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004

ON EXISTENCE OF SOLUTIONS FOR SEVERAL CLASSES OF FREE BOUNDARY PROBLEMS*

PHILIP KORMAN†

Dedicated to Ken Meyer on the occasion of his 50th birthday.

Abstract. Some general techniques are developed for treating nonstandard nonlinear elliptic problems arising when a so-called domain perturbation method is applied to free boundary problems. Our results are applied to two model problems from fluid mechanics.

Key words. free boundary problems, domain perturbation method, existence and uniqueness of solutions

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1. Introduction. Our goal is to explore a general approach to free boundary problems, based on the so-called domain perturbation method. Using this method we get a solution, which is a perturbation of a known one, by mapping (nonconformally) the unknown fluid domain back to the known domain for an unperturbed solution. The linear equations of motion then get transformed to a fully nonlinear system, but since nonlinearities are small, it can usually be treated by the contractive mapping argument. The earliest reference that we know for the domain perturbation method is Joseph [6]. It was then used by Shinbrot [10] to prove the existence of double-periodic water waves in three dimensions. Our work was motivated by that paper.

Rather than present our results in general form, we prefer to consider two model problems, whose treatment illustrates how one should approach various possibilities, and which are of considerable independent interest.

The nonlinear elliptic problems obtained by the domain perturbation method can be either coercive or noncoercive (here "coercive" means that the problem satisfies the Lopatinski–Schapiro condition at all points of the boundary). For the coercive problems we outline an approach using the Schauder-type estimates of Agmon, Douglis, and Nirenberg [1], and present it for the model Problem I. For noncoercive problems there are no Schauder's estimates. Estimates in the Sobolev spaces (which are available for Problems I and II) cannot be used, because of loss of smoothness when taking traces. For the model Problem II we present the second approach based on Λ^m spaces, which are defined and studied below. Similar spaces were used by Shinbrot [10]; however, ours have several advantages: it is easier to establish their properties, the proofs of the estimates for $m>2$ are more transparent, and finally they seem to be more natural. Problem II leads to a coercive problem, so that the first approach based on Schauder's estimates can be used as well. In § 6 we present an example of a physically significant problem, leading to a noncoercive problem which can be solved only by the second approach. We proceed to describe our model problems.

Problem I. Let $x \in R^n$. Given a 2π periodic in each variable x_i function $B(x)$ (bottom), find the functions $u(x, y)$, $H(x)$, 2π periodic in each x_i, such that

$$\begin{aligned} &u=0, \quad \frac{\partial u}{\partial n}=-1, \quad y=H(x),\\ &\Delta u=0, \qquad B(x)<y<H(x), \qquad\qquad (1.1)\\ &u=1, \qquad y=B(x), \end{aligned}$$

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† Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221.

where $\partial/\partial n$ is the outward normal derivative.

Problem II. Let $r=B(\theta)$ be a closed curve in the plane. Find another closed curve $r=H(\theta)$ outside of $B(\theta)$, and a function u on the closed region between $B(\theta)$ and $H(\theta)$ with

$$u=0, \quad \frac{\partial u}{\partial n}=-1, \quad r=H(\theta),$$

$$\Delta u=0, \qquad B(\theta)<r<H(\theta), \tag{1.2}$$

$$u=1, \qquad r=B(\theta).$$

For Problem I we start with flat bottom $B=0$, and the corresponding solution $H=1$ and $u=1-y$. Then for small bottoms $B=\varepsilon b(x)$ we are looking for the solution in the form

$$H=1+\varepsilon h(x), \qquad u=1-y+\varepsilon v(x,y), \tag{1.3}$$

and show existence if ε is sufficiently small. We use the change of variables $(x,y)\to(x,y')$, $y'=(y-\varepsilon b)/(1+\varepsilon h-\varepsilon b)$, to transform the unknown domain onto a fixed one, $0\leqq y'\leqq 1$.

For Problem II notice that if $B(\theta)=1$ then the solution is $H(r)=h_0$, $u=-h_0\log r+1$, where $h_0\simeq 1.76$ is defined by $h_0\log h_0=1$, and (r,θ) are the polar coordinates. Then we assume that $B=1+\varepsilon b(\theta)$, and look for the solution in the form

$$H=h_0+\varepsilon h(\theta), \qquad u=-h_0\log r+1+\varepsilon v(r,\theta). \tag{1.4}$$

We show the existence of such a solution for ε sufficiently small.

Problem II was considered by Hamilton [5] (and also earlier by Schaeffer [9] and Acker [11]). Hamilton proved that for every smooth convex curve B there exists a unique solution to Problem II (the curve $H(\theta)$ is also smooth and convex, and u is smooth). His result is strictly two-dimensional, since conformal mappings were used to derive a priori estimates. Our existence result complements Hamilton's in that we do not require the curve $r=B(\theta)$ to be convex. In three dimensions we were unable so far to carry out a similar approach, because of the singularities in the Laplace operator in spherical coordinates.

