ON UNIQUENESS OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR EQUATIONS

PHILIP KORMAN *

Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025

ABSTRACT. Using the technique of Adimurthy, F. Pacella and S.L. Yadava [1], we extend an uniqueness result for a class of non-autonomous semilinear equations in M.K. Kwong and Y. Li [8]. We also observe that combining the results of [1] with bifurcation theory, one can obtain a detailed picture of the global solution curve for a class of concave-convex nonlinearities.

1. INTRODUCTION

Our main result deals with uniqueness of positive solution for a class of semilinear equations

$$\Delta u + q(|x|)u + u^p = 0 \quad \text{for } |x| < R \;\; u = 0, \;\; \text{when } |x| = R. \tag{1.1}$$

Here $x \in R^n$, $u = u(x)$, and $q = q(r)$ is a given differentiable function. The constant p is assumed to be subcritical, $1 < p \le \frac{n+2}{n-2}$. This problem was studied by M.K. Kwong and Y. Li [8]. In Theorem 2 of that paper a condition for uniqueness of positive radial solution is given. Namely, they define the constants $\beta = \frac{2(n-1)(p-1)}{p+3}$ and $L = \frac{2(n-1)(np+n-2p-4)}{(p+3)^2}$, and show that the problem (1.1) has at most one positive solution, provided that $1 < p < \frac{n+2}{n-2}$, $n \ge 3$, and the function

$$G(r) := r^\beta q(r) - Lr^{\beta-2} \quad \text{has a } \wedge \text{ property,} \tag{1.2}$$

i.e. there exists some $c \in [0, R]$ so that the function $G(r)$ is increasing on $[0, c)$, and decreasing on $(c, R]$ (notice that both increasing and decreasing $G(r)$ are included in this definition). Since the condition (1.2) can be cumbersome to verify, M.K. Kwong and Y. Li [8] provide the following corollary: condition (1.2) can be replaced by

$$r^\beta q(r) \quad \text{is a nondecreasing function.} \tag{1.3}$$

Using the technique of Adimurthy, F. Pacella and S.L. Yadava [1], we prove uniqueness of positive solution of (1.1) for $n \ge 1$, $1 < p \le p^*$, where $p^* = \frac{n+2}{n-2}$ for $n \ge 3$, $p^* = \infty$ for $p = 1, 2$; and

$$\frac{d}{dr}\left[r^{2c_n} q(r)\right] \ge 0, \tag{1.4}$$

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where $c_n = n(\frac{1}{2} - \frac{1}{p+1})$. For differentiable $q(r)$ our condition (1.4) improves the condition (1.3) for all dimensions n, as one easily checks that inequality $2c_n > \beta$ is equivalent to p being subcritical. Our condition (1.4) provides also an extension of the full uniqueness result in M.K. Kwong and Y. Li [8], as the following example shows.

Example. Take $n = 4$, $p = 2$ and $R = 50$. Then one has $2c_4 = 4/3 > \beta = 6/5$. Compute also $L = 24/25$ and $\beta - 2 = -4/5$. Assume that

$$q(r) = 0.02 + \frac{20}{1 + r^{4/3}}.$$

Then $r^{4/3}q(r)$ is increasing so that condition (1.4) holds. On the other hand,

$$r^{\beta}q(r) - Lr^{\beta-2} = 0.02r^{6/5} + \frac{20r^{6/5}}{1 + r^{4/3}} - \frac{24}{25}r^{-4/5}$$

is not of type $\wedge$ on $(0, 50)$.

We remark that approaches of M.K. Kwong and Y. Li [8] and of Adimurthy, F. Pacella and S.L. Yadava [1] are similar. Both papers show that any two positive solutions cannot intersect, which is done by examining the ratio of the solutions. Adimurthy, F. Pacella and S.L. Yadava [1] use a Pohozhaev identity to complete the argument, whereas M.K. Kwong and Y. Li [8] were using the energy function.

In Section 3 we consider the problem

$$\Delta u + \lambda\left(u^q + u^p\right) = 0 \quad \text{for } |x| < R,\ u = 0 \quad \text{for } |x| = R, \tag{1.5}$$

depending on a positive parameter λ, with $0 < q < 1 < p \le p^*$, where p^* is the critical exponent, defined above. This problem was the focus of the above mentioned paper of Adimurthy, F. Pacella and S.L. Yadava [1], where it is proved that any two positive solution cannot intersect. We observe that this result implies that at any turning point any non-trivial solution of the corresponding linearized system is of one sign. This allows us to use the bifurcation theory in the spirit of P. Korman, Y. Li and T. Ouyang [7] and T. Ouyang and J. Shi [9] to obtain an exact multiplicity result, together with a detailed picture of the solution curve. Earlier a similar result was obtained in [9], but for $p \le \frac{n}{n-2}$.

