (5.4), we have

$$\frac{1}{2} \left[u'^2(\xi) - v'^2(\xi) \right] + \int_{u(\xi)}^{u(0)} \left[f\left(\frac{1}{\sqrt{\lambda}}t_1(u), u\right) - f\left(\frac{1}{\sqrt{\lambda}}t_2(u), u\right) \right] du < 0.$$

Notice that $t_2(u) > t_1(u)$ for all $u \in (u(\xi), u(0))$. Using the condition (3.3), both terms on the left are non-negative and the integral is positive, and we obtain a contradiction. \diamond

We now consider a boundary value problem arising in combustion theory, see e.g., J. Bebernes and D. Eberly [2], S.-H. Wang [15], and S.-H. Wang and F.P. Lee [16]

$$(5.6) \quad u'' + \lambda\alpha(x)e^u = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$

depending on a positive parameter λ . The given function $\alpha(x)$ is assumed to be sign changing, with $\alpha(0) > 0$ (the result of this section is known if $\alpha(x)$ is positive on $(-1, 1)$). The linearized problem corresponding to (5.6) is

$$(5.7) \quad w'' + \lambda\alpha(x)e^u w = 0, \quad -1 < x < 1, \quad w(-1) = w(1) = 0.$$

Lemma 5.1 *Assume that any non-trivial solution of (5.7) satisfies $w(x) > 0$ on $(-1, 1)$, and $\alpha(x) \in C[-1, 1]$. Then*

$$(5.8) \quad \int_{-1}^1 \alpha(x) e^u w^3 dx > 0, \quad \text{and} \quad \int_{-1}^1 \alpha(x) e^u w dx > 0.$$

Proof: Multiplying the equation (5.7) by w^2 , and integrating

$$\int_{-1}^1 \alpha(x) e^u w^3 dx = 2 \int_{-1}^1 w w'^2 dx > 0.$$

Integrating (5.7)

$$\int_{-1}^1 \alpha(x) e^u w dx = w'(-1) - w'(1) > 0,$$

as claimed. ◇

We need the following simple lemma.

Lemma 5.2 *Let $u(x)$ be a solution of the problem*

$$(5.9) \quad -u'' = \alpha(x), \quad 0 < x < 1, \quad u'(0) = u(1) = 0.$$

Assume that $\alpha(x) \in C[0, 1]$ is sign-changing, and it satisfies

$$(5.10) \quad A(x) \equiv - \int_0^x (x - \xi) \alpha(\xi) d\xi + \int_0^1 (1 - \xi) \alpha(\xi) d\xi > 0 \quad \text{for } x \in [0, 1),$$

$$(5.11) \quad \int_0^1 \alpha(\xi) d\xi > 0.$$

Then $u(x) > 0$ on $[0, 1)$, and $u'(1) < 0$.

Proof: The formulas (5.10) and (5.11) give $u(x)$ and $-u'(1)$ respectively.

◇

We shall assume that $\alpha(x) \in C^1[-1, 1]$ satisfies

$$(5.12) \quad \alpha(-x) = \alpha(x) \quad \text{for all } x \in [0, 1],$$

$$(5.13) \quad \alpha'(x) < 0 \quad \text{for almost all } x \in [0, 1].$$

As before, we know that under these conditions any non-trivial solution of (5.7) is positive on $(-1, 1)$. Observe also that under these conditions a sign-changing $\alpha(x)$ changes sign exactly once on $(0, 1)$ (and on $(-1, 0)$).

Theorem 5.2 *Consider the problem (5.6), and assume that the function $\alpha(x)$ satisfies (5.12) and (5.13). We also assume that $\alpha(x)$ is sign-changing, and it satisfies (5.10) and (5.11). Then there is a critical $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ the problem (5.6) has no positive solutions, it has exactly one positive solution for $\lambda = \lambda_0$, and exactly two positive solutions for $0 < \lambda < \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve $u(x, \lambda)$, which for $0 < \lambda < \lambda_0$ has two branches denoted by $0 < u^-(x, \lambda) < u^+(x, \lambda)$, with $u^-(x, 0) = 0$, and $\lim_{\lambda \rightarrow 0} u^+(0, \lambda) = \infty$. The maximal value of solution, $u(0, \lambda)$, serves as a global parameter on this solution curve.*

Proof: We begin with the solution $(\lambda = 0, u = 0)$. This solution is non-singular (the corresponding linearized problem (5.7) has only the trivial solution), so that by the Implicit Function Theorem we have a curve of solutions $u(x, \lambda)$ passing through the point $(\lambda = 0, u = 0)$. We claim that these solutions are positive for small λ . Indeed, $u_\lambda(x, 0) \equiv u_\lambda$ satisfies

$$-u'' = \alpha(x), \quad -1 < x < 1, \quad u(-1) = u(1) = 0.$$

Since $\alpha(x)$ is even, so is $u(x)$, and hence $u(x)$ satisfies (5.9), and so the Lemma 5.2 applies. Hence, for small λ , solution $u(x)$ of (5.6) is positive, and hence it is an even function, with $u'(x) < 0$ for $x \in (0, 1)$.