We wish to stress the generality of our approach. It can be used to attack problems with boundary conditions of arbitrary order and variable coefficients, and with nonlinear equations of motion. In contrast, a more common variational approach (see, e.g., [2, Chap. 3]) is rather restricted (but it is a global method).

Finally, we mention that Problems I and II have an interesting physical interpretation. For Problem II it is described in [5, p. 215], so that we present a similar interpretation for Problem I (with similar deficiency as mentioned in [5]). We consider fluid occupying the half space $y>0$, $x\in R^n$, and assume there is a stream flowing over the periodic bottom $y=B(x)$. The fluid is assumed to be perfect with unit density, and at rest outside a free surface boundary, so that there is a velocity jump at the free surface. Let u be the stream potential. By choosing units of length and time we can make the velocity on free surface $y=H(x)$ and circulation equal to one. This leads us to Problem I.

2. Preliminary results. Let $x=(x_1,\cdots,x_n)$, $j=(j_1,\cdots,j_n)$, $n\geqq 1$. Let the function $u=u(x,y)$ be 2π periodic in each variable x_i, $i=1,\cdots,n$, $0\leqq y\leqq 1$; $u(x,y)=\sum_{j=-\infty}^{\infty} u_j(y)\,e^{ij\cdot x}$. Define the norms $\|u(x,y)\|_0=\sum_{j=-\infty}^{\infty}\max_{0\leqq y\leqq 1}|u_j(y)|$, $\|u\|_m=\sum_{|\alpha|\leqq m}\|D^\alpha u\|_0$, where D^α is a mixed partial in x and y, $m=$ integer $\geqq 1$. Denote

$D=[0,2\pi]^n$, $V=D\times[0,1]$. Let $\Lambda^m(V)$ be the closure of trigonometric polynomials of the form $\sum u_j(y)\, e^{ij\cdot x}$, $u_j(y)\in C^\infty[0,1]$, with respect to the norm $\|\cdot\|_m$. Clearly $\Lambda^m(V)$ are Banach spaces with $\|u\|_m \leqq \|u\|_n$ if $m\leqq n$. The space $\Lambda^m(D)$ is defined in the same way for functions independent of y. The norm on $\Lambda^m(D)$ is denoted by $|\!|\!|\cdot|\!|\!|_m$. If we are given a function of polar coordinates in the plane, $u=u(r,\theta)=\sum_{n=-\infty}^{\infty} u_n(r)\, e^{in\theta}$ on an annulus $1\leqq r\leqq h_0$, $h_0=$ constant, then as before $\|u\|_0=\sum_{n=-\infty}^{\infty}\max_{1\leqq r\leqq h_0}|u_n(r)|$, and $\|u\|_m=\sum_{|\alpha|\leqq m}\|D^\alpha u\|_0$ where D^α is a mixed partial in r and θ. This time domain V is defined by $1\leqq r\leqq h_0$, $0\leqq\theta\leqq 2\pi$ domain D by $r=h_0$, $0\leqq\theta\leqq 2\pi$.

We write c for all positive constants independent of unknown functions. We write $f=f(D^2v)$ when f depends on the function v and all its partial derivatives or orders one and two.

LEMMA 2.1. *Let* $u, v\in\Lambda^m$. *Then* $uv\in\Lambda^m$, *and* $\|uv\|_m\leqq c_m\|u\|_m\|v\|_m$, $c_m=$ const $(c_0=1)$.

Proof. Let $u=\sum_{j=-\infty}^{\infty} u_j\, e^{ij\cdot x}$, $v=\sum_{k=-\infty}^{\infty} v_k\, e^{ik\cdot x}$. Then

$$\|uv\|_0=\sum_\gamma \max_y \left|\sum_\delta u_{\gamma-\delta}v_\delta\right| \leqq \sum_\delta \max_y |v_\delta| \sum_\gamma \max_y |u_{\gamma-\delta}|$$
$$=\|u\|_0\|v\|_0.$$

For $m\geqq 1$ we get

$$\|uv\|_m=\sum_{|\alpha|\leqq m}\|D^\alpha uv\|=\sum_{|\alpha|\leqq m}\sum_{0\leqq\beta\leqq\alpha} c_\beta\|D^\beta u\|_0\|D^{\alpha-\beta}v\|_0$$
$$\leqq c_m\|u\|_m\|v\|_m.$$

COROLLARIES. (i) $\|g_1\cdots g_p\|_m\leqq c_m^{p-1}\|g_1\|_m\cdots\|g_p\|_m$.

