Curves of equiharmonic solutions, and solvability of elliptic systems

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Abstract

We study solutions of the system

$$\Delta u + kf(v) = h_1(x), \quad x \in \Omega, \quad u = 0 \text{ for } x \in \partial\Omega$$
$$\Delta v + kg(u) = h_2(x), \quad x \in \Omega, \quad v = 0 \text{ for } x \in \partial\Omega$$

on a bounded smooth domain $\Omega \subset R^n$, with given functions $f(t)$, $g(t) \in C^2(R)$, and $h_1(x)$, $h_2(x) \in L^2(\Omega)$. When the parameter $k = 0$, the problem is linear, and uniquely solvable. We continue the solutions in k on curves of equiharmonic solutions. We show that in the absence of resonance the problem is solvable for any $h_1(x)$, $h_2(x) \in L^2(\Omega)$, while in case of resonance we develop necessary and sufficient conditions for existence of solutions of E.M. Landesman and A.C. Lazer [12] type, and sufficient conditions for existence of solutions of D.G. de Figueiredo and W.-M. Ni [7] type. Our approach is constructive, and computationally efficient.

Key words: Curves of equiharmonic solutions, resonance.

AMS subject classification: 35J60.

1 Introduction

We study existence of solutions for a semilinear system

$$\begin{aligned} &\Delta u + kf(v) = h_1(x), \quad x \in \Omega, \quad u = 0 \text{ for } x \in \partial\Omega \\ &\Delta v + kg(u) = h_2(x), \quad x \in \Omega, \quad v = 0 \text{ for } x \in \partial\Omega \end{aligned} \tag{1.1}$$

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on a bounded smooth domain $\Omega \subset R^n$. Here the functions $h_1(x)$, $h_2(x) \in L^2(\Omega)$ and $f(t)$, $g(t) \in C^2(R)$ are given, $k > 0$ is a parameter. There is considerable recent interest in systems of this type, see e.g., the recent surveys of D.G. de Figueiredo [6] and B. Ruf [14]. The techniques used so far, are mostly variational and involving the degree theory. We shall obtain solutions of this problem by continuation in k. When $k = 0$, the problem is linear. It has a unique solution, which can be computed by using Fourier series of the form $u(x) = \Sigma_{k=1}^{\infty} u_k \varphi_k$, $v(x) = \Sigma_{k=1}^{\infty} v_k \varphi_k$, where φ_k is the k-th eigenfunction of the Laplacian on Ω, and λ_k is the corresponding eigenvalue. We now continue in k, looking for a solution triple (k, u, v), or $(u, v) = (u(x, k), u(x, k))$. At a generic point (k, u, v), the implicit function theorem applies, allowing the continuation in k. These are the *regular* points, at which the corresponding linearized system has only the trivial solution. So until a *singular* point is encountered, we have a solution curve $(u, v) = (u(x, k), u(x, k))$. In case of one equation, we have shown in [10] that one can continue forward in k, if one stays on a curve of equiharmonic solutions. We proceed similarly here. We decompose the solution $u = \xi_1 \varphi_1 + U(\xi_1, \eta_1)$, $v = \eta_1 \varphi_1 + V(\xi_1, \eta_1)$, and the forcing terms $h_1(x) = \mu_1^0 \varphi_1 + e_1(x)$, $h_2(x) = \nu_1^0 \varphi_1 + e_2(x)$, with the functions $U(\xi_1, \eta_1)$, $V(\xi_1, \eta_1)$, $e_1(x)$ and $e_2(x)$ orthogonal to $\varphi_1(x)$. When we do continuation in k, we keep the first harmonics (ξ_1, η_1) fixed, but in return allow for μ_1 and ν_1 to vary in k, i.e., we let $(\mu_1, \nu_1) = (\mu_1(k), \nu_1(k))$, with $(\mu_1(0), \nu_1(0)) = (\mu_1^0, \nu_1^0)$. This gives us a curve of equiharmonic solutions. Under proper conditions, this process allows us to continue solutions forward in k, on a smooth curve through any point, including the nasty singular points. (I.e., a singular point of (2.2) is no longer singular, if one stays on an equiharmonic curve.) One can think of the curves of equiharmonic solutions as highways taking us through mountains and swamps.

So, we continue solutions with fixed first harmonics (ξ_1, η_1) from $k = 0$ until the desired value of k. At k, we obtain solution of the modified problem, with $\mu_1 = \mu_1(k)$, $\nu_1 = \nu_1(k)$. We now vary the point (ξ_1, η_1), to obtain a solution of the original problem (1.1), i.e., $(\mu_1(k), \nu_1(k)) = (\mu_1^0, \nu_1^0)$. Any solution of (1.1) can be constructed this way, and we define *solution manifold* to be the set of all points (μ_1^0, ν_1^0), for which solution exists. We show that the pair (ξ_1, η_1) gives a global parameterization of the solution manifold, i.e., the pair (ξ_1, η_1) uniquely identifies the solution quadruple (μ_1, ν_1, u, v) solving (1.1). We consider in detail the case when f and g are sums of linear and bounded functions. We show that in non-resonant case the solution manifold is the whole of R^2, while in case of resonance we develop necessary and sufficient conditions for existence of solutions of E.M. Landesman and A.C. Lazer [12] type, and sufficient conditions

for existence of solutions of D.G. de Figueiredo and W.-M. Ni [7] type.

Our approach is suitable for efficient numerical computation of solutions. It is easy to implement numerically the continuation process, first in k, then in (ξ_1, η_1) (by the predictor-corrector, or basically the Newton's method), since our results guarantee that this continuation will proceed on simple smooth curves (in particular, there is no turning back in parameters). We had performed such computations in [9].

2 Preliminary results

Recall that on a bounded smooth domain $\Omega \subset R^n$ the eigenvalue problem

$$\Delta u + \lambda u = 0 \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has an infinite sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$, where we repeat each eigenvalue according to its multiplicity, and the corresponding eigenfunctions we denote by φ_k. These eigenfunctions φ_k form an orthogonal basis of $L^2(\Omega)$, i.e., any $f(x) \in L^2(\Omega)$ can be written as $f(x) = \Sigma_{k=1}^{\infty} a_k \varphi_k$, with the series convergent in $L^2(\Omega)$, see e.g. L. Evans [5]. We shall normalize $||\varphi_k||_{L^2(\Omega)} = 1$ for all k.

