Exact multiplicity of positive solutions for concave-convex and convex-concave nonlinearities

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Abstract

This note gives an unified treatment of the exact multiplicity results for both S-shaped and reversed S-shaped bifurcation for positive solutions of the two-point problem

$$u'' + \lambda f(u) = 0, \text{ for } -1 < x < 1, \; u(-1) = u(1) = 0 \,,$$

depending on a positive parameter λ, for both concave-convex and convex-concave nonlinearities $f(u)$.

Key words: Exact multiplicity of positive solutions, S-shaped and reversed S-shaped bifurcation.

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1 Introduction

In this note we give an unified treatment of the exact multiplicity results for both S-shaped and reversed S-shaped bifurcation for positive solutions of the two-point problem

$$u'' + \lambda f(u) = 0, \text{ for } -1 < x < 1, \; u(-1) = u(1) = 0 \,. \tag{1.1}$$

We show that in essentially the same manner one can derive the result of the authors on S-shaped bifurcation [9], and the recent result of K.C. Hung [6] on reversed S-shaped bifurcation.

Recall that positive solutions of (1.1) are even functions, with $u'(x) < 0$ for $x > 0$, so that they take the global maximums at $x = 0$. Moreover, the maximum value $\alpha = u(0)$ is a global parameter, i.e., it uniquely identifies the solution pair $(\lambda, u(x))$, see e.g., [8]. It follows that the two-dimensional graph of (λ, α) gives a faithful representation of the solution curves.

We define $f(u)$ to be convex-concave if there is an $\gamma \in (0, \infty)$, such that

$$f''(u) > 0 \text{ for } u \in (0, \gamma), \quad f''(u) < 0 \text{ for } u \in (\gamma, \infty).$$

Similarly, $f(u)$ is called concave-convex if there is an $\gamma \in (0, \infty)$, such that

$$f''(u) < 0 \text{ for } u \in (0, \gamma), \quad f''(u) > 0 \text{ for } u \in (\gamma, \infty).$$

The following result is known, and it has proved to be quite useful, see P. Korman, Y. Li and T. Ouyang [10], or T. Ouyang and J. Shi [12] (it also holds for balls in R^n)

Theorem 1.1 **(i)** *Assume that $f(u)$ is convex-concave, and $f(0) \leq 0$. Then the global solution curve admits at most one turn in the (λ, α) plane. Moreover, the turn is to the right.*
(ii) *Assume that $f(u)$ is concave-convex, and $f(0) \geq 0$. Then the global solution curve admits at most one turns in the (λ, α) plane. Moreover, the turn is to the left.*

In this note we shall consider the cases when $f(0)$ has a "wrong sign", i.e., either $f(u)$ is concave-convex and $f(0) < 0$, or convex-concave, and $f(0) > 0$. Following P. Korman and Y. Li [9], we give conditions under which the solution curve makes exactly two turns, so that it is either S-shaped, or reversed S-shaped. We show that the argument in P. Korman and Y. Li [9] can cover both cases. We also simplify that argument in several places. We show that the reversed S-shaped bifurcation can be seen as a "dual version" of the S-shaped bifurcation. In particular, we easily recover one of the main results of K.C. Hung [6], and suggest a generalization. We also provide some extra information on the reversed S-shaped curve. K.C. Hung [6] also discusses the broken reversed S-shaped curves, in case $f(u)$ is concave-convex and has three roots (Theorems 1.2 and 2.2 in [6]). Our results imply the Theorem 1.2 from that paper (which originated in [13]), but not the stronger Theorem 2.2.

2 Solution curves with at most two turns

We shall need the linearized problem corresponding to (1.1)

$$w'' + \lambda f'(u)w = 0, \;\; -1 < x < 1, \;\; w(-1) = w(1) = 0. \tag{2.1}$$

Recall that $(\lambda, u(x))$ is called a *critical point* (or a singular solution) of (1.1), if the problem (2.1) admits non-trivial solutions. In such a case $w(x)$ is an even function, and it does not change sign (thus for the rest of this paper we assume that $w(x) > 0$ for all x), see e.g., [8]. We assume that $f(u) \in C^2[0,\infty)$, and define a function $I(u) \equiv f^2(u) - 2F(u)f'(u)$, where $F(u) = \int_0^u f(t)\,dt$. The following lemma has originated from P. Korman, Y. Li and T. Ouyang [10], and P. Korman and Y. Li [9]. (Since both $u(x)$ and $w(x)$ are even functions, we may restrict the integrals below to the interval $(0,1)$.)