(ii) $\|g^p\|_m\leqq c_m^{p-1}\|g\|_m^p$.

LEMMA 2.2. *Let B be a ball in R^p centered at the origin, $f(x_1,\cdots,x_p)$: $B\to R^1$ be a real analytic function. Let g be a vector function on V, $g=(g_1,\cdots,g_p)$, $\|g\|_m\equiv\sum_{i=1}^p\|g_i\|_m=r$. Assume that r is sufficiently small. Then $f(g_1,\cdots,g_p)\in\Lambda^m$, and*

$$\|f(g)\|_m\leqq c_0+c_1(r),$$

where $c_0=$ const >0, c_1 is analytic function of r, depending only on f and r, and $c_1(0)=0$.

Proof. Let $f=\sum_{|\alpha|\geqq 0} f_\alpha x^\alpha$ for $x=(x_1,\cdots,x_p)\in B$, $f(g)=\sum_{|\alpha|\geqq 0} f_\alpha g_\alpha$. Then by Lemma 2.1

$$\|f(g)\|_m\leqq\sum_{|\alpha|\geqq 0}|f_\alpha|\,\|g^\alpha\|_m\leqq\sum_{|\alpha|\geqq 0}|f_\alpha| c_m^{|\alpha|-1}r^{|\alpha|},$$

which is easily seen to be a convergent series for r sufficiently small.

LEMMA 2.3. *$\Lambda^m(V)(\Lambda^m(D))$ is boundedly imbedded in $C^m(V)(C^m(D))$.*

Proof. Since for any multi-index α, $|\alpha|=m$, $\max_{x,y}|D^\alpha u|\leqq\|u\|_m$ the proof follows.

By $|\cdot|_{m+\alpha}$ we denote the norm in the space $C^{m+\alpha}(V)$, $m=$ integer $\geqq 0$, $0<\alpha<1$ (see, e.g., [1] for the definition).

3. Transformation to a fixed domain. For Problem I we suppose that $B(x)=\varepsilon b(x)$, and look for solution in the form (1.3). Notice that on $y=H(x)=1+\varepsilon h(x)$

$$\frac{\partial u}{\partial n}=\nabla u\cdot n=\frac{-\varepsilon\sum_{i=1}^n u_i h_i+u_y}{\sqrt{1+\varepsilon^2|\nabla h|^2}}.$$

Substituting this and (1.3) into (1.1) we get

$$v(x,y)=h(x),\qquad \frac{-\varepsilon^2\sum_{i=1}^n v_i h_i-1+\varepsilon v_y}{\sqrt{1+\varepsilon^2|\nabla h|^2}}=-1,\qquad y=1+\varepsilon h,$$

$$\Delta v=0,\qquad \varepsilon b(x)<y<1+\varepsilon h(x), \tag{3.1}$$

$$v=b(x),\qquad y=\varepsilon b(x).$$

The change of variables $(x, y) \to (x', y')$ defined by

$$x_i' = x_i, \qquad i = 1, \cdots, n,$$

$$y' = \frac{y - \varepsilon b}{1 + \varepsilon(h - b)} \equiv y d(x) + e(x),$$

$$\left(d(x) = \frac{1}{1 + \varepsilon(h - b)}, \; e(x) = -\frac{\varepsilon b}{1 + \varepsilon(h - b)} \right)$$

will transform the unknown fluid domain onto $0 \leqq y' \leqq 1$.

By a straightforward calculation, the problem (3.1) will transform as follows (we drop primes for the independent variables)

$$\begin{aligned} &v(x, 1) = h(x), && \\ &v_y = \varepsilon g(\varepsilon, Dv, Dh, Db), && y = 1, \\ &\Delta v = \varepsilon f(\varepsilon, D^2 v, D^2 h, D^2 b), && 0 < y < 1, \\ &v = b(x), && y = 0. \end{aligned} \tag{3.2}$$

Here

$$\varepsilon g = \frac{1 - \sqrt{1 + \varepsilon^2 |Dh|^2} + \varepsilon^2 \sum_{i=1}^n [v_{x_i} + v_y(Y d_{x_i} + e_{x_i})] h_{x_i}}{\varepsilon d(x)}, \tag{3.3}$$

where $Y = y(1 + \varepsilon h - \varepsilon b) + \varepsilon b$, and

$$\begin{aligned} -\varepsilon f = (d^2 - 1) v_{yy} + \sum_{i=1}^n [2 v_{x_i y}(Y d_{x_i} + e_{x_i}) + v_{yy}(Y d_{x_i} + e_{x_i})^2 \\ + v_y(Y d_{x_i x_i} + e_{x_i x_i})]. \end{aligned} \tag{3.4}$$

We easily see that the functions f and g are analytic in their arguments for small ε.

