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NON-SINGULAR SOLUTIONS OF TWO-POINT PROBLEMS, WITH MULTIPLE CHANGES OF SIGN IN THE NONLINEARITY

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ABSTRACT. We prove that positive solutions of the two-point boundary value problem

$$u''(x) + \lambda f(u(x)) = 0, \text{ for } -1 < x < 1, \ u(-1) = u(1) = 0,$$

satisfying $\max u = u(0) > \gamma$, are non-singular, provided that $f(u)$ is predominantly negative for $u \in (0, \gamma]$, and superlinear for $u > \gamma$. This result adds a solution curve without turns to whatever is known about the solution set for $u(0) \in (0, \gamma)$. In particular, we combine it with the well-known cases of parabola-like, or S-shaped solution curves.

1. INTRODUCTION

We study global solution curves, and the exact multiplicity of positive solutions of the Dirichlet problem

$$u''(x) + \lambda f(u(x)) = 0, \text{ for } -1 < x < 1, \ u(-1) = u(1) = 0\,, \tag{1.1}$$

where λ is a positive parameter. Since positive solutions are symmetric with respect to the midpoint of the interval, it is convenient to pose the problem on the interval $(-1, 1)$. Our problem is autonomous, and so this does not restrict the generality. Any positive solution $u(x)$ is an even function, and $u'(x) < 0$ for $x \in (0, 1)$, so that $u'(0) = 0$, and $u(0) = \max_{[-1,1]} u(x)$. This follows immediately by observing that any solution is symmetric with respect to any of its critical points (of course, the theorem of B. Gidas, W.-M. Ni, and L. Nirenberg [1] is also applicable here). We shall need the corresponding linearized problem

$$w'' + \lambda f'(u(x))w = 0, \quad \text{for } -1 < x < 1, \ w(-1) = w(1) = 0\,. \tag{1.2}$$

The following lemma is known; see, e.g., P. Korman [3] or [4]. We include its proof for completeness.

Lemma 1.1. *Assume $f(u) \in C^1(\bar{R}_+)$. Let $u(x)$ be a positive solution of* (1.1), *with*

$$u'(1) < 0. \tag{1.3}$$

If the problem (1.2) *admits a non-trivial solution, then this solution does not change sign, i.e., we may assume that $w(x) > 0$ on $(-1, 1)$.*

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Proof. The function $u'(x)$ also satisfies the linear differential equation in (1.2). By the condition (1.3), $u'(x)$ is not a constant multiple of $w(x)$. Hence, by Sturm's comparison theorem, the roots of $u'(x)$ are interlaced with those of $w(x)$. If $w(x)$ had a root ξ inside the right half-interval $(0,1)$, then $u'(x)$ would have to vanish on $(\xi,1)$, which is impossible. □

We remark that the condition (1.3) holds automatically if $f(0) \geq 0$, as follows by Hopf's boundary lemma.

There are some standard conditions for solutions of (1.1) to be *non-singular*, which means that the problem (1.2) has only the trivial solution. (Non-singular solutions of (1.1) can be continued in λ, in view of the implicit function theorem.) Namely, if $f(0) \geq 0$, and

$$f'(u) > \frac{f(u)}{u} \ \ (\text{or if } f'(u) < \frac{f(u)}{u}), \ \ \text{for all } u > 0\,,$$

then the problem (1.2) has only the trivial solution. Indeed, writing the equation (1.1) in the form $u'' + \frac{f(u)}{u}u = 0$, we conclude by Sturm's comparison theorem that there is a root of $w(x) \not\equiv 0$ between the roots ± 1 of $u(x)$, which is impossible, since $w(x)$ is positive. In [3] we proved the following result (a similar result was previously given by R. Schaaf [8]).

