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Uniqueness and exact multiplicity of solutions for a class of fourth-order semilinear problems

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Using techniques of bifurcation theory, we give exact multiplicity and uniqueness results for the fourth-order Dirichlet problem, which describes deflection of an elastic beam, subjected to a nonlinear force, and clamped at the end points. The crucial part of this approach was to show positivity of non-trivial solutions of the corresponding linearized problem.

1. Introduction

We study positive solutions of the problem

$$\left.\begin{aligned} u''''(x) &= \lambda f(u(x)), \quad x \in (0,1), \\ u(0) = u'(0) &= u(1) = u'(1) = 0, \end{aligned}\right\} \tag{1.1}$$

which describes deflection $u(x)$ of an elastic beam, subjected to a nonlinear force $f(u)$, and clamped at the end points. Here, λ is a positive parameter and $f(u)$ a continuous function. We are interested in the existence and exact multiplicity of positive solutions. Throughout the paper we consider the classical solutions of (1.1), which we will denote alternatively by $u(x, \lambda)$ or $u(x)$.

While similar second-order problems have been extensively studied (see, for example, the references in [7]), relatively little is known about the problem (1.1). The likely reason is that fewer techniques are available for the fourth-order problems. In particular, the strong maximum principle does not hold here. Also, the popular quadrature method appears to be completely inapplicable. Previous works on the problem include [2–4] (see also [5]). These papers used shooting techniques, the Leray–Schauder degree and monotone iterations. In [6], a bifurcation approach was applied to elliptic systems and a result similar to theorem 1.1 below was proved in the case of boundary conditions $u(0) = u''(0) = u(1) = u''(1)$.

In the present work, for a class of convex $f(u)$ we obtain an exact multiplicity result and a complete description of the solution set. We use a bifurcation approach similar to [7]. Crucial to this approach is to show positivity of non-trivial solutions of the corresponding linearized problem. This turned out to be the most difficult part of the analysis, in which we made use of the classical paper of [8]. We state our main result next.

Theorem 1.1. *Assume that $f(u) \subset C^2(0,\infty) \cap C[0,\infty)$ satisfies $f(u) > 0$ for $u \geqslant 0$, $\lim_{u\to\infty}(f(u)/u) = \infty$, $f'(0) \geqslant 0$ and $f''(u) > 0$ for $u > 0$. Then all positive solutions of (1.1) lie on a unique smooth curve of solutions. This curve starts at $(\lambda = 0, u = 0)$, it continues for $\lambda > 0$, until a critical λ_0, where it bends back and continues for decreasing λ without any more turns, tending to infinity as $\lambda \downarrow 0$. In other words, we have exactly two, exactly one or no solution, depending on whether $0 < \lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Moreover, all solutions are symmetric with respect to the midpoint $x = \frac{1}{2}$, and the maximum value of the solution, $u(\frac{1}{2})$, is strictly monotone on the curve.*

Once we show that non-trivial solutions of the linearized problem cannot vanish, we can get other uniqueness and exact multiplicity results. For example, we get an easy proof of the following uniqueness result of [3]. Moreover, we get some extra information on the solution curve.

Theorem 1.2 (cf. [3]). *Assume that $f(u) \subset C^1(0,\infty) \cap C[0,\infty)$ satisfies*

$$0 < f(u) < uf'(u) \quad \text{for } u > 0. \tag{1.2}$$

Then, for any $\lambda > 0$, the problem (1.1) has at most one positive solution. Moreover, all positive solutions are non-degenerate (i.e. the corresponding linearized problem (2.4) has only the trivial solution), they lie on a single smooth solution curve, they are symmetric with respect to the midpoint $x = \frac{1}{2}$ and the maximum value of the solution, $u(\frac{1}{2})$, is strictly monotone on the curve.

After this work was completed, we became aware of an interesting preprint of Rynne [9], where similar results are derived for problems of order $2m$. In the case of $m = 2$ our results are more detailed.

We next recall the Crandall–Rabinowitz theorem [1], which will be used in the proof of theorem 1.1.