Lemma 2.1 **(i)** *Assume that $f(u)$ is convex-concave and there is a $\beta > \gamma$, such that*

$$I(\beta) = f^2(\beta) - 2F(\beta)f'(\beta) \geq 0. \tag{2.2}$$

Let (λ, u) be any critical point of (1.1), such that $u(0) \geq \beta$, and let $w(x)$ be any non-trivial solution of the linearized problem (2.1). Then

$$\int_0^1 f''(u(x))u'(x)w^2(x)\,dx > 0, \tag{2.3}$$

$$\int_0^1 f''(u(x))w^3(x)\,dx < 0, \tag{2.4}$$

and the solution curve turns to the right.

(ii) *Assume that $f(u)$ is concave-convex and there is a $\beta > \gamma$, such that*

$$I(\beta) = f^2(\beta) - 2F(\beta)f'(\beta) \leq 0. \tag{2.5}$$

Let (λ, u) be any critical point of (1.1), such that $u(0) \geq \beta$, and let $w(x)$ be any non-trivial solution of the linearized problem (2.1). Then

$$\int_0^1 f''(u(x))u'(x)w^2(x)\,dx < 0, \tag{2.6}$$

$$\int_0^1 f''(u(x))w^3(x)\,dx > 0, \tag{2.7}$$

and a turn to the left occurs.

Proof: We shall derive a convenient expression for the integral in (2.3). Differentiating (2.1), we get

$$w_x'' + \lambda f'(u) w_x + \lambda f''(u) u' w = 0. \tag{2.8}$$

Multiplying the equation (2.8) by w, the equation (2.1) by w_x, subtracting and integrating over $(0, 1)$, we express

$$\lambda \int_0^1 f''(u) u' w^2 \, dx = {w'}^2(1) - \lambda w^2(0) f'(u(0)). \tag{2.9}$$

By differentiation, we verify that $u''(x)w(x) - u'(x)w'(x)$ is equal to a constant for all x, and hence

$$u''(x)w(x) - u'(x)w'(x) = -\lambda w(0) f(u(0)), \text{ for all } x \in [0, 1].$$

Evaluating this expression at $x = 1$, we obtain

$$w'(1) = \frac{\lambda w(0) f(u(0))}{u'(1)}. \tag{2.10}$$

Multiplying (1.1) by u', and integrating over $(0, 1)$, we have

$${u'}^2(1) = 2\lambda F(u(0)). \tag{2.11}$$

Using (2.11) and (2.10) in (2.9), we finally express

$$\int_0^1 f''(u) u' w^2 \, dx = \frac{w^2(0)}{2F(u(0))} I(u(0)) \, . \tag{2.12}$$

Let us prove part **(ii)** of the lemma (part **(i)** is similar, and it was proved in P. Korman and Y. Li [9]). Since

$$I'(u(0)) = -2F(u(0)) f''(u(0)) < 0, \quad \text{for } u(0) > \beta,$$

we conclude that $I(u(0)) < I(\beta) \le 0$, and, in view of (2.12), the inequality (2.6) follows. (Observe that $F(u(0)) > 0$ for any positive solution.)

Turning to the proof of (2.7), consider the function $p(x) \equiv \frac{w(x)}{-u'(x)}$. Since $p(1) = 0$, and since $u''(x)w(x) - u'(x)w'(x)$ is equal to a constant for all x, we have

$$p'(x) = -\frac{u'(1)w'(1)}{{u'(x)}^2} < 0 \, ,$$

which implies that the function $p(x)$ is positive and decreasing on $(0,1)$. Let x_0 be the point where $f''(u(x))$ changes sign (i.e., $u(x_0)=\gamma$, and $f''(u(x))>0$ on $(0,x_0)$, and $f''(u(x))<0$ on $(x_0,1)$). By scaling $w(x)$, we may achieve that $w(x_0)=-u'(x_0)$, or $p(x_0)=1$. Then $w(x)>-u'(x)$ on $(0,x_0)$, and $w(x)<-u'(x)$ on $(x_0,1)$, and using (2.6), we have

$$\int_0^1 f''(u(x))w^3\,dx > \int_0^1 f''(u(x))w^2(-u'(x))\,dx > 0\,,$$

concluding the proof. ♢

We shall use the following lemma (see e.g., [8]).

