# On separated solutions of logistic population equation with harvesting 

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#### Abstract

We provide a surprising answer to a question raised in S. Ahmad and A.C. Lazer [2], and extend the results of that paper.


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## 1 An example

If one considers the logistic population model

$$
\begin{equation*}
z^{\prime}(t)=a(t) z(t)-z^{2}(t), \tag{1.1}
\end{equation*}
$$

with $a(t)$ continuous, positive and periodic function, of period $T$, then by a straightforward integration of this Bernoulli's equation one shows that there exists a unique positive solution $z_{0}(t)$ of period $T$, which attracts all other positive solutions as $t \rightarrow \infty$, see e.g., M.N. Nkashama [5]. The problem also has the trivial solution $z=0$. To account for harvesting, one may consider the model

$$
\begin{equation*}
z^{\prime}(t)=a(t) z(t)-z^{2}(t)-k \gamma(t) \tag{1.2}
\end{equation*}
$$

where $\gamma(t)$ is also a continuous, positive and periodic function, of period $T$, and $k>0$ is a parameter. It was shown in [1] and [4] that there exists a $\bar{k}$, so that for $0<k<\bar{k}$ the problem (1.2) has exactly two positive solutions of period $T$, exactly one positive $T$-periodic solution at $k=\bar{k}$, and no $T$ periodic solutions for $k>\bar{k}$. Moreover, one can show that there is a curve
of solutions beginning with $z_{0}(t)$ at $k=0$ which is decreasing in $k$, and a curve of solutions beginning with $z=0$ at $k=0$ which is increasing in $k$. At $k=\bar{k}$ these solutions coincide, and then disappear for $k>\bar{k}$.

In a very interesting recent paper S. Ahmad and A.C. Lazer [2] studied the equation (1.2) without the periodicity assumption on $a(t)$ and $\gamma(t)$. They introduced the key concept of separated solutions to take place of the periodic ones. Two solutions $z_{1}(t)>z_{2}(t)$ are called separated if

$$
\int_{0}^{\infty}\left[z_{1}(t)-z_{2}(t)\right] d t=\infty .
$$

They proved the following result (among a number of other results).
Theorem 1.1 ([2]) Let the functions a $(t)$ and $\gamma(t)$ be continuous and bounded by positive constants from above and from below on $[0, \infty)$. Then there exists a critical $\bar{k}$, so that for $0<k<\bar{k}$ the problem (1.2) has two separated positive solutions. At $k=\bar{k}$ there exists a positive bounded solution, while for $k>\bar{k}$ there are no bounded positive solutions.

The authors of [2] asked if it is possible for two separated solutions to exist at $k=\bar{k}$. Our next example shows that the answer is yes, which appears to be counter-intuitive.

We consider the equation

$$
z^{\prime}(t)=z(t)-z^{2}(t)-\frac{1}{4} k,
$$

i.e., $a(t)=1, \gamma(t)=\frac{1}{4}$. When $0<k<1$ this equations has two constant solutions, which are the roots of the quadratic equation $z-z^{2}-\frac{1}{4} k=0$. (By [2] there are no solutions separated from both of the constant ones.) At $k=1$ there are two separated solutions $z=\frac{1}{2}$ and $z=\frac{1}{2}+\frac{1}{t+1}$, while for $k>1$ all solutions go to $-\infty$ in finite time, as can be seen by writing this equation in the form

$$
z^{\prime}(t)=-\frac{1}{4}(2 z-1)^{2}-\frac{1}{4}(k-1) .
$$

Here $\bar{k}=1$.

## 2 Separated from zero solution of the logistic equation

We consider now the logistic model $(t \geq 0)$

$$
\begin{equation*}
z^{\prime}(t)=a(t) z(t)-b(t) z^{2}(t), \quad z(0)>0 . \tag{2.1}
\end{equation*}
$$

There is a zero solution $z=0$. Any solution of (2.1) is positive, and by definition it is separated from zero if $\int_{0}^{\infty} z(t) d t=\infty$.

Proposition 1 Let the functions $a(t)$ and $b(t)$ be continuous and satisfy $a(t) \leq A, 0<b_{1}<b(t)<b_{2}$ on $[0, \infty)$, with positive constants $A, b_{1}, b_{2}$. Then any solution of (2.1) is bounded and separated from zero if and only if

$$
\begin{equation*}
\int_{0}^{\infty} b(t) e^{\int_{0}^{t} a(s) d s} d t=\infty \tag{2.2}
\end{equation*}
$$

Proof: Setting $1 / z=v$ and $\mu(t)=e^{\int_{0}^{t} a(s) d s}$, we integrate (2.1) to obtain (here $c=\frac{1}{z(0)}$ )

$$
\begin{equation*}
z(t)=\frac{\mu(t)}{c+\int_{0}^{t} b(s) \mu(s) d s}=\frac{1}{b(t)} \frac{g^{\prime}(t)}{g(t)}, \tag{2.3}
\end{equation*}
$$

with $g(t)=c+\int_{0}^{t} b(s) \mu(s) d s$. Then

$$
\frac{1}{b_{2}}(\ln g(\infty)-\ln g(0)) \leq \int_{0}^{\infty} z(t) d t \leq \frac{1}{b_{1}}(\ln g(\infty)-\ln g(0)),
$$

and the proof follows.