We show next that solutions remain positive throughout the solution curve. Since for positive solutions $u'(x) < 0$ for $x \in (0, 1)$, there is only one mechanism by which solutions may stop being positive: $u'(1) = 0$ at some $\lambda = \lambda_1$ (and then solutions becoming negative near $x = 1$). We claim that for any positive solution $u(x)$ of (5.6)

$$(5.14) \quad u'(1) < 0,$$

which will rule out such a possibility. (Hopf's boundary lemma does not apply here.) Let ξ be the point where $\alpha(x)$ changes sign, i.e., $\alpha(x) > 0$ on $[0, \xi)$ and $\alpha(x) < 0$ on $(\xi, 1)$. Denote $u(\xi) = u_0$. Then from the equation (5.6)

$$-u'' > \lambda e^{u_0} \alpha(x), \quad 0 < x < 1, \quad u'(0) = u(1) = 0.$$

Comparing this to (5.9), we conclude that $u(x) > \lambda e^{u_0} A(x)$ for all $x \in (0, 1)$, and then $u'(1) \leq \lambda e^{u_0} A'(1) < 0$.

We claim next that the curve of positive solutions cannot be continued in λ beyond a certain point. With ξ denoting the root of $\alpha(x)$, as above, fix

any $\eta \in (0, \xi)$, and denote $a_0 = \alpha(\eta) > 0$. If we denote $\varphi(x) = \cos \frac{\pi}{2\eta}x$ and $\lambda_1 = \frac{\pi^2}{4\eta^2}$, then

$$\varphi'' + \lambda_1 \varphi = 0, \quad \text{on } (0, \eta), \quad \varphi'(0) = \varphi(\eta) = 0.$$

We have

$$\int_0^\eta u'' \varphi \, dx = \int_0^\eta u \varphi'' \, dx - u(\eta) \varphi'(\eta) > \int_0^\eta u \varphi'' \, dx = -\lambda_1 \int_0^\eta u \varphi \, dx.$$

Then multiplying the equation (5.6) by φ and integrating, we have

$$\lambda \int_0^\eta \alpha(x) e^u \varphi \, dx < \lambda_1 \int_0^\eta u \varphi \, dx.$$

Also

$$\int_0^\eta \alpha(x) e^u \varphi \, dx > a_0 \int_0^\eta u \varphi \, dx.$$

We conclude that

$$\lambda < \frac{\lambda_1}{a_0}.$$

The next step is to show that solutions of (5.6) remain bounded, if λ is bounded away from zero. Indeed, for large u , $\lambda \alpha(x) e^u > Mu$, with arbitrarily large constant M , when x belongs to any sub-interval of $(-\xi, \xi)$. By Sturm's comparison theorem, the length of the interval on which $u(x)$ becomes large, must tend to zero. But that is impossible, because $u(x)$ is concave on $(-\xi, \xi)$.

We now return to the curve of positive solutions, emanating from $(\lambda = 0, u = 0)$. Solutions on this curve are bounded, while the curve cannot be continued indefinitely in λ . Hence, a critical point (λ_0, u_0) must be reached on this curve, i.e., at (λ_0, u_0) the corresponding linearized problem (5.7) has a non-trivial solution. By the results on symmetric problems, reviewed above, any non-trivial solution of the corresponding linearized problem (5.7) is of one sign, i.e., we may assume that $w(x) > 0$ on $(-1, 1)$. By Lemma 5.1, the Crandall-Rabinowitz theorem applies at any critical point, and a turn to the left occurs. Hence, after the turn at (λ_0, u_0) , the curve continues for decreasing λ , without any more turns. By the Theorem 5.1, $u(0, \lambda)$ is a global parameter on the solution curve. Along the solution curve, the global parameter $u(0, \lambda)$ is increasing and tending to infinity. By above, this may happen only as $\lambda \rightarrow 0$. \diamond