Lemma 2.1 *Assume that $u(x) \in H_0^1(\Omega)$, and $u(x) = \Sigma_{k=2}^{\infty} a_k \varphi_k$. Then*

$$\int_\Omega |\nabla u|^2 \, dx \geq \lambda_2 \int_\Omega u^2 \, dx.$$

Proof: Since $u(x)$ is orthogonal to φ_1, the proof follows by the variational characterization of λ_2. $\diamondsuit$

Lemma 2.2 *Assume that $u(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, and $u(x) = \Sigma_{k=2}^{\infty} a_k \varphi_k$. Then*

$$\int_\Omega (\Delta u)^2 \, dx \geq \lambda_2 \int_\Omega |\nabla u|^2 \, dx.$$

Proof: We have, using Lemma 2.1,

$$\begin{aligned} \textstyle\int_\Omega |\nabla u|^2 \, dx &= -\textstyle\int_\Omega u \Delta u \, dx \leq \left(\int_\Omega u^2 \, dx\right)^{1/2} \left(\int_\Omega (\Delta u)^2 \, dx\right)^{1/2} \\ &\leq \tfrac{1}{\sqrt{\lambda_2}} \left(\textstyle\int_\Omega |\nabla u|^2 \, dx\right)^{1/2} \left(\int_\Omega (\Delta u)^2 \, dx\right)^{1/2}, \end{aligned}$$

and the proof follows. $\diamondsuit$

We consider the following linear system: find $(w(x), z(x), \mu_1, \nu_1)$ solving

$$\begin{aligned} \Delta w + a(x)z = \mu_1\varphi_1, \quad x \in \Omega, \quad w = 0 \ \text{ for } x \in \partial\Omega \\ \Delta z + b(x)w = \nu_1\varphi_1, \quad x \in \Omega, \quad z = 0 \ \text{ for } x \in \partial\Omega \\ \textstyle\int_\Omega w(x)\varphi_1(x)\,dx = \int_\Omega z(x)\varphi_1(x)\,dx = 0\,, \end{aligned} \tag{2.1}$$

where $a(x) \in C(\bar{\Omega})$ and $b(x) \in C(\bar{\Omega})$ are given continuous functions, while μ_1 and ν_1 are unknown constants. We assume throughout this section that solutions of (2.1) (and of (2.4) below) satisfy $w, z \in H^2(\Omega) \cap H_0^1(\Omega)$. Define

$$A = \max_{x\in\bar{\Omega}} |a(x) + b(x)|\,. \tag{2.2}$$

Lemma 2.3 *Assume that*

$$A < 2\lambda_2\,. \tag{2.3}$$

Then the only solution of the problem (2.1) is $w(x) = z(x) \equiv 0$, $\mu_1 = \nu_1 = 0$.

Proof: Multiply the first equation in (2.1) by w, the second one by z, add and integrate:

$$\int_\Omega (|\nabla w|^2 + |\nabla z|^2)\,dx = \int_\Omega (a(x) + b(x))zw\,dx\,.$$

The quantity on the left is bounded from below by $\lambda_2 \int_\Omega (w^2 + z^2)\,dx$, in view of Lemma 2.1, while the integral on the right is bounded from above by $\frac{A}{2}\int_\Omega (w^2 + z^2)\,dx$. It follows that $w(x) = z(x) \equiv 0$, and then from (2.1), $\mu_1 = \nu_1 = 0$. $\diamondsuit$

We need to consider another linear system

$$\begin{aligned} \Delta w + a(x)z = 0, \quad x \in \Omega, \quad w = 0 \ \text{ for } x \in \partial\Omega \\ \Delta z + b(x)w = 0, \quad x \in \Omega, \quad z = 0 \ \text{ for } x \in \partial\Omega\,. \end{aligned} \tag{2.4}$$

Lemma 2.4 *Assume that (with A defined by (2.2))*

$$A < 2\sqrt{\lambda_1\lambda_2}\,. \tag{2.5}$$

Then the solution space of the problem (2.4) is one dimensional.

Proof: Assume on the contrary that (w, z) and (w_1, z_1) are two non-trivial solutions of (2.4), which are not constant multiples of one another. Claim: we can find a non-trivial solution of (2.4), with $\int_\Omega w\varphi_1\,dx = 0$, i.e., with the first component orthogonal to φ_1. Indeed, consider $a \equiv \int_\Omega w\varphi_1\,dx$, and $b \equiv \int_\Omega w_1\varphi_1\,dx$. If either a or b is zero, there is nothing to prove. Otherwise, consider

$(\frac{1}{a}w - \frac{1}{b}w_1, \frac{1}{a}z - \frac{1}{b}z_1)$, which is a non-trivial solution of (2.4). Its first component is orthogonal to φ_1.

So let (w, z) be a non-trivial solution of (2.4), with $\int_\Omega w\varphi_1\, dx = 0$. Using Lemma 2.1 and the Poincare's inequality, we obtain from the corresponding equations in (2.4)

$$\int_\Omega a(x)\, w\, z\, dx = \int_\Omega |\nabla w|^2\, dx \geq \lambda_2 \int_\Omega w^2\, dx\,;$$

$$\int_\Omega b(x)\, w\, z\, dx = \int_\Omega |\nabla z|^2\, dx \geq \lambda_1 \int_\Omega z^2\, dx\,.$$

Adding,

$$A \int_\Omega |w\, z|\, dx \geq \int_\Omega (a(x) + b(x))\, w\, z\, dx \geq \lambda_2 \int_\Omega w^2\, dx + \lambda_1 \int_\Omega z^2\, dx\,. \tag{2.6}$$