2. Uniqueness for the equation $\Delta u + q(r)u + u^p = 0$

We consider positive radially symmetric solutions for the semilinear Dirichlet problem on a ball $|x| < R$ in R^n

$$\Delta u + f(r, u) = 0 \quad \text{for } |x| < R, \quad u = 0 \quad \text{on } |x| = R. \tag{2.1}$$

We set, as is customary, $F(r, u) = \int_0^u f(r, s)\, ds$, and introduce the function $H(r) = \dfrac{1}{2}ru_r^2 + \alpha u u_r + rF(r, u)$, with a constant α to be chosen shortly. Then using the equation (2.1), one easily verifies that

$$\begin{gathered} H' + \tfrac{n-1}{r}H = \\ \left(1 - \tfrac{n}{2} + \alpha\right)u_r^2 + nF(r, u) - \alpha u f(u, r) + rF_r(r, u). \end{gathered} \tag{2.2}$$

We now restrict to the problem

$$u'' + \frac{n-1}{r}u' + q(r)u + u^p = 0 \quad \text{for } r < R, \quad u = 0 \quad \text{on } |x| = R. \tag{2.3}$$

Here $q(r)$ is a positive function of class $C^1(0,R) \cap C[0,R]$, and p is a sub-critical power, $1 < p \le p^*$, where $p^* = \frac{n+2}{n-2}$, if $n > 2$ and $p^* = \infty$ for $n = 1, 2$. For this problem it is convenient to select $\alpha = \frac{n}{p+1}$. Then (2.2) takes the form

$$H' + \frac{n-1}{r} H = (1 - c_n) u_r^2 + c_n q u^2 + \frac{1}{2} r q' u^2, \tag{2.4}$$

where $c_n = n(\frac{1}{2} - \frac{1}{p+1})$. Notice that condition $1 < p \le \frac{n+2}{n-2}$ for $n \ge 3$ is equivalent to

$$0 < c_n \le 1, \tag{2.5}$$

and in the cases $n = 1, 2$ we have (2.5) for any $1 < p < \infty$.

Multiplying the equation (2.4) by the integrating factor r^{n-1}, and integrating over $[r_1, r_2] \subset (0, 1)$, we obtain the generalized Pohozhaev identity,

$$\begin{aligned} &\int_{r_1}^{r_2} \left[(1 - c_n) u_r^2 + c_n q u^2 + \tfrac{1}{2} r q' u^2 \right] r^{n-1} \, dr \\ &\quad = r_2^{n-1} H(r_2) - r_1^{n-1} H(r_1), \end{aligned} \tag{2.6}$$

which will be used later. Notice that the quantity on the left in (2.6) is positive, provided that $rq' + 2c_n q > 0$, i.e. if

$$\frac{d}{dr} \left[r^{2c_n} q(r) \right] \ge 0, \tag{2.7}$$

and in case $c_n = 1$ (i.e., $p = p^*$) the above inequality is assumed to be strict on a set of positive measure. It is easy to see that (2.7) implies that $q(r)$ is a positive function.

We need the following simple observation.

Lemma 2.1. *Assume that $q(r) > 0$ for all $r \in (0, R)$. Then any positive solution of (2.3) satisfies*

$$u'(r) < 0 \quad \textit{for all } x \in (0, R).$$

Proof: It is clear from the equation (2.3) that any critical point of the solution inside $(0, R)$ would have to be a point of local maximum, and hence $r = 0$ is the only critical point.