Theorem 1.3 (cf. [1]). *Let X and Y be Banach spaces. Let $(\bar\lambda, \bar x) \in \boldsymbol{R} \times X$ and let F be a continuously differentiable mapping of an open neighbourhood of $(\bar\lambda, \bar x)$ into Y. Let the null-space $N(F_x(\bar\lambda, \bar x)) = \operatorname{span}\{x_0\}$ be one-dimensional and $\operatorname{codim} R(F_x(\bar\lambda, \bar x)) = 1$. Let $F_\lambda(\bar\lambda, \bar x) \notin R(F_x(\bar\lambda, \bar x))$. If Z is a complement of $\operatorname{span}\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\bar\lambda, \bar x)$ near $(\bar\lambda, \bar x)$ form a curve $(\lambda(s), x(s)) = (\bar\lambda + \tau(s), \bar x + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \boldsymbol{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

2. Preliminary results

We begin by studying the corresponding linear boundary-value problem

$$\left.\begin{aligned} v''''(x) = c(x)v(x), \quad c(x) > 0, \quad x \in (0,1), \\ v(0) = v'(0) = v(1) = v'(1) = 0, \end{aligned}\right\} \tag{2.1}$$

where $c(x) \in C[0,1]$ is a given positive function.

Lemma 2.1. *If $v''(0) = 0$, then $v(x) \equiv 0$.*

Proof. Assume, on the contrary, that $v''(0) = 0$. Consider $v'''(0)$. If $v'''(0) = 0$, then $v(x) \equiv 0$ by uniqueness for initial-value problems. Otherwise, by the linearity of the problem, we may assume that $v'''(0) > 0$, and then $v(x)$ is positive for small x. Assume that

$$v(x) > 0 \quad \text{for } x \in (0, \delta), \tag{2.2}$$

where $\delta \in (0, 1]$ is the supremum of numbers for which (2.2) holds (i.e. $v(\delta) = 0$).

Since, from equation (2.1), the function $v'''(x)$ is increasing, it follows that

$$v'''(x) > 0 \quad \text{for } x \in [0, \delta].$$

Hence the function $v''(x)$ is increasing. Combining that with $v''(0) = 0$, we conclude that

$$v''(x) > 0 \quad \text{for } x \in (0, \delta].$$

It follows that the function $v'(x)$ is increasing. Combining that with $v'(0) = 0$, we conclude that

$$v'(x) > 0 \quad \text{for } x \in (0, \delta].$$

If $\delta = 1$, then the last inequality contradicts the boundary condition $v'(1) = 0$, otherwise we conclude that $v(\delta) > 0$, contradicting the maximality of δ. □

COROLLARY 2.2. *The linear space of non-trivial solutions of (2.1) is either empty or one dimensional. Indeed, in view of the lemma, the set of solutions of (2.1) can be parametrized by $v''(0)$.*

A similar result holds for the nonlinear problem (1.1).

LEMMA 2.3. *Assume that $f(u) \geqslant 0$ for $u > 0$. If $u(x)$ is a positive solution of (1.1), then*

$$u''(0) > 0 \quad \textit{and} \quad u''(1) > 0. \tag{2.3}$$

Proof. Since $u(x)$ is positive, $u''(0) \geqslant 0$. Assume that, on the contrary, $u''(0) = 0$. Again, we consider $u'''(0) \equiv \alpha$. Clearly, $\alpha \geqslant 0$. Integrating the equation (1.1), we get

$$u(x) = \int_0^x \frac{(x-\xi)^3}{3!} f(u(\xi))\, \mathrm{d}\xi + \alpha \frac{x^3}{3!}.$$

But then $u(1) > 0$, contradicting the boundary condition $u(1) = 0$. □

We shall need to consider the linearized problem for (1.1),

$$\left.\begin{aligned} w''''(x) = \lambda f'(u) w, \quad x \in (0, 1), \\ w(0) = w'(0) = w(1) = w'(1) = 0. \end{aligned}\right\} \tag{2.4}$$

LEMMA 2.4. *If $u(x)$ and $w(x)$ are solutions of problems (1.1) and (2.4), respectively, then*

$$w'''u' - wu'''' - w''u'' + w'u''' = -u''(1)w''(1) \quad \textit{for all } x \in [0, 1]. \tag{2.5}$$

Proof. Differentiating the function $w'''u' - wu'''' - w''u'' + w'u'''$, and using equations (1.1) and (2.4), we see that this function is constant on $[0, 1]$. Evaluating it at $x = 1$ gives us (2.5). □

We shall need the following lemma to verify the hypothesis of the Crandall–Rabinowitz theorem 1.3.

LEMMA 2.5. *If $u(x)$ and $w(x)$ are solutions of problems (1.1) and (2.4), respectively, then*

$$\int_0^1 f(u(x))w(x)\,\mathrm{d}x = \frac{1}{4\lambda}u''(1)w''(1). \tag{2.6}$$

Proof. Integrate (2.5) over $(0,1)$. Then integrate by parts in all four terms, shifting all derivatives on u. All boundary terms vanish, in view of our boundary conditions. Then, expressing u'''' from equation (1.1), we conclude the lemma. □

The following lemma provides an 'energy' functional for problem (1.1). We omit its straightforward proof. As usual, we write $F(u) = \int_0^u f(t)\,\mathrm{d}t$.