Proposition 2 Let the functions $a(t)$ and $b(t)$ be continuous and satisfy $a(t) \leq A, 0<b_{1}<b(t)<b_{2}$ on $[0, \infty)$, with positive constants $A, b_{1}, b_{2}$. Assume that $J \equiv \int_{0}^{\infty} b(t) e^{\int_{0}^{t} a(s) d s} d t<\infty$, and $a(t) \leq 0$ for large $t$. Then all solutions of (2.1) tend to zero as $t \rightarrow \infty$.

Proof: We have $\mu^{\prime}=a(t) \mu \leq 0$ for large $t$. Since $J<\infty$, it follows that $\mu(t) \rightarrow 0$ as $t \rightarrow \infty$. Then $z(t) \rightarrow 0$ by (2.3).

The situation is different in case of negative initial data:

$$
\begin{equation*}
z^{\prime}(t)=a(t) z(t)-b(t) z^{2}(t), \quad z(0)<0 . \tag{2.4}
\end{equation*}
$$

Proposition 3 Case 1. If the condition (2.2) holds, then all solutions of (2.4) go to $-\infty$ in finite time.

Case 2. If

$$
J \equiv \int_{0}^{\infty} b(t) e^{\int_{0}^{t} a(s) d s} d t<\infty
$$

then solutions with $z(0)<-\frac{1}{J}$ go to $-\infty$ in finite time, while solutions with $z(0) \in\left(0,-\frac{1}{J}\right]$ exist for all $t>0$, and the solution with $z(0)=-\frac{1}{J}$ is separated from zero. Moreover, under the additional assumptions that $a(t) \leq 0$ for large $t$ and $\lim _{t \rightarrow \infty} a(t)=0$, solutions with $z(0) \in\left(0,-\frac{1}{J}\right]$ tend to zero as $t \rightarrow \infty$.

Proof: $\quad$ Solutions of (2.4) are given by (2.3), with $c=\frac{1}{z(0)}<0$, from which the claims on blow up and global existence follow. When $z(0)=-\frac{1}{J}$, we have

$$
\begin{equation*}
z(t)=\frac{\mu(t)}{-J+\int_{0}^{t} b(s) \mu(s) d s} \tag{2.5}
\end{equation*}
$$

and then

$$
\begin{gathered}
\int_{0}^{\infty} z(t) d t \leq \frac{1}{b_{2}} \int_{0}^{\infty} \frac{b(t) \mu(t)}{-J+\int_{0}^{t} b(s) \mu(s) d s} d t \\
\left.\quad=\frac{1}{b_{2}} \ln \right\rvert\,-J+\int_{0}^{t} b(s) \mu(s) d s \|_{0}^{\infty}=-\infty,
\end{gathered}
$$

so that $\int_{0}^{\infty} z(t) d t=-\infty$, and $z(t)$ is separated from zero. The proof that $z(t) \rightarrow 0$ follows as in Proposition 2 in case $z(0) \in\left(0,-\frac{1}{J}\right)$, and by L'Hospital's rule for $z(0)=-\frac{1}{J}$.

If $p(t)$ is any particular solution of (1.2), defined for $t \in[0, \infty)$, and $z(t)$ is any other solution of (1.2), then $v(t)=z(t)-p(t)$ satisfies Bernoulli's equation

$$
\begin{equation*}
v^{\prime}=[a(t)-2 p(t)] v-v^{2} . \tag{2.6}
\end{equation*}
$$

It follows that any other solution of (1.2), which is larger than $p(t)$, is separated from $p(t)$ if and only if

$$
\begin{equation*}
\int_{0}^{\infty} e^{\int_{0}^{t}[a(s)-2 p(s)] d s} d t=\infty . \tag{2.7}
\end{equation*}
$$

The anonymous reviewer of this paper posed the following question.
Question. Are there always separated solutions at $\bar{k}$ ?
We shall show that the answer is affirmative. We consider first the periodic case, where the picture is simpler.

Proposition 4 In the conditions of the Theorem 1.1 (from [2]), assume additionally that $a(t)$ and $\gamma(t)$ are $T$-periodic, i.e., $a(t+T)=a(t)$ and $\gamma(t+T)=\gamma(t)$ for all $t \in[0, \infty)$, and some $T>0$. Let $p(t)$ be the unique $T$-periodic solution of (1.2) at $k=\bar{k}$. Then any other solution of (1.2), which is larger than $p(t)$, is bounded on $[0, \infty)$, and is separated from $p(t)$.