We have

$$A \int_\Omega |w\, z|\, dx = \int_\Omega \sqrt{2\lambda_2}|w|\, \frac{A}{\sqrt{2\lambda_2}}|z|\, dx \leq \lambda_2 \int_\Omega w^2\, dx + \frac{A^2}{4\lambda_2} \int_\Omega z^2\, dx\,.$$

Using this in (2.6),

$$\frac{A^2}{4\lambda_2} \int_\Omega z^2\, dx \;\geq \lambda_1 \int_\Omega z^2\, dx$$

which implies that $A^2 \geq 4\lambda_1\lambda_2$, contrary to our assumption (if $\int_\Omega z^2\, dx = 0$, then $w = z = 0$, and the solution is trivial). $\diamondsuit$

We consider the non-homogeneous version of (2.4)

$$\begin{aligned} &\Delta w + a(x)z = h_1(x), \quad x \in \Omega, \quad w = 0 \;\text{ for } x \in \partial\Omega \\ &\Delta z + b(x)w = h_2(x), \quad x \in \Omega, \quad z = 0 \;\text{ for } x \in \partial\Omega\,. \end{aligned} \tag{2.7}$$

with given $h_1(x), h_2(x) \in L^2(\Omega)$, and as before, $w, z \in H^2(\Omega) \cap H_0^1(\Omega)$. We shall need the following version of the Fredholm alternative.

Lemma 2.5 *If the homogeneous problem (2.4) has only the trivial solution, then the problem (2.7) is solvable for any $h_1(x)$ and $h_2(x)$. If the homogeneous problem (2.4) has a one-dimensional solution set spanned by $(W(x), Z(x))$, then the problem (2.7) is solvable if and only if*

$$\int_\Omega (h_1(x)Z(x) + h_2(x)W(x))\, dx = 0\,. \tag{2.8}$$

Proof: Multiply the first equation in (2.7) by Z, and subtract from that the second equation in (2.4) (written for $(W(x), Z(x))$) by w, then integrate:

$$\int_\Omega (a(x)zZ - b(x)wW)\, dx = \int_\Omega h_1(x)Z(x)\, dx\,.$$

Similarly, from the second equation in (2.7) and the first equation in (2.4)

$$-\int_\Omega (a(x)zZ - b(x)wW)\, dx = \int_\Omega h_2(x)W(x)\, dx\,.$$

Adding, we see that the condition (2.8) is necessary for existence of solutions.

By the elliptic theory, the solution map of (2.7) $(h_1, h_2) \to (w, z)$ is a Fredholm operator of index zero. Hence, this map is onto, if the kernel is zero. If the kernel is one-dimensional, the range has co-dimension one. If the problem (2.7) was not solvable for some right hand sides satisfying (2.8), the range would have co-dimension greater than one, which is impossible. $\diamondsuit$

3 Continuation of solutions in k, with first harmonics fixed

We now consider a nonlinear problem, depending on a parameter k

$$\begin{aligned} \Delta u + kf(v) &= \mu_1\varphi_1 + e_1(x), \quad x \in \Omega, \quad u = 0 \ \text{ for } x \in \partial\Omega \\ \Delta v + kg(u) &= \nu_1\varphi_1 + e_2(x), \quad x \in \Omega, \quad v = 0 \ \text{ for } x \in \partial\Omega \\ \textstyle\int_\Omega u(x)\varphi_1(x)\, dx &= \xi_1; \quad \textstyle\int_\Omega v(x)\varphi_1(x)\, dx = \eta_1\,. \end{aligned} \tag{3.1}$$

The functions f and g are assumed to be of the form

$$f(t) = \lambda t + b_1(t), \quad g(t) = \bar\lambda t + b_2(t)\,, \tag{3.2}$$

with positive constants λ and $\bar\lambda$, and bounded functions $b_1(t)$ and $b_1(t)$ of class $C^1(R)$. We assume that solutions of (3.1) satisfy $u, v \in H^2(\Omega) \cap H^1_0(\Omega)$.
In accordance with the third line of (3.1), we decompose solution in the form $u(x) = \xi_1\varphi_1 + U$, $v(x) = \eta_1\varphi_1 + V$, with both $U = U(\xi_1, \eta_1)$ and $V = V(\xi_1, \eta_1)$ orthogonal to φ_1. Our goal is to continue the solution of (3.1) in k, while keeping ξ_1 and η_1 fixed.

Theorem 3.1 *Assume that $e_1(x)$ and $e_2(x)$ are given functions in $L^2(\Omega)$, with $\int_\Omega e_1(x)\varphi_1(x)\, dx = \int_\Omega e_2(x)\varphi_1(x)\, dx = 0$, and the following conditions hold, with some constants M_1 and M_2,*

$$|f'(t)| = |\lambda + b_1'(t)| < M_1, \quad |g'(t)| = |\bar\lambda + b_2'(t)| < M_2, \quad \textit{for all } t \in R\,; \tag{3.3}$$

$$kM_1 + kM_2 < 2\sqrt{\lambda_1\lambda_2}\,. \tag{3.4}$$

Then for any pair $(\xi_1, \eta_1) \in R^2$ there exists a unique quadruple (μ_1, ν_1, u, v) solving (3.1).

Proof: Assume first that $\xi_1 = \eta_1 = 0$. Define $H^2_{\mathbf{0}}$ to be the subspace of $H^2(\Omega) \cap H^1_0(\Omega)$ with zero first harmonic:

$$H^2_{\mathbf{0}} = \left\{ u \in H^2(\Omega) \cap H^1_0(\Omega) \mid \int_\Omega u\varphi_1\, dx = 0 \right\}.$$