Example The function $\alpha(x) = 1 - cx^2$ for $1 < c < 3$ is sign-changing on $(-1, 1)$, and it satisfies all of the above conditions. Indeed

$$\int_0^1 \alpha(t) dt = 1 - c/3 > 0,$$

and

$$A(x) = \frac{cx^4}{12} - \frac{x^2}{2} - \frac{c}{12} + \frac{1}{2} > 0, \text{ for } 0 < x < 1,$$

because $A(1) = 0$, and

$$A'(x) = \frac{cx^3}{3} - x < 0 \text{ for } 0 < x < 1.$$

6 Autonomous problems

In case the nonlinearity does not depend explicitly on x , i.e., $f = f(u)$, the main result on the direction of turn is the following theorem from P. Korman, Y. Li and T. Ouyang [7] and T. Ouyang and J. Shi [13]. We present a little simpler proof of this key result.

Autonomous problems can be posed on any interval. We use the interval $(-1, 1)$ for convenience, i.e., we consider positive solutions of

$$(6.1) \quad u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0.$$

Corresponding linearized problem is

$$(6.2) \quad w'' + \lambda f'(u)w = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0.$$

Both $u(x)$ and $w(x)$ are even functions (see e.g., [6]), and so the direction of bifurcation at a critical point (λ_0, u_0) is governed by

$$I = \frac{\int_0^1 f''(u_0)w^3 dx}{\int_0^1 f'(u_0)w dx}.$$

Recall that $f(u) \in C^2(\bar{R}_+)$ is called convex-concave if $f''(u) > 0$ on $(0, u_0)$, and $f''(u) < 0$ on (u_0, ∞) for some $u_0 > 0$, and the definition of concave-convex functions is similar.

Theorem 6.1 ([7], [13]) **(i)** Assume $f(0) \leq 0$, and $f(u)$ is convex-concave. Then at any critical point, with $u_0(0) > u_0$ and $u'_0(1) < 0$, we have $I < 0$, and hence a turn to the right occurs.

(ii) Assume $f(0) \geq 0$, and $f(u)$ is concave-convex. Then at any critical point, with $u_0(0) > u_0$, we have $I > 0$, and hence a turn to the left occurs.

Proof: In case $f(0) \geq 0$, we have $u'_0(1) < 0$ by Hopf's boundary lemma. We shall write (λ, u) instead of (λ_0, u_0) . It is known that any non-trivial solution of the linearized problem (6.2) is of one sign, see e.g., [7] or [6], and so we may assume that $w(x) > 0$ on $(-1, 1)$, which implies that

$$(6.3) \quad w'(1) < 0.$$

From the equations (6.1) and (6.2) it is straightforward to verify the following identities

$$(6.4) \quad u'(x)w'(x) - u''(x)w(x) = \text{constant} = u'(1)w'(1);$$

$$(6.5) \quad (u''w' - u'w'')' = \lambda f''(u)u'^2w.$$

Assume that the first set of conditions hold. Integrating (6.5),

$$(6.6) \quad \lambda \int_0^1 f''(u)u'^2w \, dx = u''(1)w'(1) = -\lambda f(0)w'(1) \leq 0.$$

Consider the function $p(x) \equiv \frac{w(x)}{-u'(x)}$. Since $p(1) = 0$, and by (6.4)

$$p'(x) = -\frac{u'(1)w'(1)}{u'(x)^2} < 0,$$

the function $p(x)$ is positive and decreasing on $(0, 1)$. The same is true for $p^2(x) = \frac{w^2(x)}{u'^2(x)}$. Let x_0 be the point where $f''(u(x))$ changes sign (i.e., $f''(u(x)) < 0$ on $(0, x_0)$, and $f''(u(x)) > 0$ on $(x_0, 1)$). By scaling $w(x)$, we may achieve that $w^2(x_0) = u'^2(x_0)$. Then $w^2(x) > u'^2(x)$ on $(0, x_0)$, and the inequality is reversed on $(x_0, 1)$. Using (6.6), we have

$$(6.7) \quad \int_0^1 f''(u(x))w^3 \, dx < \int_0^1 f''(u(x))u'^2w \, dx \leq 0.$$

Integrating (6.4), we have

$$(6.8) \quad \int_0^1 f(u)w \, dx = \frac{1}{2\lambda}u'(1)w'(1) > 0.$$

The formulas (6.7) and (6.8) imply that $I < 0$. The second part of the theorem is proved similarly. \diamond

Observe that, in case $f = f(u)$, this theorem and the Lemma 2.1 have intersecting domains of applicability, but neither one is more general than the other.