For Problem II we suppose that $B = 1 + \varepsilon b(\theta)$, and look for a solution in the form (1.4). By an elementary computation on $r = H(\theta) = h_0 + \varepsilon h(\theta)$ we have

$$\frac{\partial u}{\partial n} = \frac{r}{\sqrt{r^2 + \varepsilon^2 h'^2}} u_r - \frac{\varepsilon h'}{r\sqrt{r^2 + \varepsilon^2 h'^2}} u_\theta. \tag{3.5}$$

Using this formula and (1.4) in (1.2) we get

$$\begin{aligned} &v = \frac{-1 + h_0 \log r}{\varepsilon}, && r = h_0 + \varepsilon h, \\ &\frac{r}{\sqrt{r^2 + \varepsilon^2 h'^2}} \left(-\frac{h_0}{r} + \varepsilon v_r \right) - \frac{\varepsilon^2 h'}{r\sqrt{r^2 + \varepsilon^2 h'^2}} v_\theta = -1, && r = h_0 + \varepsilon h, \\ &\Delta v = 0, \qquad 1 + \varepsilon b(\theta) < r < h_0 + \varepsilon h(\theta), && \\ &v = \frac{h_0 \log r}{\varepsilon}, && r = 1 + \varepsilon b(\theta). \end{aligned} \tag{3.6}$$

The change of variables $(r, \theta) \to (r', \theta')$

$$r' = (h_0 - 1) \frac{r - 1 - \varepsilon b}{\varepsilon(h - b) + h_0 - 1} + 1, \qquad \theta' = \theta$$

maps the fluid domain $1+\varepsilon b \leqq r \leqq h_0+\varepsilon h$ onto the annulus $1 \leqq r' \leqq h_0$. The problem (3.5) will transform as follows (dropping the primes)

$$\begin{aligned} &v = h+\varepsilon r(\varepsilon, h), && r = h_0,\\ &v_r = -\frac{h}{h_0}+\varepsilon g(\varepsilon, Dv, Dh, Db), && r = h_0,\\ &\Delta v = \varepsilon f(\varepsilon, D^2v, D^2h, D^2b), && 1 < r < h_0,\\ &v = h_0 b+\varepsilon q(\varepsilon, b), && r = 1. \end{aligned} \tag{3.7}$$

Here

$$\varepsilon r = \frac{h_0 \log(h_0+\varepsilon h)-1}{\varepsilon} - h,$$

$$\varepsilon g = \frac{\varepsilon}{(h_0+\varepsilon h)^2 p}[v_\theta h' + v_r(Rp_\theta+q_\theta)h' - h^2 - (h_0+\varepsilon h)^2 r_1] + \frac{1}{\varepsilon}\left(\frac{h}{h_0} - \frac{h_0 h}{r^2 p}\right)$$

with

$$p = \frac{h_0-1}{\varepsilon(h-b)+h_0-1}, \qquad q = -\frac{(h_0-1)(1+\varepsilon b)}{\varepsilon(h-b)+h_0+1}+1,$$

$$R = \frac{r-q}{p}, \qquad r_1 = \frac{1}{\varepsilon^2}\left(\sqrt{1+\varepsilon^2\frac{h'^2}{R^2}}-1\right),$$

$$-\varepsilon f = (p^2-1)v_{rr} + \left(\frac{p}{R}-\frac{1}{r}\right)v_r + \left(\frac{1}{R^2}-\frac{1}{r^2}\right)v_{\theta\theta}$$

$$+\frac{1}{R^2}[2v_{\theta r}(Rp_\theta+q_\theta)+v_{rr}(Rp_\theta+q_\theta)^2+v_r(Rp_{\theta\theta}+q_{\theta\theta})],$$

$$\varepsilon q = \frac{h_0}{\varepsilon}(\log(1+\varepsilon b)-\varepsilon b).$$

We verify that the functions r, g, f, q are analytic in their arguments for small ε.

4. A priori estimates for the linear problem. Consider the problem $(x \in R^n)$

$$\begin{aligned} &u_y = g(x), && y = 1,\\ &\Delta u = f(x, y), && 0 < y < 1,\\ &u = b(x), && y = 0, \end{aligned} \tag{4.1}$$

where f, g, and b are given functions, 2π periodic in each variable x_i, $i = 1, \cdots, n$.