Lemma 2.2 *Assume that $f''(u)<0$ ($f''(u)>0$) for $u\in(0,\delta)$, for some $\delta>0$. Then only turns to the right (left) are possible on the curve of positive solutions of (1.1), while $u(0)\in(0,\delta)$.*

Recall that the maximum value of solution, $\alpha=u(0)$, uniquely identifies the solution pair $(\lambda,u(x))$ of (1.1), and the solution set of (1.1) can be faithfully depicted by the planar curves in the (λ,α) plane. It is natural to ask: which way the solution curve travels through a given point (λ,α)? Define

$$h(u)=2F(u)-uf(u)\,,$$

where, as usual, $F(u)=\int_0^u f(t)\,dt$. The following result is from P. Korman [7], see also the discussion in P. Korman [8] of the preceding results in [11], [2], and [14].

Theorem 2.1 **(i)** *Assume that*

$$h(\alpha)<h(u),\quad \text{for } 0<u<\alpha. \tag{2.13}$$

Then the positive solution of (1.1), with maximum value $u(0)=\alpha$, travels to the left in the (λ,α) plane, i.e., $\lambda'(\alpha)<0$. (This solution is unstable, see P. Korman [8] for the definition and details.)

(ii) *Assume that*

$$h(\alpha)>h(u),\quad \text{for } 0<u<\alpha. \tag{2.14}$$

Then the positive solution of (1.1), with maximum value $u(0)=\alpha$, travels to the right in the (λ,α) plane, i.e., $\lambda'(\alpha)>0$. (This solution is stable.)

The following is the central result of this paper.

Theorem 2.2 **(i)** *Assume that $f(u)$ is convex-concave, and $f(0) > 0$. Assume that $h(\gamma) \le 0$, and $f(u) > 0$ for $u > \gamma$. Then the global solution curve admits at most two turns in the (λ, α) plane. Moreover, only turns to the right are possible if $u(0) > \gamma$.*

(ii) *Assume that $f(u)$ is concave-convex, and $f(0) < 0$. Assume that $h(\gamma) \ge 0$, and $f(u) > 0$ for $u > \gamma$. Then the global solution curve admits at most two turns in the (λ, α) plane. Moreover, only turns to the left are possible if $u(0) > \gamma$.*

Proof: Let us prove the part **(ii)** first. We have $h(0) = 0$, $h'(u) = f(u) - uf'(u)$, $h''(u) = -uf''(u)$. Since $h''(u) > 0$ on $(0, \gamma)$, and $h(\gamma) > 0$, there exists $u_1 \in (0, \gamma)$ so that $h'(u) < 0$ on $(0, u_1)$ and $h'(u) > 0$ on (u_1, γ), see Figure 1.

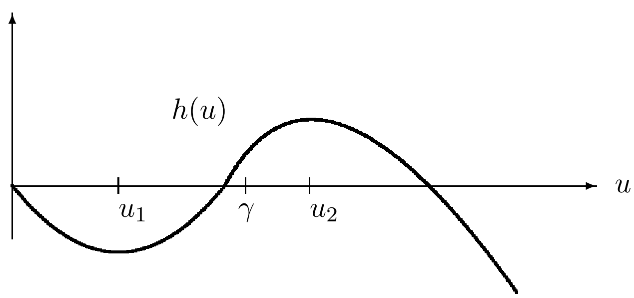

Figure 1: The function $h(u)$ for part **(ii)**

On (γ, ∞) we have $h''(u) < 0$, so that either $h'(u) > 0$ on (γ, ∞), or there is a point u_2 where $h'(u_2) = 0$. In the first case, the solution curve travels to the right for all $u(0) > \gamma$, in view of the Theorem 2.1. In that case the global solution curve has at most one turn, a turn to the right occurring where $u(0) < \gamma$ ($f(u)$ is concave in that range, see Lemma 2.2). Turning to the second case, we have by our assumptions $f(u_2) > 0$, and then $h'(u_2) = 0$ implies that $f'(u_2) > 0$. Since $h(u_2) > 0$, we have $f(u_2)u_2 < 2F(u_2)$ and then

$$I(u_2) = f^2(u_2) - 2F(u_2)f'(u_2) < f^2(u_2) - f(u_2)u_2 f'(u_2) = 0 \,.$$

By the second part of Lemma 2.1 (with $\beta = u_2$) it follows that only turns to the left are possible when $u(0) > u_2$. Since only turns to the right are possible when $u(0) < \gamma$, and the curve travels to the right when $u(0) \in (\gamma, u_2]$, the proof follows. (At most one turn, to the right, is possible for $u(0) \le u_2$.)