We recast the system (3.1) in the operator form

$$F(u, v, \mu_1, \nu_1, k) = e(x)\,, \tag{3.5}$$

where $F(u, v, \mu_1, \nu_1, k) = \begin{bmatrix} \Delta u + kf(v) - \mu_1\varphi_1 \\ \Delta v + kg(u) - \nu_1\varphi_1 \end{bmatrix}$, $F : H^2_{\mathbf{0}} \times H^2_{\mathbf{0}} \times R^3 \to L^2(\Omega) \times L^2(\Omega)$, and $e(x) = \begin{bmatrix} e_1(x) \\ e_2(x) \end{bmatrix}$. When $k = \mu_1 = \nu_1 = 0$, the problem (3.5) has a unique solution, which is easily found using Fourier series. We now continue this solution in k, i.e., we solve for (u, v, μ_1, ν_1) as functions of k. Compute the Frechet derivative

$$F_{(u,v,\mu_1,\nu_1)}(u, v, \mu_1, \nu_1, k)(w, z, \mu_1^*, \nu_1^*) = \begin{bmatrix} \Delta w + kf'(v)z - \mu_1^*\varphi_1 \\ \Delta z + kg'(u)w - \nu_1^*\varphi_1 \end{bmatrix}.$$

To apply the implicit function theorem, we need to check that this map is both injective and surjective. In view of the assumptions (3.3) and (3.4), the Lemma 2.3 applies, and hence the only solution of the linearized problem

$$\begin{gathered} \Delta w + kf'(v)z - \mu_1^*\varphi_1 = 0, \quad x \in \Omega, \quad w = 0, \quad \text{for } x \in \partial\Omega \\ \Delta z + kg'(u)w - \nu_1^*\varphi_1 = 0, \quad x \in \Omega, \quad z = 0, \quad \text{for } x \in \partial\Omega \\ \textstyle\int_\Omega w\varphi_1\, dx = \int_\Omega z\varphi_1\, dx = 0 \end{gathered} \tag{3.6}$$

is $(w, z, \mu_1^*, \nu_1^*) = (0, 0, 0, 0)$, proving the injectivity.

Turning to the surjectivity, we need to show that for any $e^*(x) = \begin{bmatrix} e_1^*(x) \\ e_2^*(x) \end{bmatrix} \in L^2(\Omega) \times L^2(\Omega)$ the problem

$$\begin{gathered} \Delta w + kf'(v)z - \mu_1^*\varphi_1 = e_1^*(x), \quad x \in \Omega, \quad w = 0, \quad \text{for } x \in \partial\Omega \\ \Delta z + kg'(u)w - \nu_1^*\varphi_1 = e_2^*(x), \quad x \in \Omega, \quad z = 0, \quad \text{for } x \in \partial\Omega \\ \textstyle\int_\Omega w\varphi_1\, dx = \int_\Omega z\varphi_1\, dx = 0 \end{gathered} \tag{3.7}$$

has a solution (w, z, μ_1^*, ν_1^*). Consider an auxiliary problem

$$\begin{aligned} \Delta w + kf'(v)z &= \mu_1^*\varphi_1 + e_1^*(x), \quad x \in \Omega, \quad w = 0, \quad \text{for } x \in \partial\Omega \\ \Delta z + kg'(u)w &= \nu_1^*\varphi_1 + e_2^*(x), \quad x \in \Omega, \quad z = 0, \quad \text{for } x \in \partial\Omega\,, \end{aligned} \tag{3.8}$$

which differs from (3.7) only in not requiring vanishing first harmonics. Denote $L\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} \Delta w + kf'(v)z \\ \Delta z + kg'(u)w \end{bmatrix}$.

Case 1 The operator L, subject to the zero boundary conditions in (3.8) is invertible. Then we express from (3.8)

$$\begin{bmatrix} w \\ z \end{bmatrix} = \mu_1^* L^{-1}\begin{bmatrix} \varphi_1 \\ 0 \end{bmatrix} + \nu_1^* L^{-1}\begin{bmatrix} 0 \\ \varphi_1 \end{bmatrix} + L^{-1}\begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix}.$$

We now select the constants μ_1^* and ν_1^*, so that the orthogonality conditions in the last line of (3.7) are satisfied. Denote $L^{-1}\begin{bmatrix} \varphi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_{11} \\ \psi_{12} \end{bmatrix}$, $L^{-1}\begin{bmatrix} 0 \\ \varphi_1 \end{bmatrix} = \begin{bmatrix} \psi_{21} \\ \psi_{22} \end{bmatrix}$, and $L^{-1}\begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$. Then we need

$$\begin{aligned} \mu_1^* \textstyle\int_\Omega \psi_{11}\varphi_1\,dx + \nu_1^* \int_\Omega \psi_{21}\varphi_1\,dx &= -\int_\Omega \gamma_1\varphi_1\,dx \\ \mu_1^* \textstyle\int_\Omega \psi_{12}\varphi_1\,dx + \nu_1^* \int_\Omega \psi_{22}\varphi_1\,dx &= -\int_\Omega \gamma_2\varphi_1\,dx\,. \end{aligned}$$

This system has a unique solution, unless the columns of its matrix are proportional, i.e.,

$$\begin{bmatrix} \int_\Omega \psi_{11}\varphi_1\,dx \\ \int_\Omega \psi_{12}\varphi_1\,dx \end{bmatrix} = \theta \begin{bmatrix} \int_\Omega \psi_{21}\varphi_1\,dx \\ \int_\Omega \psi_{22}\varphi_1\,dx \end{bmatrix}$$

for some number θ. I.e., the vector $\begin{bmatrix} \psi_{11} \\ \psi_{12} \end{bmatrix} - \theta\begin{bmatrix} \psi_{21} \\ \psi_{22} \end{bmatrix} \equiv \begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix}$ has both components orthogonal to φ_1. This vector satisfies $\begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix} = L^{-1}\begin{bmatrix} \phi_1 \\ -\theta\phi_1 \end{bmatrix}$, i.e.,

$$\begin{gathered} \Delta\bar{w} + kf'(v)\bar{z} = \varphi_1, \quad x \in \Omega, \quad \bar{w} = 0, \quad \text{for } x \in \partial\Omega \\ \Delta\bar{z} + kg'(u)\bar{w} = -\theta\varphi_1, \quad x \in \Omega, \quad \bar{z} = 0, \quad \text{for } x \in \partial\Omega \\ \textstyle\int_\Omega \bar{w}\varphi_1\,dx = \int_\Omega \bar{z}\varphi_1\,dx = 0\,, \end{gathered}$$

which is impossible by Lemma 2.3 (here $\mu_1 = 1 \neq 0$, $\nu_1 = -\theta$).