Theorem 1.1. *Assume that $f(u) \in C^1(\bar{R}_+)$, $f(u) < 0$ on $(0,\gamma)$, while $f(u) > 0$ on (γ,∞), for some $\gamma > 0$. Assume also that*

$$f'(u) > \frac{f(u)}{u}, \quad \textit{for all } u > \gamma\,.$$

Then any positive solution of (1.1)*, satisfying* (1.3)*, is non-singular, i.e., the problem* (1.2) *admits only the trivial solution.*

Recall that the solution set of of (1.1) can be faithfully represented by curves in the $(\lambda, u(0))$ plane, with $u(0)$ being the maximum value of the solution $u(x)$; see e.g., [3], or [4]. The above theorem implies that there are no turns on the solution curves, once $u(0) > \gamma$. What is remarkable here is that no additional assumptions are imposed on $f(u)$ on the interval where it is negative. In this note we present a much stronger result, replacing the condition $f(u) < 0$ on $(0,\gamma)$ by an integral condition. The new result adds a solution curve without turns to whatever is known about the solution set for $u(0) \in (0,\gamma)$, thus providing exact multiplicity results in cases where there are exactly three or exactly four solutions; see the bifurcation diagrams for Theorems 3.2, 3.3, and 3.4. Such exact multiplicity results are rare.

2. The main result

As usual, we denote $F(u) = \int_0^u f(t)\,dt$.

Theorem 2.1. *Assume that $f(u) \in C^1(\bar{R}_+)$, and for some $\gamma > 0$, it satisfies*

$$f(\gamma) = 0, \ \ \textit{and} \ \ f(u) > 0 \ \textit{on} \ (\gamma,\infty), \tag{2.1}$$

$$f'(u) > \frac{f(u)}{u}, \quad \textit{for } u > \gamma\,, \tag{2.2}$$

$$F(\gamma) - F(u) = \int_u^\gamma f(t)\,dt < 0, \ \ \textit{for } u \in (0,\gamma)\,. \tag{2.3}$$

Then any positive solution of (1.1), *satisfying*

$$u(0) > \gamma, \quad \textit{and} \;\; u'(1) < 0\,, \tag{2.4}$$

is non-singular, which means that the problem (1.2) *admits only the trivial solution.*

Proof. Assume, on the contrary, that the problem (1.2) admits a non-trivial solution $w(x) > 0$. Let $x_0 \in (0,1)$ denote the point where $u(x_0) = \gamma$, so that the condition (2.2) holds for $0 \le x < x_0$. From the equations (1.1) and (1.2), we get

$$(w'u - wu')' = -\left[f'(u) - \frac{f(u)}{u}\right] uw < 0, \;\; \text{for } 0 \le x < x_0\,.$$

Integrating this over $(0, x_0)$,

$$w'(x_0)u(x_0) - w(x_0)u'(x_0) < 0\,. \tag{2.5}$$

We claim that

$$(x_0 - 1)u'(x_0) - u(x_0) > 0\,. \tag{2.6}$$

Indeed, denoting $q(x) \equiv (x-1)u'(x) - u(x)$, we see that $q(1) = 0$, while

$$q'(x) = (x-1)u''(x) = (x-1)\left[u'(x) - u'(x_0)\right]'\,.$$

We integrate this formula over $(x_0, 1)$, and perform integration by parts

$$-q(x_0) = \int_{x_0}^1 (x-1)\left[u'(x) - u'(x_0)\right]' \, dx = -\int_{x_0}^1 \left[u'(x) - u'(x_0)\right] \, dx\,,$$

so that

$$q(x_0) = \int_{x_0}^1 \left[u'(x) - u'(x_0)\right] \, dx > 0\,,$$

which is the same as the desired inequality (2.6), provided we can prove that

$$u'(x) - u'(x_0) > 0, \;\; \text{for } x \in (x_0, 1)\,. \tag{2.7}$$

The "energy" $E(x) = \frac{1}{2}{u'}^2(x) + F(u(x))$ is easily seen to be constant, so that $E(x) = E(x_0)$, or

$$\frac{1}{2}{u'}^2(x) + F(u(x)) = \frac{1}{2}{u'}^2(x_0) + F(\gamma)\,.$$

By the assumption (2.3),

$$\frac{1}{2}\left[{u'}^2(x) - {u'}^2(x_0)\right] = F(\gamma) - F(u(x)) < 0, \;\; \text{for } x \in (x_0, 1)\,.$$

It follows that ${u'}^2(x) < {u'}^2(x_0)$, justifying (2.7), and then giving (2.6).