Theorem 2.1. *Assume that $0 < q(r) \in C^1(0,R) \cap C[0,R]$ satisfies the condition (2.7), and $p \le p^*$ is subcritical. Then any two positive solutions of (2.1) do not intersect (i.e. they are strictly ordered on $(0, R)$).*

Proof: We follow closely Adimurthy et al [1], and use their notation. Let $v_1(r)$ and $v_2(r)$ be two positive solutions of (2.1) with $v_1(0) < v_2(0)$. Assume on the contrary that solutions intersect. It follows that the function $z(r) \equiv \frac{v_1(r)}{v_2(r)}$ becomes greater than 1 somewhere on $(0, R)$, while $z(0) < 1$. Let $\xi_0 \in (0, R]$ be the first local maximum of $z(r)$ with $z(\xi_0) > 1$, and denote $t_0 = z(\xi_0)$. If $\xi_0 = R$, then

$$t_0 = \frac{v_1'(R)}{v_2'(R)}. \tag{2.8}$$

If $\xi_0 \in (0, R)$, then $z'(\xi_0) = 0$. In both cases we have

$$1 < t_0 = \frac{v_1(\xi_0)}{v_2(\xi_0)} = \frac{v_1'(\xi_0)}{v_2'(\xi_0)}, \tag{2.9}$$

$$v_1(r) < t_0 v_2(r) \quad \text{for all } r \in [0, \xi_0). \tag{2.10}$$

We claim that

$$\frac{v_1'(r)}{v_2'(r)} < t_0 \quad \text{for all } r \in [0, \xi_0). \tag{2.11}$$

Denote $p = v_1' v_2 - v_1 v_2'$. Then from the equation (2.1) written at both v_1 and v_2 we obtain

$$\frac{d}{dr}\left(r^{n-1}p(r)\right) + r^{n-1}v_1 v_2\left(v_1^{p-1} - v_2^{p-1}\right) = 0. \tag{2.12}$$

From the definition of ξ_0 as the *first* point of maximum of $z(r)$ (with maximum value above one), it follows that $v_1(r)$ and $v_2(r)$ intersect exactly once on $(0, \xi_0)$. Let $r_1 < \xi_0$ be such that $v_1(r_1) = v_2(r_1)$. Then the function $r^{n-1}p(r)$ vanishes at $r = 0$ and $r = \xi_0$, it is increasing on $(0, r_1)$, and decreasing on (r_1, ξ_0). It follows that $p(r)$ is positive on $(0, \xi_0)$, i.e., using Lemma 2.1,

$$\frac{v_1'(r)}{v_2'(r)} < \frac{v_1(r)}{v_2(r)} \quad \text{for all } r \in [0, \xi_0). \tag{2.13}$$

From (2.13) and (2.10) the claim (2.11) follows.

We now distinguish between two cases.

Case 1. $\xi_0 = R$. Taking a linear combination of two Pohozhaev identities, the ones for $v_1(r)$ and $v_2(r)$ respectively, we have

$$\begin{aligned}(1-c_n)\int_0^R \left({v_1'}^2 - t_0^2 {v_2'}^2\right) r^{n-1}\,dr + \int_0^R Q(r)\left(v_1^2 - t_0^2 v_2^2\right) r^{n-1}\,dr \\ = \tfrac{1}{2}\left({v_1'}^2(R) - t_0^2 {v_2'}^2(R)\right) R^n,\end{aligned} \tag{2.14}$$

where $Q(r) = \frac{d}{dr}\left[r^{2c_n}q(r)\right] \geq 0$. In view of (2.8) the quantity on the right is zero, while the one on the left is negative by (2.10) and (2.11), a contradiction.

Case 2. $\xi_0 < R$. Similarly to the above we have

$$\begin{aligned}(1-c_n)\int_0^{\xi_0} \left({v_1'}^2 - t_0^2 {v_2'}^2\right) r^{n-1}\,dr + \int_0^{\xi_0} Q(r)\left(v_1^2 - t_0^2 v_2^2\right) r^{n-1}\,dr \\ = \xi_0^{n-1}\left[H_{v_1}(\xi_0) - t_0^2 H_{v_2}(\xi_0)\right],\end{aligned} \tag{2.15}$$

where by $H_{v_1}(\xi_0)$ we denote the function $H(r)$ defined above, with $v_1(r)$ replacing $u(r)$. The quantity on the left is negative as before. Using the definition of t_0 and the formula (2.9), we see that two pairs of terms cancel on the right. What remains is

$$\begin{aligned}\xi_0^{n-1}\left(H_{v_1}(\xi_0) - t_0^2 H_{v_2}(\xi_0)\right) &= \xi_0^n\left[F(v_1, r) - t_0^2 F(v_2, r)\right] \\ &= \tfrac{\xi_0^n}{(p+1)}\left(v_1^{p+1}(\xi_0) - \tfrac{v_1^2(\xi_0)}{v_2^2(\xi_0)} v_2^{p+1}(\xi_0)\right) > 0,\end{aligned}$$

since $v_1(\xi_0) > v_2(\xi_0)$. We reached a contradiction again. It follows that $v_1(r)$ and $v_2(r)$ cannot intersect.