Case 2 The operator L, subject to the zero boundary conditions in (3.8) is not invertible. According to Lemma 2.4, the null space of L is one dimensional, spanned by some $\begin{bmatrix} \bar{U} \\ \bar{V} \end{bmatrix}$, and according to the Fredholm alternative, the problem

$$L \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

is solvable if and only if

$$\int_\Omega \left[h_1 \bar{V} + h_2 \bar{U} \right] \, dx = 0 \, .$$

According to the Lemma 2.3, either $\int_\Omega \bar{U} \varphi_1 \, dx \neq 0$ or $\int_\Omega \bar{V} \varphi_1 \, dx \neq 0$ (or both inequalities hold). Let us assume that $\int_\Omega \bar{U} \varphi_1 \, dx \neq 0$. Then by the Fredholm alternative, we can find the constants δ_1 and δ_2, so that $\begin{bmatrix} \varphi_1 \\ \delta_1 \varphi_1 \end{bmatrix} \in R(L)$, and $\begin{bmatrix} e_1^* \\ e_2^* + \delta_2 \varphi_1 \end{bmatrix} \in R(L)$, where $R(L)$ denotes the range of L.

Our goal is to find a constant t, such that the problem

$$\begin{gathered} L \begin{bmatrix} w \\ z \end{bmatrix} = t \begin{bmatrix} \varphi_1 \\ \delta_1 \varphi_1 \end{bmatrix} + \begin{bmatrix} e_1^* \\ e_2^* + \delta_2 \varphi_1 \end{bmatrix} \\ w = z = 0 \ \text{ on } \partial\Omega \\ \textstyle\int_\Omega w \varphi_1 \, dx = \int_\Omega z \varphi_1 \, dx = 0 \end{gathered} \tag{3.9}$$

is solvable. Indeed, then the pair (w, z) gives us a solution of (3.7), corresponding to $\mu_1^* = t$ and $\nu_1^* = t\delta_1 + \delta_2$. We express from (3.9)

$$\begin{bmatrix} w \\ z \end{bmatrix} = t L^{-1} \begin{bmatrix} \varphi_1 \\ \delta_1 \varphi_1 \end{bmatrix} + L^{-1} \begin{bmatrix} e_1^* \\ e_2^* + \delta_2 \varphi_1 \end{bmatrix} + s \begin{bmatrix} \bar{U} \\ \bar{V} \end{bmatrix} ,$$

where s is an arbitrary constant. Denoting $L^{-1} \begin{bmatrix} \varphi_1 \\ \delta_1 \varphi_1 \end{bmatrix} = \begin{bmatrix} \psi_{11} \\ \psi_{12} \end{bmatrix}$, and $L^{-1} \begin{bmatrix} e_1^* \\ e_2^* + \delta_2 \varphi_1 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$, we have

$$\begin{bmatrix} w \\ z \end{bmatrix} = t \begin{bmatrix} \psi_{11} \\ \psi_{12} \end{bmatrix} + s \begin{bmatrix} \bar{U} \\ \bar{V} \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} .$$

To satisfy the last line in (3.9), we need

$$t\int_\Omega \psi_{11}\varphi_1\,dx + s\int_\Omega \bar U\varphi_1\,dx = -\int_\Omega \gamma_1\varphi_1\,dx$$
$$t\int_\Omega \psi_{12}\varphi_1\,dx + s\int_\Omega \bar V\varphi_1\,dx = -\int_\Omega \gamma_2\varphi_1\,dx\,.$$

This system has a unique solution (and we are done), unless the columns of its matrix are proportional, i.e.,

$$\begin{bmatrix} \int_\Omega \psi_{11}\varphi_1\,dx \\ \int_\Omega \psi_{12}\varphi_1\,dx \end{bmatrix} = \theta \begin{bmatrix} \int_\Omega \bar U\varphi_1\,dx \\ \int_\Omega \bar V\varphi_1\,dx \end{bmatrix}$$

for some number θ. I.e., the vector $\begin{bmatrix} \psi_{11} \\ \psi_{12} \end{bmatrix} - \theta \begin{bmatrix} \bar U \\ \bar V \end{bmatrix} \equiv \begin{bmatrix} \bar w \\ \bar z \end{bmatrix}$ has both components orthogonal to φ_1. This vector satisfies $L\begin{bmatrix} \bar w \\ \bar z \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \delta_1\phi_1 \end{bmatrix}$, i.e.,

$$\Delta\bar w + kf'(v)\bar z = \varphi_1,\quad x\in\Omega,\quad \bar w = 0,\quad \text{for } x\in\partial\Omega$$
$$\Delta\bar z + kg'(u)\bar w = \delta_1\varphi_1,\quad x\in\Omega,\quad \bar z = 0,\quad \text{for } x\in\partial\Omega$$
$$\int_\Omega \bar w\varphi_1\,dx = \int_\Omega \bar z\varphi_1\,dx = 0\,,$$

which is impossible by Lemma 2.3 (here $\mu_1 = 1 \neq 0$, $\nu_1 = \delta_1$).

We now consider the case of general first harmonics (ξ_1, η_1), and reduce it to the case $(\xi_1, \eta_1) = (0, 0)$, by setting $\bar u(x) = u(x) - \xi_1\varphi_1$, and $\bar v(x) = v(x) - \eta_1\varphi_1$. Then $(\bar u, \bar v)$ satisfies

$$\Delta\bar u + kf(\bar v + \eta_1\varphi_1) = (\mu_1 + \lambda_1\xi_1)\varphi_1 + e_1(x),\quad \bar u = 0 \ \text{ for } x\in\partial\Omega \tag{3.10}$$
$$\Delta\bar v + kg(\bar u + \xi_1\varphi_1) = (\nu_1 + \lambda_1\eta_1)\varphi_1 + e_2(x),\quad \bar v = 0 \ \text{ for } x\in\partial\Omega$$
$$\int_\Omega \bar u(x)\varphi_1(x)\,dx = \int_\Omega \bar v(x)\varphi_1(x)\,dx = 0\,.$$

Even though this problem is slightly different from (3.1), the bounds on the derivatives of f and g remain the same, so that we can repeat the above argument, and obtain a curve of solutions of (3.10). Then $(u(x,k), v(x,k))$ is a solution curve of the original problem (3.1).