LEMMA 2.6. *If $u(x)$ is any solution of (1.1), then*

$$u'u''' - \tfrac{1}{2}u''^2 - \lambda F(u) = \text{const.} \quad \textit{for all } x \in [0,1]. \tag{2.7}$$

The following lemma gives a condition for any positive solution of (1.1) to be unimodular.

LEMMA 2.7. *Assume that $f(u) > 0$ for $u > 0$. Then any positive solution of (1.1) has exactly one local (and hence global) maximum.*

Proof. Since $(u'')'' = f(u) > 0$, we see, by the minimum principle, that the function u'' is negative between any two of its consecutive roots, i.e. $u(x)$ is concave between any two consecutive points of inflection. The proof follows. □

COROLLARY 2.8. *The solution $u(x)$ has exactly two points of inflection. Indeed, by lemma 2.3, $u(x)$ is convex near the endpoints and by lemma 2.7, there are at most two points of inflection.*

The following lemma gives conditions for any positive solution of (1.1) to be symmetric. A more general symmetry result (in the case of $f = f(x, u, u'')$) was given previously by Dalmasso [2]. For the particular case of (1.1), our proof is much easier. Moreover, a similar argument is used in a lemma to follow.

LEMMA 2.9. *For problem (1.1), assume that $f(u) \subset C^1(0,\infty) \cap C[0,\infty)$ satisfies $f(u) > 0$ and $f'(u) \geqslant 0$. Then any positive solution of (1.1) is symmetric with respect to $x = \frac{1}{2}$. Moreover, $u'(x) > 0$ on $(0, \frac{1}{2})$.*

Proof. By lemma 2.7, any positive solution $u(x)$ is unimodular, with a unique point of maximum, which we denote by x_0. Assuming, on the contrary, that $u(x)$ is not symmetric, we observe that $v(x) \equiv u(1-x)$ is a different solution of (1.1). We have $u(0) = v(0) = 0$, $u'(0) = v'(0) = 0$ and, by lemma 2.6, $u''(0) = v''(0)$. By the uniqueness for the initial-value problems, the third derivatives must be different at 0, so that we may assume, for definiteness, that $v'''(0) > u'''(0)$. It follows that $v(x) > u(x)$ for small x. Let ξ denote the first point where the graphs of $u(x)$ and $v(x)$ intersect, and if they never intersect, we let ξ be the point of maximum of $v(x)$ (i.e. $\xi = 1 - x_0$). Let $w(x) = v(x) - u(x)$. Then $w(x)$ satisfies

$$w'(\xi) \leqslant 0 \tag{2.8}$$

and

$$\left.\begin{aligned} w'''' &= c(x)w \quad \text{on } (0,\xi), \\ w(0) = w'(0) &= w''(0) = 0, \quad w'''(0) > 0, \end{aligned}\right\} \tag{2.9}$$

where

$$c(x) = \lambda \int_0^1 f'(\theta u + (1-\theta)v)\,\mathrm{d}\theta > 0.$$

Since $w(x)$ is positive for small x, it follows that $w'''(x)$ is increasing, and hence positive. The same is true for $w''(x)$ and $w'(x)$ for all $x \in (0,\xi)$. In particular, $w'(\xi) > 0$, contradicting (2.8).

Turning to the last claim, since $u(x)$ has only one point of inflection on $(0,\frac{1}{2})$, say ξ, it follows that $u'(x)$ is increasing on $(0,\xi)$ and decreasing on $(\xi,\frac{1}{2})$, while $u'(0) = u'(\frac{1}{2}) = 0$. It follows that $u'(x) > 0$ for $x \in (0,\frac{1}{2})$. □

By the last lemma, any positive solution of (1.1) has a global maximum at $x = \frac{1}{2}$. It turns out that the maximal value uniquely identifies the solution.

Lemma 2.10. *Assume the conditions of lemma 2.9. Then the positive solutions of (1.1) are globally parametrized by their maximum values $u(\frac{1}{2},\lambda)$, i.e. for every $p > 0$, there is at most one $\lambda > 0$ and at most one solution $u(x,\lambda)$ of (1.1) for which $u(\frac{1}{2},\lambda) = p$.*