Next, we observe that the function $u''(x)w(x) - u'(x)w'(x)$ is constant over $[0,1]$. (Just differentiate this function, and use the corresponding equations.) Evaluating this function at $x = x_0$, and at $x = 1$, and observing that $u''(x_0) = -f(u(x_0)) = -f(\gamma) = 0$, we have

$$-u'(x_0)w'(x_0) = -u'(1)w'(1)\,, \tag{2.8}$$

which implies, in particular, that

$$w'(x_0) < 0\,. \tag{2.9}$$

Using the assumption (2.4), we also have

$$u'(x)w'(x) - u''(x)w(x) = u'(1)w'(1) > 0, \quad \text{for all } x \in (0,1)\,. \tag{2.10}$$

The function $z(x) \equiv xu'(x)$ is easily seen to satisfy

$$z'' + f'(u)z = -2f \, .$$

Combining this equation with (1.2), we express

$$(z'w - w'z)' = -2fw = -2\left[F(u(x)) - F(\gamma)\right]' \frac{w(x)}{u'(x)} \, .$$

Integrating this over $(x_0, 1)$, we get (observe that $z' = u' + xu''$, $z'(x_0) = u'(x_0)$)

$$\begin{aligned} M &\equiv -u'(1)w'(1) - u'(x_0)w(x_0) + x_0 u'(x_0)w'(x_0) \\ &= -2\int_{x_0}^{1} \left[F(u(x)) - F(\gamma)\right]' \frac{w(x)}{u'(x)} \, dx \\ &= 2\int_{x_0}^{1} \left[F(u(x)) - F(\gamma)\right] \frac{w'(x)u'(x) - w(x)u''(x)}{u'^2(x)} \, dx > 0 \, , \end{aligned}$$

in view of (2.3) and (2.10).

On the other hand, using (2.8) and (2.5), and then (2.6) and (2.9), we have

$$\begin{aligned} M &< -u'(x_0)w'(x_0) - w'(x_0)u(x_0) + x_0 u'(x_0)w'(x_0) \\ &= w'(x_0)\left[(x_0 - 1)u'(x_0) - u(x_0)\right] < 0 \, , \end{aligned} \tag{2.11}$$

a contradiction. ☐

Observe that we only needed $\int_u^\gamma f(t)\, dt \leq 0$ for $u \in (0, \gamma)$, with the inequality being strict on a set of positive measure.

3. Applications

Our condition (2.3) implies that $F(\gamma) \leq 0$. Let $\gamma_1 \geq \gamma$ be defined by

$$F(\gamma_1) = 0 \, .$$

(In case $F(\gamma) = 0$, we have $\gamma_1 = \gamma$.) Our condition (2.2) implies that $\frac{f(u)}{u}$ is increasing for $u > \gamma$, and so the limit $L = \lim_{u \to \infty} \frac{f(u)}{u} \leq \infty$ exists.

We consider positive solutions of the problem

$$u''(x) + \lambda f(u(x)) = 0, \;\; \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0 \, , \tag{3.1}$$

where λ is a positive parameter. Recall that $u(0)$ gives the maximum value of any solution. Moreover, the value of $u(0)$ gives the global parameter on the solution curves, uniquely identifying the solution pair $(\lambda, u(x))$; see e.g., [3]. Hence, the set of positive solutions of (3.1) can be represented by curves in the $(\lambda, u(0))$ plane, giving us the *solution curves.*

Theorem 3.1. *Assume that $f(u) \in C^1(\bar{R}_+)$; satisfies the conditions* (2.1), (2.2) *and* (2.3). *Then all positive solutions of* (3.1), *satisfying $u(0) > \gamma$, lie on a unique solution curve joining (δ, ∞) to (δ_1, γ_1) in the $(\lambda, u(0))$ plane, with some $0 \leq \delta < \delta_1 \leq \infty$. If $L = \infty$, then $\delta = 0$, and in case $L < \infty$, we have $\delta > 0$. If $F(u) < 0$ for all $u \in (0, \gamma)$, then $\delta_1 = \frac{1}{2}\left(\int_0^{\gamma_1} \frac{du}{\sqrt{-F(u)}}\right)^2 < \infty$, and $\delta_1 = \infty$ in case $\max_{u \in (0,\gamma)} F(u) \geq 0$.*