To show that the local solution curve $(u(x,k), v(x,k))$ continues for all k, satisfying (3.3) and (3.4), we need to show that this curve cannot go to infinity, i.e., we need an a priori estimate. Since the values of ξ_1 and η_1 are kept fixed on the curve, we only need to estimate U and V. We claim that there is a constant $c > 0$, independent of the solution $(u(x,k), v(x,k))$, so that

$$||U||_{H^2(\Omega)} + ||V||_{H^2(\Omega)} \le c\,. \tag{3.11}$$

Writing $u(x) = \xi_1\varphi_1 + U$, $v(x) = \eta_1\varphi_1 + V$, we have from (3.1) and (3.2)

$$\begin{aligned} \Delta U + k\lambda V + kb_1(\eta_1\varphi_1 + V) &= (\mu_1 + \lambda_1\xi_1 - k\lambda\eta_1)\varphi_1 + e_1(x), \\ \Delta V + k\bar{\lambda}U + kb_2(\xi_1\varphi_1 + U) &= (\nu_1 + \lambda_1\eta_1 - k\bar{\lambda}\xi_1)\varphi_1 + e_2(x)\,. \end{aligned} \tag{3.12}$$

We multiply the first equation by U, the second one by V, and add

$$\begin{gathered} \textstyle\int_\Omega |\nabla U|^2 + \int_\Omega |\nabla V|^2\,dx - k(\lambda + \bar{\lambda}) \int_\Omega U\,V\,dx = \\ k\int_\Omega b_1 U\,dx + k\int_\Omega b_2 V\,dx - \int_\Omega e_1 U\,dx - \int_\Omega e_2 V\,dx\,. \end{gathered} \tag{3.13}$$

Using Lemma 2.1, and (3.4), we estimate the left hand side of (3.13) from below by

$$(2\lambda_2 - k\lambda - k\bar{\lambda}) \int_\Omega (U^2 + V^2)\,dx \geq c_0 \int_\Omega (U^2 + V^2)\,dx\,,$$

for some $c_0 > 0$. Combining that with an estimate of the right hand side of (3.13), we conclude a bound on $\int_\Omega (U^2 + V^2)\,dx$. Returning to (3.13) again, we conclude a bound on $\int_\Omega (|\nabla U|^2 + |\nabla V|^2)\,dx$. Next, we multiply the first equation in (3.12) by ΔU, and integrate over Ω. Observe that $\int_\Omega \Delta U\varphi_1\,dx = 0$, giving us

$$\int_\Omega (\Delta U)^2\,dx + k\lambda \int_\Omega \Delta U V\,dx + k\int_\Omega \Delta U b_1\,dx = \int_\Omega \Delta U e_1(x)\,dx\,.$$

Integrating by parts in the second term, and using the estimate just obtained, we estimate $\int_\Omega (\Delta U)^2\,dx$. We estimate $\int_\Omega (\Delta V)^2\,dx$ similarly, and by the standard elliptic estimates we conclude (3.11). $\diamondsuit$

Remark Observe that the a priori estimate (3.11) is uniform in ξ_1 and η_1.

4 Continuation in (ξ_1, η_1) for k fixed

The Theorem 3.1 implies that the pair (ξ_1, η_1) uniquely identifies the solution quadruple $(\mu_1, \nu_1, u, v)(k)$ solving (3.1). We call (ξ_1, η_1) to be the *signature of solution*. The solution set of the original problem (1.1) can be faithfully described by the map $(\xi_1, \eta_1) \in R^2 \to (\mu_1, \nu_1) \in R^2$, which we call the *solution manifold*. For example, if this map is *onto*, then the original problem (1.1) is solvable for any $h_1(x)$ and $h_2(x)$. We show next that the solution manifold is connected.

Theorem 4.1 *In the conditions of Theorem 3.1, we can continue solutions of (1.1) of any signature $(\bar{\xi}_1, \bar{\eta}_1)$ to solution of arbitrary signature $(\hat{\xi}_1, \hat{\eta}_1)$, by following any continuous curve $(\xi_1(t), \eta_1(t))$ joining $(\bar{\xi}_1, \bar{\eta}_1)$ and $(\hat{\xi}_1, \hat{\eta}_1)$, and obtaining a continuous curve of solutions $(u(t), v(t), \mu_1(t), \nu_1(t))$.*

Proof: Let $\xi_1 = \alpha(t)$, $\eta_1 = \beta(t)$, $0 \le t \le 1$, be any continuous curve joining these points, with $(\alpha(0), \beta(0)) = (\bar{\xi}_1, \bar{\eta}_1)$, and $(\alpha(1), \beta(1)) = (\hat{\xi}_1, \hat{\eta}_1)$. Arguing as in the proof of Theorem 3.1, we show that continuation in t can be performed similarly to the continuation in k above. In particular, the linearized problem is the same as (3.6), and the implicit function theorem applies the same way (see [9], [10], where more details were given on the continuation in the first harmonic). By the a priori estimate (3.11), which is is uniform in ξ_1 and η_1, solutions on the curve remain bounded in H^2 norm. $\diamondsuit$