Proof. By shifting and then stretching of the x variable, we can replace problem (1.1) by

$$\left.\begin{aligned} u''''(x) &= \lambda f(u(x)), \quad x \in (-1,1), \\ u(-1) = u'(-1) &= u(1) = u'(1) = 0, \end{aligned}\right\} \tag{2.10}$$

with the maximum value $u(0,\lambda) = p$. Then

$$v(x) = u\left(\frac{1}{\sqrt[4]{\lambda}}x, \lambda\right)$$

satisfies

$$\left.\begin{aligned} v'''' &= f(v) \quad \text{on } (0,\sqrt[4]{\lambda}), \\ v(0) = p, \quad v'(0) &= v'''(0) = 0, \quad v(\sqrt[4]{\lambda}) = v'(\sqrt[4]{\lambda}) = 0. \end{aligned}\right\} \tag{2.11}$$

Assume there is a different solution $u_1(x,\mu)$ of (2.10) with $u_1(0,\mu) = p$. Assume first that $\mu \neq \lambda$. Then

$$v_1(x) = u_1\left(\frac{1}{\sqrt[4]{\mu}}x, \mu\right)$$

is another solution of the equation in (2.11), which has its first root at $\sqrt[4]{\mu}$. By the uniqueness for the initial-value problems, $v''(0) \neq v_1''(0)$, so that we may assume that $v''(0) > v_1''(0)$. It follows that $v(x) > v_1(x)$ for $|x|$ small. Then $w(x) = v(x) - v_1(x)$ satisfies

$$\left.\begin{aligned} w'''' &= c(x)w \quad \text{for } x > 0, \\ w(0) = w'(0) &= w'''(0) = 0, \quad w''(0) > 0, \end{aligned}\right\} \tag{2.12}$$

where

$$c(x) = \lambda \int_0^1 f'(\theta v + (1-\theta)v_1)\,\mathrm{d}\theta > 0.$$

We see from (2.12) that $w(x)$ and its first three derivatives are positive for $x > 0$. On the other hand, either $v(x)$ and $v_1(x)$ intersect at some $\xi > 0$, where $w'(\xi) \leqslant 0$, or else the smaller solution $v_1(x)$ reaches its root before $v(x)$ does, so that $w'(\sqrt[4]{\mu}) \leqslant 0$. In both cases, we have a contradiction.

In the case $\mu = \lambda$, we argue similarly. In that case, either $v(x)$ and $v_1(x)$ intersect at some ξ, where we have $w'(\xi) \leqslant 0$, or else these functions reach their first root at $\sqrt[4]{\lambda}$, where $w'(\sqrt[4]{\lambda}) = 0$, leading to the same contradiction. □

The next lemma shows that solutions of the linearized problem can change monotonicity only after changing sign.

LEMMA 2.11. *Let $a(x)$ be a positive continuous function. Then any non-trivial solution of the problem*

$$w'''' = a(x)w, \qquad w(0) = w(1) = 0$$

cannot have non-negative local minimums and non-positive local maximums. In particular, under the conditions of lemma 2.9, any non-trivial solution $w(x)$ of the linearized problem (2.4) cannot have non-negative local minimums and non-positive local maximums.

Proof. Since $w(x)$ vanishes at the endpoints of the interval $(0,1)$, for any point of non-negative local minimum x_0 we can find two inflection points of $w(x)$, say α and β, with $\alpha < x_0 < \beta$. On (α,β), $(w'')'' > 0$, which, together with $w''(\alpha) = w''(\beta) = 0$, implies that $w(x)$ is concave, a contradiction. The other claim is proved similarly ($w(x)$ would have to be convex near the point of maximum). □

We need the following lemma, which is a special case of theorem 3.1 of the classical paper of Leighton and Nehari [8]. We present a proof for completeness.

LEMMA 2.12. *Let $a(x)$ be a positive continuous function. Let $u(x)$ and $v(x)$ be non-trivial solutions of*

$$v''''(x) = a(x)v(x), \quad a(x) > 0, \quad x \in (0,1), \tag{2.13}$$

which are not multiples of one another, such that $u(0) = v(0) = u(1) = v(1) = 0$. Then the zeros of $u(x)$ and $v(x)$ in $(0,1)$ separate each other.

Proof. Assume, on the contrary, that $0 < \alpha < \beta < 1$ are two consecutive roots of $u(x)$, i.e. $u(\alpha) = u(\beta) = 0$, while $v(x) > 0$ on $[\alpha,\beta]$. We claim that $p(x) \equiv u'v - uv'$ has to vanish on (α,β). Indeed, assuming that, say, $p(x) > 0$ on (α,β), we integrate the identity

$$\left(\frac{u}{v}\right)' = \frac{p}{v^2} > 0,$$

obtaining

$$0 = \frac{u(\beta)}{v(\beta)} - \frac{u(\alpha)}{v(\alpha)} = \int_\alpha^\beta \frac{p}{v^2}\,\mathrm{d}x > 0,$$

a contradiction, proving the claim. So let $p(x_0) = 0$ at some $x_0 \in (\alpha, \beta)$. We may regard $p(x_0)$ as a 2×2 Wronskian determinant. Since it vanishes, its columns are proportional, i.e.