LEMMA 4.1. *Assume that* $f \in C^\alpha(V)$, $g \in C^{1+\alpha}(D)$. *Then* (4.1) *has a unique* 2π *periodic in each* x_i *solution, and*

$$|u|_{2+\alpha} \leqq c(|f|_\alpha + |g|_{1+\alpha} + |b|_{2+\alpha}). \tag{4.2}$$

Proof. Existence of solutions follows by elementary Fourier analysis, uniqueness from the estimate

$$|u|_0 \leqq c(|f|_0 + |g|_0 + |b|_0), \tag{4.3}$$

which easily follows by the maximum principle. It remains to show how one adapts Schauder's estimates for our problem (4.1). Redefine f, g, b as functions of compact support outside $0 \leqq x_i \leqq 2\pi$, $i = 1, \cdots, n$, and call the extensions $\bar{f}, \bar{g}, \bar{b}$, respectively. Clearly, this can be done with say $|\bar{f}|_\alpha \leqq 2|f|_\alpha$, $|\bar{g}|_\alpha \leqq 2|g|_\alpha$, $|\bar{b}|_\alpha \leqq 2|b|_\alpha$. Let $\xi_1(y)$, $\xi_2(y)$ be C^∞ functions on $[0, 1]$, such that $\xi_1 \equiv 1$ near $y = 1$ and $\xi_1 \equiv 0$ near $y = 0$, and $\xi_2 = 1 - \xi_1$. Write $u = \xi_1 u + \xi_2 u \equiv u_1 + u_2$. Multiplying (4.1) by ξ_1 and ξ_2, we easily get

$$\begin{aligned} u_{1y} &= \bar{g}, && y = 1, \\ \Delta u_1 &= \xi_1'' u + 2\xi_1' u_y + \xi_1 \bar{f}, && -\infty < y < 1, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \Delta u_2 &= \xi_2'' u + 2\xi_2' u_y + \xi_2 \bar{f}, && 0 < y < \infty, \\ u_2 &= \bar{b}(x), && y = 0. \end{aligned} \tag{4.5}$$

Using usual Schauder's estimates (see [1, Thm. 7.3]), we get (for arbitrary small ε)

$$\begin{aligned} |u|_{2+\alpha} &\leqq c\left(\sum_{i=1}^{2} |\xi_i'' u + 2\xi_i' u_y + \xi_i \bar{f}|_\alpha + |\bar{g}|_{1+\alpha} + |\bar{b}|_{2+\alpha} \right) \\ &\leqq c(\varepsilon |u|_{2+\alpha} + c_\varepsilon |u|_0 + |f|_\alpha + |g|_{1+\alpha} + |b|_{2+\alpha}), \end{aligned}$$

and by (4.3) the lemma follows.

Next, in the plane (r, θ) consider the problem ($h_0 \log h_0 = 1$)

$$\begin{aligned} u_r + \sigma u &= g(\theta), && r = h_0 \quad (\sigma = \text{const} \geqq 0), \\ \Delta u &= f(r, \theta), && 1 < r < h_0, \\ u &= b(\theta), && r = 1. \end{aligned} \tag{4.6}$$

Here, g, f, b are given functions 2π periodic in θ.

LEMMA 4.2. *Assume $b \in \Lambda^{m+2}$, $f \in \Lambda^m$, $g \in \Lambda^{m+1}$, $m \geqq 0$. Then* (4.6) *has a unique solution and*

$$\|u\|_{m+2} + \overline{\|u\|}_{m+2} \leqq c(\|f\|_m + \|g\|_{m+1} + \|b\|_{m+2}). \tag{4.7}$$

Proof. Express $f = \sum_{n=-\infty}^{\infty} f_n(r)\, e^{in\theta}$, $g = \sum_{n=-\infty}^{\infty} g_n\, e^{in\theta}$, $b = \sum_{n=-\infty}^{\infty} b_n\, e^{in\theta}$, $u = \sum_{n=-\infty}^{\infty} u_n(r)\, e^{in\theta}$. Substituting these into (4.6) and suppressing the subscript n (i.e., writing f for f_n, g for g_n, etc.) and letting $r = e^x$, we get

$$u_{xx} - n^2 u = e^{2x} f(e^x), \quad u(0) = b, \quad \frac{1}{h_0} u_x(h_1) + \sigma u(h_1) = g, \tag{4.8}$$

where $h_1 = \log h_0 \simeq 0.57$. Set $F(t) = e^{2t} f(e^t)$. The solution of (4.8) is

$$u(x) = \gamma \sinh nx + b \cosh nx + \frac{1}{n} \int_0^x F(t) \sinh n(x - t)\, dt, \tag{4.9}$$

where the constant γ is determined from

$$\begin{aligned} &\gamma A + b\left(\frac{n \sinh nh_1}{h_0} + \sigma \cosh nh_1 \right) \\ &\quad + \int_0^{h_1} F(t) \left[\frac{1}{h_0} \cosh n(h_1 - t) + \frac{\sigma}{n} \sinh n(h_1 - t) \right] dt = g. \end{aligned} \tag{4.10}$$