5 Solution manifold and existence of solutions

We now return to the original problem (1.1)

$$\begin{aligned} \Delta u + f(v) &= h_1(x), \quad x \in \Omega, \quad u = 0 \ \text{ for } x \in \partial\Omega \\ \Delta v + g(u) &= h_2(x), \quad x \in \Omega, \quad v = 0 \ \text{ for } x \in \partial\Omega\,, \end{aligned} \tag{5.1}$$

where $h_1(x) = \mu_1^0\varphi_1 + e_1(x)$, and $h_2(x) = \nu_1^0\varphi_1 + e_2(x)$, with $e_1(x)$ and $e_2(x)$ orthogonal to φ_1, and the functions $f(t)$ and $g(t)$ satisfy (3.2), i.e., $f(t) = \lambda t + b_1(t)$ and $g(t) = \bar{\lambda} t + b_2(t)$, with bounded $b_1(t)$ and $b_2(t)$. Here we set $k = 1$, which does not restrict the generality, since we may redefine λ, $\bar{\lambda}$ and b_1, b_2. This problem is solvable if and only if the pair (μ_1^0, ν_1^0) belongs to the solution manifold. To make the presentation easier, we assume that $b_1(t)$ and $b_2(t)$ have finite limits at $\pm\infty$, and

$$b_i(-\infty) < b_i(t) < b_i(\infty), \quad \text{for } t \in (-\infty, \infty),\ i = 1, 2\,. \tag{5.2}$$

Theorem 5.1 *Assume that the conditions of the Theorem 3.1 hold, with $k = 1$. If $\lambda\bar{\lambda} \neq \lambda_1^2$, then the system (5.1) has a solution for any $h_1(x)$, $h_2(x) \in L^2(\Omega)$. In the resonance case $\lambda\bar{\lambda} = \lambda_1^2$, the system (5.1) has a solution if and only if*

$$(\lambda_1 b_1(-\infty) + \lambda b_2(-\infty)) \int_\Omega \varphi_1\, dx < \lambda_1\mu_1^0 + \lambda\nu_1^0 < (\lambda_1 b_1(\infty) + \lambda b_2(\infty)) \int_\Omega \varphi_1\, dx\,. \tag{5.3}$$

Proof: As before, we decompose $u(x) = \xi_1\varphi_1 + U(\xi_1, \eta_1)$ and $v(x) = \eta_1\varphi_1 + V(\xi_1, \eta_1)$, with U and V orthogonal to φ_1. By the Theorem 3.1, for any pair (ξ_1, η_1) we can find (μ_1, ν_1, u, v) solving (5.1). Our goal is to find a pair (ξ_1, η_1), so that $(\mu_1, \nu_1) = (\mu_1^0, \nu_1^0)$. Multiplying each equation in (5.1) by φ_1, and integrating, we have

$$\begin{aligned} \mu_1^0 &= -\lambda_1\xi_1 + \lambda\eta_1 + \textstyle\int_\Omega b_1(\eta_1\varphi_1 + V(\xi_1, \eta_1))\varphi_1\, dx \\ \nu_1^0 &= \bar{\lambda}\xi_1 - \lambda_1\eta_1 + \textstyle\int_\Omega b_2(\xi_1\varphi_1 + U(\xi_1, \eta_1))\varphi_1\, dx\,. \end{aligned} \tag{5.4}$$

Let us denote $I_1(\xi_1,\eta_1)=\int_\Omega b_1(\eta_1\varphi_1+V(\xi_1,\eta_1))\varphi_1\,dx$, $I_2(\xi_1,\eta_1)=\int_\Omega b_2(\xi_1\varphi_1+U(\xi_1,\eta_1))\varphi_1\,dx$, and $A=\begin{bmatrix}-\lambda_1 & \lambda\\ \bar\lambda & -\lambda_1\end{bmatrix}$. Assume first that the matrix A is invertible, i.e.,

$$\lambda\bar\lambda\neq\lambda_1^2\,. \tag{5.5}$$

Then we rewrite (5.4) as

$$\begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}=A^{-1}\begin{bmatrix}\mu_1^0\\ \nu_1^0\end{bmatrix}-A^{-1}\begin{bmatrix}I_1(\xi_1,\eta_1)\\ I_2(\xi_1,\eta_1)\end{bmatrix}. \tag{5.6}$$

The right hand side of (5.6) gives a continuous map of a sufficiently large ball around the origin in the plane (ξ_1,η_1) into itself, and hence existence of solutions for (5.6) follows by Brouwer's fixed point theorem. We present next another proof of solvability of (5.6), which gives an indication as to where one should search for the solution numerically, and it also introduces the method of sliding lines that we shall use in the resonance case. Let $A^{-1}=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$, with some α, β, γ and δ. Denoting $\begin{bmatrix}A_0\\ B_0\end{bmatrix}=A^{-1}\begin{bmatrix}\mu_1^0\\ \nu_1^0\end{bmatrix}$, we rewrite (5.6) as

$$\begin{aligned}A(\xi_1,\eta_1)&\equiv\xi_1+\textstyle\int_\Omega(\alpha b_1+\beta b_2)\,\varphi_1\,dx=A_0\\ B(\xi_1,\eta_1)&\equiv\eta_1+\textstyle\int_\Omega(\gamma b_1+\delta b_2)\,\varphi_1\,dx=B_0\,.\end{aligned} \tag{5.7}$$

In (ξ_1,η_1) plane consider a vertical line $\xi_1=N$, $-\infty<\eta_1<\infty$. On each such line we can find a point where the second equation in (5.7) holds (by continuity, since $B(\xi_1,\eta_1)\to\infty$ $(-\infty)$, as $\eta_1\to\infty$ $(-\infty)$), while at this point $A(\xi_1,\eta_1)$ is large and negative (positive) if N is large and negative (positive). Sliding this line, i.e., varying N from $-\infty$ to ∞, we obtain a solution of (5.7).

Next, consider the case of resonance

$$\lambda\bar\lambda=\lambda_1^2\,. \tag{5.8}$$

We then conclude from (5.4)

$$\lambda_1\mu_1^0+\lambda\nu_1^0=\int_\Omega\left[\lambda_1 b_1(\eta_1\varphi_1+V)+\lambda b_2(\xi_1\varphi_1+U)\right]\varphi_1\,dx\,, \tag{5.9}$$

from which it follows that (5.3) is a necessary condition for solvability. (By the a priori estimate (3.11), which is uniform in ξ_1 and η_1, it follows that $\int_\Omega b_1(\eta_1\varphi_1+V)\varphi_1\,dx\to b_1(\pm\infty)\int_\Omega\varphi_1\,dx$ as $\eta_1\to\pm\infty$, and $\int_\Omega b_2(\xi_1\varphi_1+U)\varphi_1\,dx\to b_2(\pm\infty)\int_\Omega\varphi_1\,dx$ as $\xi_1\to\pm\infty$, see [10] for a detailed argument.)