7 Numerical computation of the solution curves

In this section we present computations of the global curves of positive solutions for the problem

$$(7.1) \quad u'' + \lambda f(x, u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0.$$

We assume that the function $f(x, u)$ satisfies the conditions (3.2) and (3.3), so that the Theorem 5.1 applies, which tells us that $\alpha \equiv u(0)$ is a global parameter. Since any positive solution $u(x)$ is an even function, we shall compute it on the half-interval $(0, 1)$, by solving

$$(7.2) \quad u'' + \lambda f(x, u) = 0 \quad \text{for } 0 < x < 1, \quad u'(0) = u(1) = 0.$$

The standard approach to numerical computation involves curve following, i.e., continuation in λ by using the predictor-corrector methods, see e.g., E.L. Allgower and K. Georg [1]. These methods are well developed, but not easy to implement, as the solution curve $u = u(x, \lambda)$ may consist of several parts, each having multiple turns. Here λ is a local parameter, but not a global one, because of the turning points.

Since $\alpha = u(0)$ is a global parameter, we shall compute the solution curve of (7.2) in the form $\lambda = \lambda(\alpha)$. If we solve the initial value problem

$$(7.3) \quad u'' + \lambda f(x, u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0,$$

then we need to find λ , so that $u(1) = 0$, in order to obtain the solution of (7.2). Rewrite the equation (7.2) in the integral form

$$u(x) = \alpha - \lambda \int_0^x (x-t)f(t, u(t)) dt,$$

and then the equation for λ is

$$(7.4) \quad F(\lambda) \equiv u(1) = \alpha - \lambda \int_0^1 (1-t)f(t, u(t)) dt = 0.$$

We solve this equation by using Newton's method

$$\lambda_{n+1} = \lambda_n - \frac{F(\lambda_n)}{F'(\lambda_n)}.$$

We have

$$F(\lambda_n) = \alpha - \lambda_n \int_0^1 (1-t)f(t, u(t, \lambda_n)) dt,$$

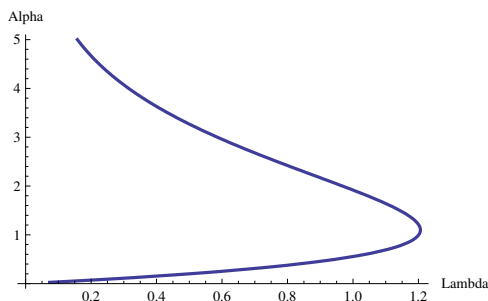


Figure 1: The solution curve for a problem from combustion theory

$$F'(\lambda_n) = - \int_0^1 (1-t) f(t, u(t, \lambda_n)) dt - \lambda_n \int_0^1 (1-t) f_u(t, u(t, \lambda_n)) u_\lambda dt ,$$

where $u(x, \lambda_n)$ and u_λ are respectively the solutions of

$$(7.5) \quad u'' + \lambda_n f(x, u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0 ;$$

$$(7.6) \quad u_\lambda'' + \lambda_n f_u(x, u(x, \lambda_n)) u_\lambda + f(x, u(x, \lambda_n)) = 0, \quad u_\lambda(0) = 0, \quad u_\lambda'(0) = 0 .$$

(As we vary λ , we keep $u(0) = \alpha$ fixed, that is why $u_\lambda(0) = 0$.) This method is very easy to implement. It requires repeated solutions of the initial value problems (7.5) and (7.6) (using the NDSolve command in *Mathematica*).

Example Consider a problem from combustion theory with sign-changing potential

$$u'' + \lambda(1 - 2x^2)e^u = 0 \quad \text{for } 0 < x < 1, \quad u'(0) = u(1) = 0 .$$

The global solution curve is presented in Figure 1. For any point (λ, α) on this curve, the actual solution $u(x)$ is easily computed by shooting (or NDSolve command), see (7.3). In Figure 2 we present the solution $u(x)$ for $\lambda \approx 1.1955$, when $u(0) = 1.25$. (This solution lies on the upper branch, shortly after the turn.)

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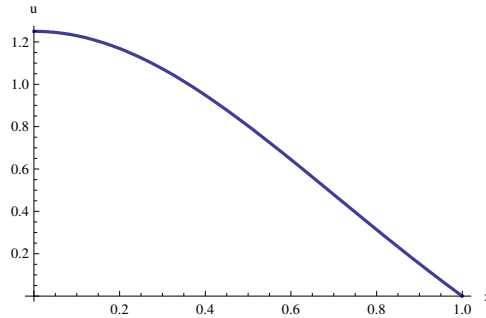


Figure 2: The solution $u(x)$, corresponding to $\alpha = u(0) = 1.25$

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