Proof. We begin by showing the existence of positive solutions, with $u(0) > \gamma$. We use "shooting", considering for $x > 0$ the solutions of

$$u''(x) + f(u(x)) = 0, \quad u(0) = u_0, \quad u'(0) = 0\,. \tag{3.2}$$

The "energy" $\frac{1}{2}{u'}^2(x) + F(u(x))$ is constant, and so

$$\frac{1}{2}{u'}^2(x) + F(u(x)) = F(u_0), \qquad \text{for all } x\,. \tag{3.3}$$

Let $u_0 > \gamma$ be such that $F(u_0) > F(u)$ for all $u \in (0, u_0)$. Then the solution of (3.2) is decreasing, and $u'(x)$ cannot become zero, or tend to zero, by (3.3). It follows that $u(x)$ becomes zero at some x_0, and then by rescaling we get a solution of (3.1) at some value of λ.

This solution is non-singular by Theorem 2.1. We now continue this solution in λ, using the implicit function theorem. For decreasing λ, the solution curve goes to infinity, as described in the statement of the theorem, by standard results; see e.g., [3]. For increasing λ, the solution curve either continues for all λ, or at some $\lambda = \delta_1$, and the corresponding $u = \bar{u}(x)$, we have $\bar{u}'(1) = 0$, and the solutions become sign-changing for $\lambda > \delta_1$. Then

$$\frac{1}{2}\bar{u}'^2(x) + \delta_1 F(\bar{u}(x)) = 0\,. \tag{3.4}$$

It follows that $F(\bar{u}(0)) = 0$, and so $\bar{u}(0) = \gamma_1$, and $F(u) < 0$ for $u \in (0, \gamma_1)$. We conclude that in case $\max_{u\in(0,\gamma)} F(u) \geq 0$, the solution $\bar{u}(x)$ with $\bar{u}'(1) = 0$ is not possible, and therefore the curve of positive solutions continues for all λ. In case $F(u) < 0$ for all $u \in (0, \gamma)$, the existence of $\bar{u}(x)$ follows by shooting and scaling, as above (with $u_0 = \gamma_1$), and the value of δ_1 is computed by integration of (3.4). ☐

This result can be used in many situations. It adds a solution curve, which has no turns, to whatever is known about the solution set for $u(0) \in (0, \gamma)$. In particular, it can be used with the well-known cases of parabola-like (see P. Korman, Y. Li, and T. Ouyang [7], or S.-H. Wang [10]), or S-shaped solution curves (see P. Korman and Y. Li [5], or S.-H. Wang [11]), giving the following results.

Theorem 3.2. *Assume that the function $f(u) \in C^2(\bar{R}_+)$ has three positive roots $0 < a < b < \gamma$, and*

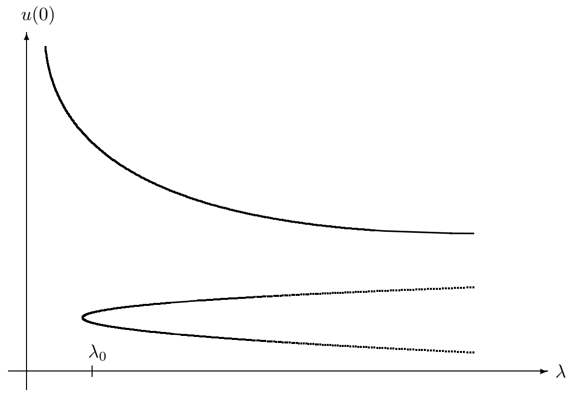
$$f(0) = 0, \;\; f(u) < 0 \;\; \text{on } (0,a) \cup (b,\gamma), \qquad f(u) > 0 \;\; \text{on } (a,b) \cup (\gamma,\infty)\,,$$

$$F(b) = \int_0^b f(u)\, du > 0\,.$$

Assume that there is an $\alpha \in (a, b)$, so that

$$f''(u) > 0 \;\; \text{for } u \in (0,\alpha), \;\; f''(u) < 0 \;\; \text{for } u \in (\alpha, b)\,.$$

Assume, finally, that $f(u)$ satisfies the conditions (2.2) and (2.3). Also, assume for definiteness that $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$. Then the set of positive solutions of (3.1) consists of two solution curves. The lower curve is parabola-like, opening to the right, and the upper one is given by Theorem 3.1, with $\delta = 0$, and $\delta_1 = \infty$. In particular, there is a critical λ_0, so that the problem (3.1) has exactly one positive solution for $\lambda \in (0, \lambda_0)$, exactly two positive solutions at $\lambda = \lambda_0$, and exactly three positive solutions for $\lambda > \lambda_0$.