Turning to the sufficiency of the condition (5.3), we recall that by the Theorem 3.1, for any pair (ξ_1, η_1) we can find (μ_1, ν_1, u, v) solving

$$(5.10)\quad \begin{aligned} &\Delta u + \lambda v + b_1(\eta_1\varphi_1 + V) = \mu_1\varphi_1 + e_1, \quad x \in \Omega, \quad u = 0 \ \text{ for } x \in \partial\Omega \\ &\Delta v + \bar{\lambda} u + b_2(\xi_1\varphi_1 + U) = \nu_1\varphi_1 + e_2, \quad x \in \Omega, \quad v = 0 \ \text{ for } x \in \partial\Omega\,, \end{aligned}$$

and our goal is to find a point (ξ_1^0, η_1^0) at which $(\mu_1, \nu_1) = (\mu_1^0, \nu_1^0)$. Multiplying each equation in (5.10) by φ_1, and integrating, we have

$$(5.11)\qquad \begin{aligned} \mu_1 &= -\lambda_1\xi_1 + \lambda\eta_1 + \textstyle\int_\Omega b_1(\eta_1\varphi_1 + V)\varphi_1\,dx \\ \nu_1 &= \bar{\lambda}\xi_1 - \lambda_1\eta_1 + \textstyle\int_\Omega b_2(\xi_1\varphi_1 + U)\varphi_1\,dx\,. \end{aligned}$$

Then

$$(5.12)\qquad \lambda_1\mu_1 + \lambda\nu_1 = \int_\Omega \left[\lambda_1 b_1(\eta_1\varphi_1 + V) + \lambda b_2(\xi_1\varphi_1 + U)\right]\varphi_1\,dx\,.$$

In (ξ_1, η_1) plane we consider a line $L_A : -\lambda_1\xi_1 + \lambda\eta_1 = A$, where A is a constant. Along this line $\eta_1 \to +\infty$ $(-\infty)$, when $\xi_1 \to +\infty$ $(-\infty)$. Hence, the right hand side in (5.12) varies along L_A between the lower and upper limits in (5.3). By (5.3), along each line L_A we can find a point P_{L_A}, so that

$$(5.13)\qquad \lambda_1\mu_1 + \lambda\nu_1 = \lambda_1\mu_1^0 + \lambda\nu_1^0\,.$$

We see from the first equation in (5.11) that when A is large and positive, μ_1 is also large and positive, i.e., $\mu_1 > \mu_1^0$ at the point P_{L_A}. Similarly, when A is large and negative, $\mu_1 < \mu_1^0$ at P_{L_A}. By continuity, we can find an A, so that $\mu_1 = \mu_1^0$ at P_{L_A}. But then from (5.13), we also have $\nu_1 = \nu_1^0$ at P_{L_A}. $\diamondsuit$

Next, we obtain an analog of D.G. de Figueiredo and W.-M. Ni [7] result, where existence of solution for a single equation at resonance is proved without the Landesman - Lazer condition. We consider the system

$$(5.14)\quad \begin{aligned} &\Delta u + \lambda v + b_1(v) = \mu_1^0\varphi_1(x) + e_1(x), \quad x \in \Omega, \quad u = 0 \ \text{ for } x \in \partial\Omega \\ &\Delta v + \tfrac{\lambda_1^2}{\lambda} v + b_2(u) = \nu_1^0\varphi_1(x) + e_2(x), \quad x \in \Omega, \quad v = 0 \ \text{ for } x \in \partial\Omega\,, \end{aligned}$$

where as before $\int_\Omega e_1(x)\varphi_1(x)\,dx = \int_\Omega e_2(x)\varphi_1(x)\,dx = 0$, and λ is a positive number.

Theorem 5.2 *Assume that the bounded functions $b_1(t)$ and $b_2(t) \in C^1(R)$ satisfy $b_1(t)t > 0$ and $b_2(t)t > 0$, for all $t \in R$, and assume also that the functions $f(t) \equiv \lambda t + b_1(t)$ and $g(t) \equiv \frac{\lambda_1^2}{\lambda}t + b_2(t)$ satisfy*

$$|f'(t)| = |\lambda + b_1'(t)| < M_1, \quad |g'(t)| = |\frac{\lambda_1^2}{\lambda} + b_2'(t)| < M_2, \quad \textit{for all } t \in R\,;$$

$$M_1 + M_2 < 2\sqrt{\lambda_1\lambda_2}\,.$$

Then the problem (5.14) is solvable for any (μ_1^0, ν_1^0)*, which lies on the line*

$$\lambda_1\mu_1^0 + \lambda\nu_1^0 = 0\,. \tag{5.15}$$

Proof: We have the formulas (5.11) and (5.12) as before, and again we wish to show that $(\mu_1, \nu_1) = (\mu_1^0, \nu_1^0)$ for some choice of (ξ_1, η_1). Consider the lines L_A: $-\lambda_1\xi_1 + \lambda\eta_1 = A$ in (ξ_1, η_1) plane. As ξ_1, $\eta_1 \to \infty$ along any such line, $\lambda_1\mu_1 + \lambda\nu_1$ is positive, as follows from (5.12) and our conditions. On the same line, as ξ_1, $\eta_1 \to -\infty$, $\lambda_1\mu_1 + \lambda\nu_1$ is negative. Hence, on any line L_A we can find a point P_{L_A}, where $\lambda_1\mu_1 + \lambda\nu_1 = 0$. We now vary the line L_A. If A is large in absolute value and positive (negative), we see from the first equation in (5.11) that μ_1 is large in absolute value and positive (negative) at the point P_{L_A}. Hence, we can find a line, such that $\mu_1 = \mu_1^0$ at the point P_{L_A}, and then, from (5.15), $\mu_2 = \mu_2^0$. $\diamondsuit$

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