The part **(i)** is proved similarly. The function $h(u)$ for this case is given in Figure 2. This time only turns to the left are possible when $u(0) < \gamma$, the solution curve travels to the left when $u(0) \in (\gamma, u_2]$, and only turns to the right are possible when $u(0) > u_2$, see [9] for more details.

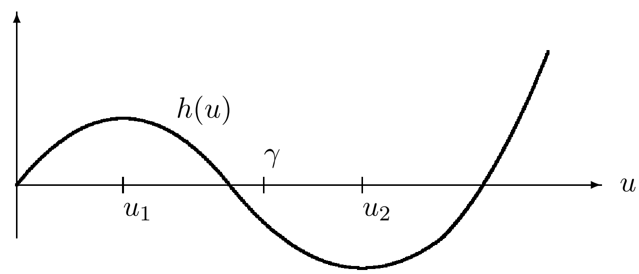

Figure 2: The function $h(u)$ for part **(i)**

The following exact multiplicity result follows easily.

Theorem 2.3 *Assume that $f(u)$ is concave-convex, and $f(0) < 0$. Assume that $f(u)$ has exactly one root, i.e., $f(u) < 0$ on $[0, a)$, $f(u) > 0$ on (a, ∞) for some $a > 0$, and*

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty \,. \tag{2.15}$$

Assume also that $\gamma > a$, and we have $F(\gamma) > 0$ and $h(\gamma) \geq 0$. Define $\theta \in (0, \gamma)$ by $F(\theta) = 0$. Then all positive solutions of (1.1) lie on a unique solution curve, which is reversed S-shaped in the $(\lambda, u(0))$ plane. Namely, one end of this curve starts at $\lambda_1 = \frac{1}{2}\left(\int_0^\theta \frac{du}{\sqrt{-F(u)}}\right)^2$, $u(0) = \theta$ (and also $u'(\pm 1) = 0$). From the point (λ_1, θ), the curve travels to the left, it makes exactly two turns, and it tends to infinity as $\lambda \to 0$.

Proof: We begin with the positive solution of (1.1) satisfying $u'(\pm 1) = 0$. Since $\frac{1}{2}u'^2 + \lambda F(u) = constant$, for that solution we have

$$\frac{1}{2}u'^2 + \lambda F(u) = 0 \,,$$

and, in particular, $u(0) = \theta$. (Existence of such solution follows by solving $u'' + f(u) = 0$, $u(0) = \theta$, $u'(0) = 1$, and then scaling, so that the first root occurs at $x = 1$.) On $(0,1)$ we have $\frac{du}{dx} = -\sqrt{2\lambda}\sqrt{-F(u)}$, and integrating over $(0,1)$, we calculate $\lambda = \lambda_1 = \frac{1}{2}\left(\int_0^\theta \frac{du}{\sqrt{-F(u)}}\right)^2$. This solution, call it $u_1(x)$, is non-singular in the class of even functions. Indeed the function $u_1'(x)$ is an odd solution of the linearized problem (2.1) (computed at $u_1(x)$). Hence, (2.1) has no non-trivial even solutions. By the Implicit Function Theorem, we can continue the solution point $(\lambda_1, u_1(x))$ in λ to obtain even solutions. It turns out that we get positive solutions for $\lambda < \lambda_1$ (and sign changing even solutions for $\lambda > \lambda_1$). Indeed, differentiating (1.1) in λ, we get

$$u_\lambda'' + \lambda f'(u)u_\lambda + f(u) = 0, \quad \text{for } -1 < x < 1, \quad u_\lambda(-1) = u_\lambda(1) = 0\,,$$

