REFERENCE

[1] A. D. BANDRAUK, *Molecules in Laser Fields*, Marcel Dekker, New York, 1994.

Average Temperature in a Reaction-Diffusion Process

Problem 97-8, *by* PHILIP KORMAN (University of Cincinnati).

The boundary value problem

$$(1) \qquad u'' + u^3 = 0 \quad \text{for} \quad 0 < x < L, \qquad u(0) = u(L) = 0$$

has a unique positive solution, $u(x) > 0$ for $x \in (0, L)$ (as can be seen by phase-plane analysis, for example).

(i) Show that for any L this solution satisfies

$$(2) \qquad \int_0^L u(x)\,dx = \frac{\pi}{\sqrt{2}}.$$

The function $u(x)$ represents the steady-state temperature for the corresponding reaction-diffusion equation with the reaction term $f(u) = u^3$. Formula (2) allows one to compute the average temperature.

(ii) Generalize for the reaction term $f(u) = u^p$ with any real $p > 1$.

Solution

Following [1] and [3], set $y = \int_0^x u(s)\,ds$, and then $h = u^2$. The problem (1) transforms to

$$(3) \qquad h'' + 2h = 0\,, \quad \text{for } 0 < y < R\,, \qquad h(0) = h(R) = 0\,,$$

where

$$(4) \qquad R = \int_0^L u(s)\,ds\,.$$

A solution of (3) satisfying $h(0) = 0$ is $h(y) = c \sin \sqrt{2}y$, where c is an arbitrary constant. Condition $h(R) = 0$ combined with the positivity of $h(y)$ implies that $\sqrt{2}R = \pi$, and by (4) the proof of (2) follows.

Later this argument was generalized in [2].

1. T.F. Chen, H.A. Levine, and P.E. Sacks, Analysis of a convective reactiondiffusion equation, *Nonlinear Analysis* **12** (1988), 1349-1370.

2. P. Korman and Y. Li, Generalized averages for solutions of two-point Dirichlet problems, *J. Math. Anal. Appl.* **239** (1999), no. 2, 478-484.

3. H.A. Levine, L.E. Payne, P.E. Sacks, and B. Straughan, Analysis of a convective reaction diffusion equation II, *SIAM J. Math. Anal.* **20** (1989), 133-147.