The following uniqueness result is then an immediate corollary.

Theorem 2.2. *In the conditions of the preceding theorem the problem (2.1) has at most one positive radial solution.*

Proof: By the preceding theorem any two solutions $v_1(r)$ and $v_2(r)$ of (2.1) are strictly ordered. Multiplying the equation for $v_1(r)$ by $v_2(r)$, and the equation for $v_2(r)$ by $v_1(r)$, integrating over $(0, R)$ and subtracting, we obtain a contradiction.

3. Exact multiplicity and the global solution curve for the equation $\Delta u + \lambda(u^q + u^p) = 0$

We begin by recalling some facts on the more general equations of the type

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < R, \;\; u = 0 \;\text{ on }\; |x| = R, \tag{3.1}$$

depending on a positive parameter λ. By the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [5] positive solutions of (3.1) are radially symmetric, which reduces (3.1) to

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0 \;\text{ for }\; 0 < r < R, \;\; u'(0) = u(R) = 0. \tag{3.2}$$

We shall also need the corresponding linearized equation

$$w'' + \frac{n-1}{r}w' + \lambda f'(u)w = 0 \;\text{ for }\; 0 < r < R, \;\; w'(0) = w(R) = 0. \tag{3.3}$$

The following lemma was proved in [6].

Lemma 3.1. *Assume that the function $f(u) \in C^2(\bar{R}_+)$, and the problem (3.3) has a nontrivial solution w at some λ. Then*

$$\int_0^1 f(u) w r^{n-1}\, dr = \frac{1}{2\lambda} u'(1) w'(1). \tag{3.4}$$

The following lemma is known, see e.g. E.N. Dancer [4]. We present its proof for completeness.

Lemma 3.2. *Positive solutions of the problem (3.2) are globally parameterized by their maximum values $u(0,\lambda)$. I.e., for every $p > 0$ there is at most one $\lambda > 0$, for which $u(0,\lambda) = p$.*

Proof. If $u(r,\lambda)$ is a solution of (3.2) with $u(0,\lambda) = p$, then $v \equiv u(\frac{1}{\sqrt{\lambda}}r)$ solves

$$v'' + \frac{n-1}{r}v' + f(v) = 0, \;\; v(0) = p, \; v'(0) = 0. \tag{3.5}$$

If $u(0,\mu) = p$ for some $\mu \neq \lambda$, then $u(\frac{1}{\sqrt{\mu}}r)$ is another solution of the same problem. This is a contradiction in view of the uniqueness of solutions for initial value problems of the type (3.5), see [10].

Let (λ_0, u_0) be a solution of (3.2). We call this solution to be non-singular if $w(r) \equiv 0$ is the only solution of the linearized problem (3.3). By the implicit function theorem we can then continue (λ_0, u_0) to nearby λ's. If (λ_0, u_0) is a singular solution of (3.2), i.e. the problem (3.3) has a nontrivial solution $w(r)$, then in view of Lemma 3.1, the Crandall-Rabinowitz [3] theorem applies at (λ_0, u_0) (see [7] for more details). According to that theorem either we can still continue the solution to nearby λ's, or a simple turn occurs, i.e., near the point (λ_0, u_0) we have two solutions to one side of λ_0, and no solutions to the other side. In the latter case we shall refer to (λ_0, u_0) as a *turning point.*

We study exact multiplicity of positive solutions for the semilinear Dirichlet problem on a ball $|x| < R$ in R^n

$$\Delta u + \lambda \left(u^q + u^p\right) = 0 \quad \text{for } |x| < R, \quad u = 0 \quad \text{on } |x| = R, \tag{3.6}$$

depending on a positive parameter λ. We assume that the constants p and q satisfy

$$0 < q < 1 < p \leq p^*, \tag{3.7}$$

where $p* = \frac{n+2}{n-2}$ for $n \geq 3$, $p^* = \infty$ for $n = 1, 2$. Notice that positive solutions of (3.6) are radial, so that they satisfy (3.2), and the corresponding linearized equation is given by (3.3).