$$\begin{pmatrix} u(x_0) \\ u'(x_0) \end{pmatrix} = \gamma \begin{pmatrix} v(x_0) \\ v'(x_0) \end{pmatrix}, \tag{2.14}$$

with a constant γ. We now consider $w(x) \equiv u(x) - \gamma v(x)$, which is also a solution of (2.13), with $w(0) = w(1) = 0$. In view of (2.14), $w(x)$ has a double root at x_0, i.e. $w(x_0) = w'(x_0) = 0$. If $w''(x_0) > 0$ (less than 0), then x_0 is a point of non-negative minimum (non-positive maximum), which is impossible by lemma 2.11. Hence $w''(x_0) = 0$. We now consider $w'''(x_0)$. If $w'''(x_0) = 0$, then $w(x) \equiv 0$, contradicting $w(\alpha) \neq 0$. Hence we may assume that $w'''(x_0) > 0$. Then $w(x)$ is positive and convex to the right of x_0. But $w(1) = 0$. Hence we can find an inflection point $\delta \in (x_0, 1)$ where $w(x)$ becomes concave, i.e. $w'''(\delta) \leqslant 0$. From equation (2.13), $w'''(x)$ is increasing on (x_0, δ), a contradiction. □

The following is the main result of this section. It is crucial for our uniqueness and exact multiplicity results.

THEOREM 2.13. *Assume the conditions of lemma 2.9. Then any non-trivial solution of the linearized problem (2.4) cannot vanish inside* $(0, 1)$, *i.e. we may assume that* $w(x) > 0$ *on* $(0, 1)$.

Proof. By lemma 2.11, it suffices to prove that $w(x)$ is non-negative, i.e. it cannot change sign. Assuming the contrary, let us ignore any roots of $w(x)$ where it 'touches' the x-axis and consider only the roots where $w(x)$ changes sign. We claim that $w(x)$ cannot change sign an odd number of times. If the contrary is true, we may assume for definiteness that $w(x)$ is negative near $x = 0$ and positive near $x = 1$. In view of lemma 2.1, we then have $w''(0) < 0$ and $w''(1) > 0$. Setting $x = 0$ in (2.5), we have

$$u''(0)w''(0) = u''(1)w''(1).$$

This gives us a contradiction, since, in view of lemma 2.3, the quantity on the left is negative, while the one on the right is positive.

So, assume that $w(x)$ changes sign an even number of times. We claim that $w(x)$ is then even with respect to $x = \frac{1}{2}$. Indeed, since, by lemma 2.9, $u(x)$ is even with respect to $x = \frac{1}{2}$, it follows that $w(1-x)$ is a solution of the linearized problem (2.4). Since, by lemma 2.1, the null-space of (2.4) is one dimensional, we have

$$w(1 - x) = cw(x), \tag{2.15}$$

with a constant $c \neq 1$. (If $c = 1$, $w(x)$ is even, and the claim is proved.) It follows from (2.15) that $w(\frac{1}{2}) = 0$. We claim that $w'(\frac{1}{2}) \neq 0$. Indeed, assume on the contrary that $w'(\frac{1}{2}) = 0$. From (2.15), we conclude that $w''(\frac{1}{2}) = 0$. Then, evaluating (2.5) at $x = \frac{1}{2}$, we obtain $0 = u''(1)w''(1)$, which is a contradiction, since the quantity on the right is non-zero. Hence $w'(\frac{1}{2}) \neq 0$, i.e. $w(x)$ changes sign at $x = \frac{1}{2}$. It follows that the number of sign changes for $w(x)$ is different on the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, which makes (2.15) impossible. Hence $w(x)$ is even.

We show next that $w(x)$ cannot change sign twice. Assuming the contrary, we know that $w(x)$ is even with respect to $x = \frac{1}{2}$. Assuming, for definiteness, that $w(x)$

is positive near $x = 1$, we have $w''(1) > 0$, and $w(\frac{1}{2}) < 0$, $w'(\frac{1}{2}) = 0$, $w''(\frac{1}{2}) \geqslant 0$. (The last inequality follows by lemma 2.11.) Then, from (2.5), we conclude

$$-w(\tfrac{1}{2})f(u(\tfrac{1}{2})) - u''(\tfrac{1}{2})w''(\tfrac{1}{2}) = -u''(1)w''(1). \tag{2.16}$$

The first term on the left in (2.16) is positive, the second one is non-negative, while the right-hand side is negative, a contradiction.