Here we denoted

$$A = \frac{n \cosh nh_1}{h_0} + \sigma \sinh nh_1 \geqq cn\, e^{nh_1}. \tag{4.11}$$

Multiplying (4.9) by A, using (4.10) and the standard identities for hyperbolic functions, we easily derive

$$
\begin{aligned}
Au(x) &= g \sinh nx + b\left(\frac{n}{h_0}\cosh n(h_1-x)+\sigma \sinh n(h_1-x)\right) \\
&\quad -\int_x^{h_1} F(t)\left[\frac{1}{h_0}\cosh n(h_1-t)\sinh nx+\frac{\sigma}{n}\sinh n(h_1-t)\sinh nx\right]dt \qquad (4.12)\\
&\quad -\int_0^{x} F(t)\left[\frac{1}{h_0}\sinh nt\cosh n(h_1-x)+\frac{\sigma}{n}\sinh nt\sinh n(h_1-x)\right]dt.
\end{aligned}
$$

Then in view of (4.11) we easily estimate

$$
\begin{aligned}
|u(r)| &\leqq c\left[\frac{|g|}{n}+|b|+\frac{1}{n}\int_x^{h_1}|F(t)|\,e^{n(x-t)}\,dt+\frac{1}{n}\int_0^x |F(t)|\,e^{n(t-x)}\,dt\right] \\
&\leqq c\left[\frac{|g|}{n}+|b|+\frac{1}{n}\sup_{1\leqq r\leqq h_0}|f(r)|\left(\frac{2}{n}-\frac{e^{n(x-h_1)}}{n}-\frac{e^{-nx}}{n}\right)\right] \qquad (4.13)\\
&\leqq c\left[\frac{|g|}{n}+|b|+\frac{1}{n^2}\max_{1\leqq r\leqq h_0}|f(r)|\right].
\end{aligned}
$$

Differentiating (4.9) and going through the same steps, we estimate

$$
|u'(r)|\leqq c\left[|g|+n|b|+\frac{1}{n}\max_{1\leqq r\leqq h_0}|f(r)|\right]. \qquad (4.14)
$$

Combining (4.13) with (4.14), and estimating $|u''(r)|$ from the equation, we conclude the estimate (4.7) with $m=0$. The higher estimates are easily proved by induction.

An a priori estimate for (4.1) is given by the following lemma whose proof is similar to the above.

LEMMA 4.3. *Assume* $f\in\Lambda^m$, $g\in\Lambda^{m+1}$, $b\in\Lambda^{m+2}$, $m=\text{integer}\geqq 0$. *Then* (4.1) *has a unique* 2π *periodic in each* x_i *solution, and*

$$
\|u\|_{m+2}+|\!|\!|u|\!|\!|_{m+2}\leqq c(\|f\|_m+\|g\|_{m+1}+\|b\|_{m+2}).
$$

5. Existence and uniqueness of solutions for Problems I and II.

THEOREM 5.1. *For* (3.2) *assume that* $b(x)\in C^{m+\alpha}(D)$, *and* $\varepsilon|b|_{m+\alpha}$ *is sufficiently small,* $m=\text{integer}\geqq 2$, $0<\alpha<1$. *Then there exists a pair of functions* $(v,h)\in C^{m+\alpha}(V)\times C^{m+\alpha}(D)$ *satisfying* (3.2).

Proof. Define a map T: $(w,k)\to(v,h)$ from $C^{m+\alpha}(V)\times C^{m+\alpha}(D)$ to itself by solving

$$
\begin{aligned}
v_y &= \varepsilon g(\varepsilon, Dw, Dk, Db), && y=1,\\
\Delta v &= \varepsilon f(\varepsilon, D^2w, D^2k, D^2b), && 0<y<1,\\
v &= b(x), && y=0,
\end{aligned}
$$

and then computing $h(x)=v(x,1)$. By Lemma 4.1 we easily conclude that the map T is well defined, takes a ball $|w|_{m+\alpha}+|k|_{m+\alpha}\leqq R$, with say $R=2|b|_{m+\alpha}$, into itself, and is a contraction for $\varepsilon|b|_{m+\alpha}$ sufficiently small.

A similar proof could be given for Problem II. Instead, we give an existence proof in Λ^m spaces based on Lemma 4.2, which provides a more general approach, as will be seen in § 6.

THEOREM 5.2. *For the problem* (3.7) *assume that* $b(\theta)\in\Lambda^m$, *and* $\varepsilon\|b\|_m$ *is sufficiently small,* $m=\text{integer}\geqq 2$. *Then there exists a pair of functions* $(v,h)\in\Lambda^m(V)\times\Lambda^m(D)$ *satisfying* (3.6).