and then it is easy to verify that $u_\lambda = \frac{1}{2}xu'(x)$. (Both functions satisfy the same equation, and are zero at $x = 1$.) Since $u_\lambda < 0$ for $x \in (0,1)$, we obtain positive solutions for $\lambda < \lambda_1$. We now continue this solution curve (which at first travels to the left). The graph of $h(u)$ is as in the Figure 1 (at least for $u \in (0,\gamma)$). By the Theorem 2.1, the solution curve travels to the right, by the time $u(0) = \gamma$. Hence the solution curve has made exactly one turn to the right before that (recall that $f''(u) < 0$ on $(0,\gamma)$, see Lemma 2.2). Since $f(u)$ is superlinear, the solution curve cannot travel to the right indefinitely, see e.g., [8]. By the Theorem 2.2, only turns to the left are possible for $u(0) > \gamma$, so that the solution curve will make exactly one turn to the left, and then tend to infinity. Using (2.15) again, we conclude that $\lambda \to 0$. $\diamondsuit$

We remark that the graph of $h(u)$ is exactly as in the Figure 1, i.e., there is u_2 such that $h'(u_2) = f(u_2) - u_2 f'(u_2) = 0$. Observe that u_1 and u_2 are the points where a straight line out of the origin is tangent to the graph of $f(u)$. Our conditions on $f(u)$ imply existence of two such points.

The case when $\gamma \le a$ is covered by the following result.

Theorem 2.4 *Assume that $f(u)$ is concave-convex, and $f(0) < 0$. Assume that $f(u)$ has exactly one root, i.e., $f(u) < 0$ on $[0,a)$, $f(u) > 0$ on (a,∞) for some $a > 0$, and the condition (2.15) holds. Assume that $\gamma \le a$. Then any positive solution of (1.1) is non-singular, i.e., the corresponding linearized problem (2.1) has only the trivial solution. Let $\theta > a$ be such that $F(\theta) = 0$. Then all positive solutions of (1.1) lie on a unique solution curve in the $(\lambda, u(0))$ plane. One end of this curve starts at $\lambda_1 = \frac{1}{2}\left(\int_0^\theta \frac{du}{\sqrt{-F(u)}}\right)^2$, $u(0) = \theta$ (and also $u'(\pm 1) = 0$). From (λ_1, θ) the curve travels to the left, it makes no turns, and it tends to infinity as $\lambda \to 0$.*

Proof: With $k(u) \equiv f(u) - uf'(u)$, we have $k(a) = -af'(a) \le 0$, $k'(u) = -uf''(u) < 0$ for $u > a$. It follows that $k(u) < 0$ for $u > a$, i.e., $f'(u) > \frac{f(u)}{u}$ for $u > a$. By the Theorem 3.1 in [8], any positive solution of (1.1) is non-singular, and the proof follows. ♢

We now recall the result of P. Korman and Y. Li [9], which follows from the Theorem 2.2 in the same way as the Theorem 2.3.

Theorem 2.5 *Assume that $f(u) > 0$ for all $u \ge 0$, $f(u)$ is convex-concave and $\lim_{u\to\infty} \frac{f(u)}{u} = 0$. Assume also that $h(\gamma) \le 0$. Then the solution curve of (1.1) is exactly S-shaped. Namely, it starts at $(\lambda = 0, u = 0)$, it makes exactly two turns, and then continues for all λ without any more turns.*

We now present generalizations of the above results, allowing for multiple changes of sign for $f''(u)$.

Theorem 2.6 **(i)** *In the conditions of the Theorem 2.3, let $\mu > u_2$ be such that $h(\mu) < h(u)$ for all $u \in [0, \mu)$, see Figure* 1*. Then the Theorem 2.3 remains true if for $u > \mu$ we no longer require that $f''(u) > 0$, replacing this by a weaker condition that $h'(u) = f(u) - uf'(u) < 0$.*

(ii) *In the conditions of the Theorem 2.5, let $\mu > u_2$ be such that $h(\mu) > h(u)$ for all $u \in [0, \mu)$, see Figure* 2*. Then the Theorem 2.5 remains true if for $u > \mu$ we no longer require that $f''(u) < 0$, replacing this by a weaker condition that $h'(u) = f(u) - uf'(u) > 0$.*

Proof: Let us prove the case **(i)**, and the other one is similar. We know from the proof of the Theorem 2.3 that the solution curve is exactly reversed S-shaped for $u(0) \in [0, \mu)$. By the Theorem 2.1, the solution curve continues to travel to the left for all $u(0) > \mu$. ♢

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