We conclude from the above discussion that $w(x)$ has to change sign at least four times, and it is an even function. Hence $w(x)$ has at least two roots on $(0, \frac{1}{2})$. Differentiating equation (1.1), we conclude that $u'(x)$ is a solution of the linearized equation (2.4), different from $w(x)$ ($u'(x)$ is odd while $w(x)$ is even). Also, $u'(0) = w(0) = u'(1) = w(1)$. By lemma 2.12, the roots of $w(x)$ and $u'(x)$ separate each other. This implies that $u'(x)$ must vanish inside $(0, \frac{1}{2})$, which is impossible. We conclude that $w(x)$ cannot change sign, and hence is non-negative on $(0, 1)$. By lemma 2.11, $w(x)$ is actually positive. □

Next we need an *a priori* estimate, which was proved in [4]. For completeness, we present a simpler proof, although we retain an idea from [4]. Notice that conditions of the lemma do not imply the symmetry of solutions.

LEMMA 2.14. *Assume that $f(u) > 0$ for $u > 0$, and $\lim_{u\to\infty}(f(u)/u) = \infty$, while λ belongs to a compact interval $I \subset (0, \infty)$. Then there exists a constant $M > 0$ such that any solution of (1.1) satisfies*

$$|u|_{C^4(0,1)} \leqslant M. \tag{2.17}$$

Proof. We claim there is a constant $m > 0$ such that any positive solution of (1.1) satisfies

$$u''(0) \leqslant m. \tag{2.18}$$

Assuming the claim for the moment, we complete the proof. Evaluating (2.7) at $x = 0$, we obtain

$$u'u''' - \tfrac{1}{2}u''^2 - \lambda F(u) = -\tfrac{1}{2}u''^2(0) \quad \text{for all } x \in [0, 1]. \tag{2.19}$$

Observe that $F(u) > 0$ for $u > 0$. Integrate (2.19) over $(0, 1)$. After integration by parts, we have

$$\frac{3}{2}\int_0^1 u''^2 \, dx \leqslant \tfrac{1}{2}m^2. \tag{2.20}$$

By the Sobolev embedding theorem, $|u|_{C^1(0,1)}$ is bounded and, using equation (1.1), we obtain a bound on $|u|_{C^4(0,1)}$, concluding the lemma.

We now turn to proving (2.18). Let $\mu_1 > 0$ and $\phi_1(x) > 0$ be the principal eigenpair of

$$\left.\begin{aligned} \phi'''' &= \mu\phi \quad \text{in } (0, 1), \\ \phi(0) = \phi'(0) &= \phi(1) = \phi'(1) = 0. \end{aligned}\right\} \tag{2.21}$$

Following [4], we observe that, by our conditions for any $c_1 > 0$, one can choose $c_2 > 0$, so that $f(u) \geqslant c_1 u - c_2$ for all $u > 0$. We choose c_1 such that $c_1\lambda > 2\mu_1$ for all $\lambda \in I$, and then select a corresponding c_2. Multiplying equation (1.1) by ϕ_1 and integrating, we get

$$\mu_1 \int_0^1 u\phi_1 \, dx \geqslant c_1\lambda \int_0^1 u\phi_1 \, dx - c_3 \geqslant 2\mu_1 \int_0^1 u\phi_1 \, dx - c_3,$$

with some constant $c_3 > 0$. It follows that

$$\int_0^1 u\phi_1 \, dx \leqslant \frac{c_3}{\mu_1}. \tag{2.22}$$

Assume now that we have a sequence of unbounded (in $C(0,1)$) solutions of (1.1) along some sequence of λs in I. (Otherwise there is nothing to prove.) By above, $u''(0)$ would have to tend to infinity along this sequence of λs. We show next that $u(x)$ must tend to infinity on a subinterval of $(0,1)$, contradicting (2.22).

Observe that, by lemma 2.7, $u(x)$ changes concavity exactly once between 0 and the point of global maximum, say x_0, and $u(x)$ is strictly increasing on $(0, x_0)$. We may assume that $x_0 \geqslant \frac{1}{2}$, otherwise we can argue from the right end. Consider, say, $u(\frac{1}{4})$. If $u(\frac{1}{4})$ is large, then $u(x)$ is large on the interval $(\frac{1}{4}, \frac{1}{2})$, and then the integral $\int_0^1 u\phi_1 \, dx$ would get large, contradicting (2.22). So $u(\frac{1}{4})$ cannot get large. We claim that $u''(\frac{1}{4})$ cannot get large (positive) either. Indeed, $u''(x)$ is a convex function on $(0,1)$, and it is negative in the middle part of that interval. So, if $u''(\frac{1}{4})$ was large, and with $u''(0)$ being large, $u''(x)$ would have to be large on $(0, \frac{1}{4})$, and then $u(x)$ would get large on, say, $(\frac{1}{8}, \frac{1}{4})$, resulting in the same contradiction with (2.22). But, if $u''(\frac{1}{4})$ is bounded, by, say, M, we see (by convexity of u'') that $u''(x)$ is bounded by M on $(\frac{1}{4}, x_0)$, and we conclude that $u(x_0)$ is bounded, contradicting our assumption. □