Proof. Substituting the first equation in (3.7) into the second we get

$$v_r+\frac{1}{h_0}v=\frac{\varepsilon}{h_0}r(\varepsilon,h)+\varepsilon g\equiv\varepsilon\bar g(\varepsilon,Dv,Dh,Db).$$

Next, we define a map $T:(w,k)\to(v,h)$ from $\Lambda^m(V)\times\Lambda^m(D)$ to itself by solving

$$\begin{aligned}&v_r+\frac{1}{h_0}v=\varepsilon\bar g(\varepsilon,Dw,Dk,Db), && r=h_0,\\ &\Delta v=\varepsilon f(\varepsilon,D^2w,D^2k,D^2b), && 1<r<h_0,\\ &v=h_0b+\varepsilon q(\varepsilon,b), && r=1,\end{aligned}\tag{5.1}$$

and then solving for $h(\theta)$ from

$$v(h_0,\theta)=h+\varepsilon r(\varepsilon,h)\left(h=\frac{h_0}{\varepsilon}(e^{\varepsilon v/h_0}-1)\right).\tag{5.2}$$

Notice that the map T is well defined, i.e., it takes $\Lambda^m(V)\times\Lambda^m(D)$ into itself, provided $\varepsilon\|b\|_m$ is sufficiently small. Indeed, in view of the estimate (4.7) of Lemma 4.2, it suffices to show that $\bar g\in\Lambda^{m-1}(D)$, $f\in\Lambda^{m-2}(V)$. For this we use the special structure of f and $\bar g$. Indeed, consider $\bar g$. By Lemma 2.3, $\log(h_0+\varepsilon h)$, p_θ, q_θ, $1/(h_0+\varepsilon h)^2\in\Lambda^{m-1}(D)$ for ε small (if smallness of $\varepsilon\|b\|_m$ comes from $\|b\|_m$, then work in small balls), and then by Lemma 2.2, $\bar g\in\Lambda^{m-1}(D)$. Similarly, $f\in\Lambda^{m-2}$ by Lemma 2.2.

Then one easily sees that map T takes the ball $\|w\|_m+\|k\|_m\leqq 2\|b\|_m$ into itself, and is a contraction.

Remark. A similar argument is valid for Problem I.

Next we prove uniqueness results, using techniques similar to [3] and [4].

THEOREM 5.3. *Problem* I *can have at most one solution* (*in the class of free surfaces satisfying interior sphere condition*).

Proof. Assume that this is not true, i.e., there are two solutions $(u(x,y),h(x))$ and $(\bar u(x,y),\bar h(x))$. By the maximum principle we conclude that $0\leqq u\leqq 1$ for $b\leqq y\leqq h$, and $0<u<1$ for $b<y<h$, and also that h and $\bar h$ are different. Consider first the special case, when one free surface is above the other, touching at some point, say, $h(x)\geqq\bar h(x)$, $h(x_0)=\bar h(x_0)$. Consider $w=u-\bar u$ in the domain $b\leqq y\leqq\bar h$. Then $\Delta w=0$, $w=0$ for $y=b(x)$, $w\geqq 0$ for $y=\bar h(x)$ with $w(x_0)=0$. Hence x_0 is a point of minimum for w. Since h and $\bar h$ have the same normal at x_0, by Hopf's lemma we have

$$0>\frac{\partial w}{\partial n}(x_0)=\frac{\partial u}{\partial n}(x_0)-\frac{\partial\bar u}{\partial n}(x_0)=0,$$

a contradiction.

Turning to the general case, we introduce translation of solution (u,h) downward, by considering

$$u_\tau(x,y)=u(x,y+\tau),\quad h_\tau(x)=h(x)-\tau,\quad \tau\geqq 0.$$

Clearly $\Delta u_\tau=0$ for $b-\tau<y<h-\tau$. Choose $\tau=\tau_0$ so that $h_{\tau_0}\leqq\bar h$, and $h_{\tau_0}(x_0)=\bar h(x_0)$ for some x_0. (If h and $\bar h$ intersect, we translate either of two solutions, if $h>\bar h$ then translate (u,h).) Let $\bar D=\{x\,|\,x_0\in D$ and $h_{\tau_0}(x)>b(x)\}$. By periodicity either $\bar D=(-\infty,\infty)$ or D is a bounded interval. In the first case consider $w=\bar u-u_{\tau_0}$ with $\bar u$, u_{τ_0}

restricted to $b(x) \leqq y \leqq h_{\tau_0}(x)$. Then $w > 0$ on $y = b(x)$, $w \geqq 0$ on $y = h_{\tau_0}(x)$, $w(x_0) = 0$, and as before we get contradiction at the point x_0. In the second case we consider the same w, with $\bar{u}$, u_{τ_0} restricted to $(x, y) = \{x \in \bar{D}, b(x) \leqq y \leqq h_{\tau_0}(x)\}$. Again we get the same contradiction at x_0.