Lemma 3.3. *Assume that (λ_0, u_0) is a turning point of (3.6). Then*

$$w(r) > 0 \quad \text{for all } r \in [0, R). \tag{3.8}$$

Proof: In view of Lemma 3.1 the Crandall-Rabinowitz theorem applies at (λ_0, u_0) (see [7] for more details). According to that theorem near the point (λ_0, u_0) the difference of two solutions is asymptotic to a factor of $w(r)$, which implies that $w(r) > 0$ for all $r \in [0, R)$, if and only if for λ close to λ_0 any two solutions on the solution curve passing through (λ_0, u_0) do not intersect. By Lemma 2.1 in [1] any two positive solutions of (3.2) do not intersect, completing the proof.

Theorem 3.1. *Assume the condition (3.7) is satisfied. Then there is a critical $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ the problem (3.6) has no positive solutions, it has exactly one positive solution for $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda < \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve, which for $\lambda < \lambda_0$ has two branches denoted by $0 < u^-(r, \lambda) < u^+(r, \lambda)$, with $u^-(r, \lambda)$ strictly monotone increasing in λ and $\lim_{\lambda \to 0} u^+(0, \lambda) = \infty$.*

Proof: According to A. Ambrosetti et al [2] there is a $\lambda^* > 0$, such that for $\lambda \in (0, \lambda^*)$ the problem (3.6) has at least two positive solutions. On the other hand, it is easy to see that no positive solutions are possible for large λ. Indeed, we can find an $\alpha > 0$ so that $u^q + u^p > \alpha u$ for all $u > 0$. Then from (3.6)

$$\lambda_1 \int_{B_R} u\phi_1 = \lambda \int_{B_R} (u^q + u^p) > \lambda\alpha \int_{B_R} u\phi_1,$$

where (λ_1, ϕ_1) is the principal eigenpair of $-\Delta$ on B_R. It follows that no positive solutions exist for $\lambda > \frac{\lambda_1}{\alpha}$. In view of Lemma 3.1 we can continue any solution of (3.6), i.e. at each point (λ, u) either implicit function theorem or the Crandall-Rabinowitz theorem applies, see e.g. [7] for more details.

Let now $u(r, \lambda)$ be an arbitrary solution of (3.6), $\lambda < \lambda^*$. We continue this solution for increasing λ. Since by above remarks this curve cannot be continued indefinitely, it has to reach a turning point (λ_0, u_0), at which the Crandall-Rabinowitz theorem applies. In view of Lemma 3.3 we have $w(r) > 0$ at (λ_0, u_0), where $w(r)$ is a solution of the linearized equation. Observe that the function $u^q + u^p$ is concave for small $u > 0$, and convex for large u, with exactly one point of inflection. Arguing exactly as in [7], we see that the function $\tau(s)$ defined in the Crandall-Rabinowitz theorem satisfies

$$\tau''(0) < 0 \tag{3.9}$$

at (λ_0, u_0), as well at any other turning point, see also [9], where this result is explicitly stated in Theorem 2.3.3. (In [7] the formula (3.9) followed immediately from the inequality $\int_0^R f''(u)w^3 r^{n-1}\,dr < \int_0^R f''(u)wu_r^2 r^{n-1}\,dr$, which is true because the integrand on the right is pointwise larger than the one on the left. Here $f(u) = u^q + u^p$. While $f''(u)$ has a singularity of order $(R-r)^{q-2}$ near $r = R$, by Hopf's boundary lemma $w(r)$ is of order $R - r$ near $r = R$, and hence both integrals above converge.) It follows that only turns "to the left" are admissible. At (λ_0, u_0) we have two solution branches $0 < u^-(r, \lambda) < u^+(r, \lambda)$. We now continue both branches for decreasing λ. In view of (3.9) both branches continue without any turns. By Lemma 3.2 $u(0, \lambda)$ is increasing on the upper branch and decreasing on the lower one. As $\lambda \downarrow 0$ the lower branch has to tend to zero (there is no other place for it to go since at $\lambda = 0$ there are no positive solutions). Similarly, on the upper branch $u(0, \lambda)$ has to go to infinity for $\lambda \downarrow \lambda_1$, $\lambda_1 \geq 0$. We claim that $\lambda_1 = 0$. Indeed, on our solution curve $u(0, \lambda)$ varies from 0 to ∞. By Lemma 3.2 the problem (3.6) cannot have any solutions, not lying on this curve (since all possible values of $u(0, \lambda)$ are taken). But by [2] there are at least two solutions for $\lambda \in (0, \lambda_1)$, a contradiction.

It follows that the solution set of (3.6) consists of one curve, which makes exactly one turn. This implies all the statements of the theorem, except for the monotonicity of the lower branch, which is proved similarly to [7].

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E-mail address: kormanp@math.uc.edu