3. Exact multiplicity of solutions

We now prove our main results, stated in § 1.

Proof of theorem 1.1. We begin with the trivial solution ($\lambda = 0, u = 0$). At this point, the implicit function theorem applies, giving us a solution for small $\lambda > 0$. This solution is positive, since our problem has a positive Green function (see, for example, [5]). We now continue the curve for increasing λ. However, we cannot continue this curve indefinitely. Indeed, since $f(u)$ is positive, increasing and superlinear, we see that $f(u) \geqslant \alpha u$ for some $\alpha > 0$ and all $u > 0$. Multiplying equation (1.1) by ϕ_1 and integrating, we get an upper bound for λ. Let λ_0 be the supremum of the λs for which we can continue the curve for increasing λ. Since, by lemma 2.14, solutions are bounded for all $\lambda < \lambda_0$, passing to the limit in the integral form of our problem (1.1), we see that our problem has a solution u_0 at λ_0. Thus the solution curve reaches a critical solution (λ_0, u_0), at which the linearized problem (2.4) has no non-trivial solution (and hence the implicit function theorem is not applicable there).

We now recast our problem (1.1) in an operator form. Let $C^4_{00}(0,1)$ denote the subspace of functions of $C^4(0,1) \cap C[0,1]$ that satisfy

$$u(0) = u'(0) = u(1) = u'(1) = 0.$$

Let $F(\lambda, u) : C^4_{00}(0,1) \to C(0,1)$ be defined as

$$F(\lambda, u) = u''''(x) - \lambda f(u(x)).$$

Then $F(\lambda_0, u_0) = 0$, and we apply the Crandall–Rabinowitz theorem 1.3 to continue the solution curve locally around the critical point (λ_0, u_0). By lemma 2.1, we see that the null-space $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w(x)\}$ is one dimensional, and then $\operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, since $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero. In order to apply the Crandall–Rabinowitz theorem 1.3, it remains to check that $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$. Assuming the contrary would imply the existence of $v(x) \not\equiv 0$ such that

$$\left.\begin{aligned} v''''(x) - \lambda_0 f_u(u_0(x))v = f(u_0(x)), \quad x \in (0,1),\\ v(0) = v'(0) = v(1) = v'(1) = 0. \end{aligned}\right\} \tag{3.1}$$

From (3.1) and the linearized problem (2.4), we easily conclude that

$$\int_0^1 f(u_0(x))w(x)\,\mathrm{d}x = 0.$$

But, by lemma 2.5,

$$\int_0^1 f(u_0(x))w(x)\,\mathrm{d}x = \frac{1}{4\lambda}u''(1)w''(1),$$

and $u''(1) \neq 0$ by lemma 2.3, while $w''(1) \neq 0$ by lemma 2.1. This is a contradiction, and hence the Crandall–Rabinowitz theorem 1.3 applies at (λ_0, u_0). The same analysis applies at any other critical solution. We conclude that all solution curves can be continued globally, i.e. at any point, either the implicit function theorem or the Crandall–Rabinowitz theorem applies.

We show next that at (λ_0, u_0), and any other critical point, a turn to the left must occur. Near the point (λ_0, u_0) according to the Crandall–Rabinowitz theorem, we have $\lambda = \lambda(s)$, $u = u(s)$, with $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$ and $u(0) = u_0$, $u_s(0) = w$. Differentiating equation (1.1) twice in s, and then setting $s = 0$, we have

$$\left.\begin{aligned} u_{ss}''''(x) = \lambda_0 f_u u_{ss} + \lambda_0 f_{uu} w^2 + \lambda''(0) f(u_0), \quad x \in (0,1),\\ u_{ss}(0) = u_{ss}'(0) = u_{ss}(1) = u_{ss}'(1) = 0. \end{aligned}\right\} \tag{3.2}$$

Multiplying the equation in (3.2) by w, the equation in (2.4) by u_{ss}, subtracting the results and integrating, we have

$$\lambda''(0) = -\frac{\int_0^1 f_{uu}(u_0)w^3\,\mathrm{d}x}{\int_0^1 f(u_0)w\,\mathrm{d}x} < 0,$$

since $f(u)$ is convex and $w(x)$ is positive.