THEOREM 5.4. *Problem* II *can have at most one solution* (*assuming* $h(\theta)$ *satisfying interior sphere condition*).

Proof. This time we introduce contraction of the solution by considering $u_a = u(ar, \theta) = u(ax, ay)$, $h_a = h/a$, $a > 1$. Clearly, $\Delta u_a = 0$. By contracting one of the free surfaces, until it is inside the other touching it at some point x_0, we get the same contradiction at x_0 as in the previous theorem. (Again, if the surfaces intersect, contract either one; if one is outside the other, contract the outside one. Also, notice that $\partial u_a/\partial n|_{h=h_a} = -a < -1$, as is clear from (3.5).)

6. General noncoercive problems. We discuss the problem ($x \in R^n$, $0 \leqq y \leqq 1$)

$$
\begin{aligned}
&u_y + \sum_{|\alpha| \leqq k} a_\alpha D^\alpha u = \varepsilon g(\varepsilon, D^k u, D^k h, D^k b), && y = 1,\\
&\Delta u = \varepsilon f(\varepsilon, D^2 u, D^2 h, D^2 b), && 0 < y < 1,\\
&u_y + \sum_{|\alpha| \leqq l} b_\alpha D^\alpha u = b(x), && y = 0.
\end{aligned} \tag{6.1}
$$

Here $h = u(x, 1)$, g, f, and b are given functions, 2π periodic in each variable x_i, $i = 1, \cdots, n$; $\alpha = (\alpha_1, \cdots, \alpha_n, 0)$, ε, a_α, b_α are constants, and k, l are integers whose magnitudes are not restricted. We are looking for 2π periodic in each x_i solution $u(x, y)$, assuming ε is small.

Solving a problem of type (6.1) was the key ingredient in solving Problems I and II, as well as in Shinbrot's proof of existence of water waves in three dimensions. If the boundary condition at $y = 1$ is coercive, i.e., satisfies the Lopatinski-Schapiro condition, then one should be able to prove existence based on Schauder's estimates, as we did for Problem I. In particular, in Shinbrot's paper one has the boundary operators (with $u = u(x, y, z)$) $u_y - \tau(u_{yxx} + u_{yzz}) + F u_{xx}$ at $y = 1$, and u_y at $y = 0$, which are both coercive. Hence, it appears that Schauder's estimates can be used, considerably simplifying the proof.

Using Λ^m spaces one can treat more general problems, including noncoercive ones. In particular, we have the following theorem, whose proof is similar to that of Theorem 5.2.

THEOREM 6.1. *Assume the following estimate for the problem* (6.1) (*with* $g = g(x)$, $f = f(x, y)$)

$$
\|u\|_m + \overline{\|u\|}_m \leqq c(\|f\|_{m-2} + \|g\|_{m-k} + \|b\|_m),
$$

with integer $m \geqq \max(2, k)$. *Assume that the functions* g *and* f *are analytic in their arguments and small if either* ε *or* $\|b\|_m$, $\|u\|_m$, $\|h\|_m$ *are sufficiently small. Then for* $\varepsilon \|b\|_m$ *sufficiently small the problem* (6.1) *has a solution.*

Example. In Shinbrot's water wave model assume that the surface tension $\tau = 0$, and the gravity is pointing up, i.e., g and $F = U^2/g$ are negative numbers, say $F = -f$, $f \geqq 0$. Assuming for simplicity $u = 0$ at $y = 0$, we consider the problem (different from the one in [10])

$$
\begin{aligned}
&u_y - f u_{xx} = \varepsilon g(\varepsilon, Du, Dh, Db), && y = 1,\\
&\Delta u = \varepsilon f(\varepsilon, D^2 u, D^2 h, D^2 b), && 0 < y < 1,\\
&u = 0, && y = 0.
\end{aligned} \tag{6.2}
$$

Here $u = u(x, y, z)$, $h = u(x, 1, z)$. The boundary condition at $y = 1$ is noncoercive (see [7]); hence Schauder's estimates are not valid for (6.2). However, by an argument similar to that of Lemma 4.2 we can estimate (with $g = g(x, z)$, $f = f(x, y, z)$)

$$\|u\|_{m+2} + |\!|\!|u|\!|\!|_{m+2} \leqq c(\|f\|_m + \|g\|_{m+1}),$$

and Theorem 6.1 applies, giving existence for (6.2). (We do not know any other way to prove existence for (6.2).)

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