Proof: As we mentioned above, there exists a $\bar{k}$, so that for $0<k<\bar{k}$ the equation (1.2) has exactly two positive solutions of period $T$, exactly one positive $T$-periodic solution at $k=\bar{k}$, and no $T$-periodic solutions for $k>\bar{k}$. So that $(\bar{k}, p(t))$ is a "turning point" of $T$-periodic solutions of (1.2). It follows that the corresponding linearized problem

$$
\begin{equation*}
w^{\prime}(t)=[a(t)-2 p(t)] w(t), \quad w(t+T)=w(t) \tag{2.8}
\end{equation*}
$$

has non-trivial solutions, which happens if and only if

$$
\begin{equation*}
\int_{0}^{T}[a(t)-2 p(t)] d t=0 \tag{2.9}
\end{equation*}
$$

(If (2.8) had only the trivial solution, the equation (1.2) would have $T$ periodic solution for $k>\bar{k}$, by the implicit function theorem, see e.g., [3] for similar arguments.) We claim that there is a constant $\alpha$, such that

$$
\int_{0}^{t}[a(t)-2 p(t)] d t \geq \alpha, \text { for all } t>0
$$

Indeed, for any $t>0$, we can find an integer $n \geq 0$, so that $n T \leq t \leq$ $(n+1) T$. Using (2.9),

$$
\int_{0}^{t}[a(t)-2 p(t)] d t=\int_{n T}^{t}[a(t)-2 p(t)] d t
$$

and the integral of the periodic function $a(t)-2 p(t)$ over an interval of length $<T$, is bounded below by some constant $\alpha$. Then (2.7) holds, and the proof follows.

We now consider the general case.
Theorem 2.1 Let $p(t)$ be a positive bounded solution of (1.2) at $k=\bar{k}$ from the Theorem 1.1. Calculate $I=\int_{0}^{\infty} e^{\int_{0}^{t}(a(s)-2 p(s)) d s} d t$.

Case 1. $I=\infty$. Then all solutions of (1.2), lying above $p(t)$ (i.e., z $(0)>$ $p(0))$ are bounded for all $t>0$ and separated from $p(t)$, while all solutions below $p(t)$ go to $-\infty$ in finite time.

Case 2. $I<\infty$, $a(t)-2 p(t) \leq 0$ for large $t$ and $\lim _{t \rightarrow \infty}(a(t)-2 p(t))=0$. Then for $z(0) \in\left[p(0)-\frac{1}{I}, \infty\right)$ solutions tend to $p(t)$, and at $z(0)=p(0)-\frac{1}{I}$ one has a positive bounded solution, call it $p_{1}(t)$, which is separated from $p(t)$ (and tending to $p(t)$ ).

Proof: $\quad$ Since $v(t)=z(t)-p(t)$ satisfies (2.6), all of the claims, except for positivity of $p_{1}(t)$ follow by the Propositions 1,2 and 3 . If $p_{1}(t)$ was negative at some point, we could find a point $t_{0}$ at which $p_{1}\left(t_{0}\right)=0$ and $p_{1}^{\prime}\left(t_{0}\right) \geq 0$, since $p_{1}(t)$ tends to $p(t)>0$. But then we have a contradiction in the equation (1.2) at $t=t_{0}$.

Both of the cases described in this theorem actually occur. The periodic equations, considered above, provide examples for the first case. The second case occurs in the following example.

Example Consider

$$
\begin{equation*}
z^{\prime}=z-z^{2}-k\left(\frac{1}{4}-\frac{2}{(t+5)^{2}}\right) \tag{2.10}
\end{equation*}
$$

with $f(t)=\frac{1}{4}-\frac{2}{(t+5)^{2}}>0$. At $k=1$ there is a bounded solution $p(t)=$ $\frac{1}{2}+\frac{2}{t+5}$. To see that $\bar{k}=1$, we write this equation as

$$
z^{\prime}=-\left(z-\frac{1}{2}\right)^{2}-\frac{k-1}{4}+\frac{2 k}{(t+5)^{2}}
$$

Then for $k>1$, the right hand side is smaller than, say $-\frac{k-1}{8}$ for $t$ large, and hence all solutions go to $-\infty$ in finite time. Here $a(t)=1, a(t)-2 p(t)=$ $-\frac{4}{t+5}$. Compute $\mu(t)=e^{-\int_{0}^{t} \frac{4}{s+5} d s}=\frac{5^{4}}{(t+5)^{4}}, \int_{0}^{\infty} \mu(t) d t=\frac{5}{3}$. We have $p_{1}(t)=p(t)+v(t)$, where by $(2.5)$

$$
v(t)=\frac{\mu(t)}{-\int_{0}^{\infty} \mu(s) d s+\int_{0}^{t} \mu(s) d s}=\frac{\mu(t)}{-\int_{t}^{\infty} \mu(s) d s}=-\frac{3}{t+5}
$$

Hence, at $k=1, p_{1}(t)=\frac{1}{2}-\frac{1}{t+5}$ is the bounded positive solution, separated from $p(t)$. At $k=1$, any solution of (2.10), with $z(0) \in\left[\frac{3}{10}, \infty\right)$ tends to $p(t)$ as $t \rightarrow \infty$, while solutions with $z(0)<\frac{3}{10}$ go to $-\infty$ in finite time $\left(p_{1}(0)=\frac{3}{10}\right)$.
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