We are now in a position to complete the proof. The curve of solutions, starting at $\lambda = 0$, $u = 0$ after turning back at (λ_0, u_0), continues without any more turns. Since, by lemma 2.10, the maximum value of the solution, $u(\frac{1}{2}, \lambda)$, is strictly increasing on the curve, and the curve continues globally, we see that $u(\frac{1}{2}, \lambda)$ tends to infinity along the curve. By lemma 2.14, this may happen only when $\lambda \downarrow 0$. Observe that

$u(\frac{1}{2}, \lambda)$ changes from zero to infinity along this curve, and hence, by lemma 2.10, this curve exhausts the set of all possible solutions (since it takes up all possible values of $u(\frac{1}{2}, \lambda)$). □

Proof of theorem 1.2. By theorem 2.13, any non-trivial solution of the linearized problem (2.4) does not change sign. Multiplying the equation in (2.4) by u, the equation in (1.1) by w, integrating and subtracting, we get

$$\int_0^1 (uf'(u) - f(u))w \, dx = 0.$$

In view of condition (1.2), the integrand on the left is non-negative, and hence $w \equiv 0$. It follows that all solutions of (1.1) are non-degenerate. We can then use the implicit function theorem to continue the solutions.

Assuming that problem (1.1) has a solution (otherwise there is nothing to prove), let us continue this solution for increasing λ. We claim that either $u(\frac{1}{2}, \lambda)$ tends to zero or to infinity as we increase λ (it is possible for $u(\frac{1}{2}, \lambda)$ to become zero at a finite λ, which does not matter for us, as will be clear shortly). Since, by lemma 2.10, $u(\frac{1}{2}, \lambda)$ is monotone, the only alternative is for $u(\frac{1}{2}, \lambda)$ to tend to a finite non-zero limit, say u_∞. But then $u(x, \lambda)$ is bounded for large λ. Rewrite (1.1) as

$$u(x) = \lambda \int_0^1 G(x, \xi) f(u(\xi)) \, d\xi, \tag{3.3}$$

where $G(x, \xi)$ is Green's function, which is known to be strictly positive for all x and ξ. Observe that $f(0) = 0$ by (1.2). We see from (3.3) that the only way its right-hand side will remain bounded as $\lambda \to \infty$ is for $u(x, \lambda)$ to become very small over most of the interval $(0, 1)$, except for a quick transition to values near u_∞ around the midpoint $x = \frac{1}{2}$.

We claim that $u''(0, \lambda) > 0$ has to be bounded for λ large. Assuming otherwise, we conclude that $u'''(0)$ would have to be negative and large in the absolute value, since $u(x)$ is small over most of the interval. But since $f(0)$ is also small over most of the interval, integrating the equation in (1.1), we conclude that $u'''(x)$ would have to be negative and large in the absolute value over some subinterval, which would make $u(x)$ negative, a contradiction. (Over most of the interval $(0, \frac{1}{2})$, $u(x)$ is a small and increasing function, which is inconsistent with large third derivatives.)

By lemma 2.6, we have

$$u'(x)u'''(x) - \tfrac{1}{2}u''(x)^2 - \lambda F(u(x)) = -\tfrac{1}{2}u''(0)^2 \quad \text{for all } x \in [0, 1]. \tag{3.4}$$

Setting $x = \frac{1}{2}$, we conclude from (3.4) that

$$\lambda F(u(\tfrac{1}{2})) < \tfrac{1}{2}u''(0)^2.$$

We have a contradiction, since $u(\frac{1}{2}) \to u_\infty$ and $u''(0)$ is bounded while $\lambda \to \infty$.

We conclude that $u(\frac{1}{2}, \lambda)$ tends to either zero or infinity for increasing λ. Clearly, the same is true for decreasing λ, since now the possibility of $u(\frac{1}{2}, \lambda)$ tending to a finite non-zero limit as $\lambda \downarrow 0$ is clearly excluded by the integral form (3.3).

By lemma 2.10, if the solution curve tends to infinity (zero) as λ increases, then it has to tend to zero (infinity) as λ decreases, since once some value of $u(\frac{1}{2}, \lambda)$

is *taken* it cannot be repeated for any other solution. Hence, along our curve of solutions, all values of $u(\frac{1}{2}, \lambda)$ from zero to infinity are taken, and so there is only one curve of solutions. Since all solutions are non-degenerate, the curve does not turn, and hence at any λ there is at most one solution. $\square$

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