Bifurcation diagram for Theorem 3.2

Theorem 3.3. *Assume that the function* $f(u) \in C^2(\bar{R}_+)$ *has two positive roots* $0 < a < \gamma$*, and*

$$f(u) > 0 \ \ on\ [0,a) \cup (\gamma,\infty), \quad f(u) < 0 \ \ on\ (a,b)\,.$$

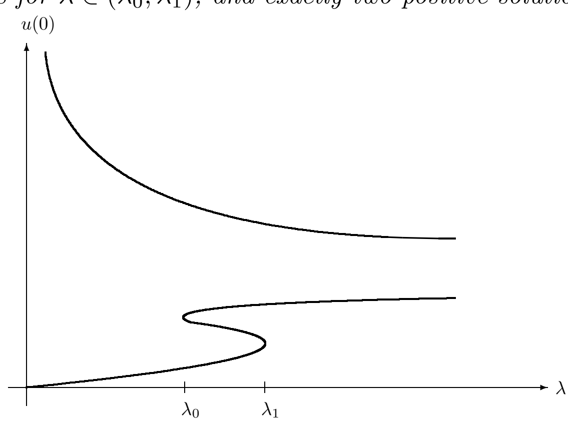
Assume that there is an $\alpha \in (0,a)$*, so that*

$$f''(u) > 0 \ \ for\ u \in (0,\alpha), \ \ f''(u) < 0 \ \ for\ u \in (\alpha,a)\,.$$

Defining $h(u) = 2F(u) - uf(u)$*, we assume that*

$$h(\alpha) < 0\,.$$

Assume, finally, that $f(u)$ *satisfies the conditions* (2.2) *and* (2.3)*. Also, assume for definiteness that* $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$*. Then the set of positive solutions of* (3.1) *consists of two solution curves. The lower curve is* S*-shaped, and the upper one is given by Theorem* 3.1*, with* $\delta = 0$*, and* $\delta_1 = \infty$*. In particular, there are two critical numbers* $\lambda_0 < \lambda_1$*, so that the problem* (3.1) *has exactly two positive solutions for* $\lambda \in (0,\lambda_0)$*, exactly three positive solutions at* $\lambda = \lambda_0$ *and* $\lambda = \lambda_1$*, exactly four positive solutions for* $\lambda \in (\lambda_0,\lambda_1)$*, and exactly two positive solutions for* $\lambda > \lambda_0$*.*



Bifurcation diagram for Theorem 3.3

Finally, we consider the case of broken reverse S-shaped curves, considered previously by J. Shi and R. Shivaji [9], and K.-C. Hung [2]. We have the following result.

Theorem 3.4. *Assume that the function* $f(u) \in C^2(\bar{R}_+)$ *has three positive roots* $0 < a < b < \gamma$, *and*

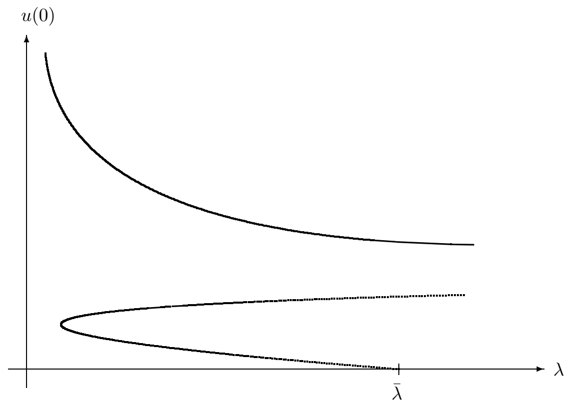
$$f(u) < 0 \ \text{ on } [0, a) \cup (b, \gamma), \quad f(u) > 0 \ \text{ on } (a, b) \cup (\gamma, \infty),$$

$$\int_0^b f(u)\, du > 0, \quad \int_a^\gamma f(u)\, du < 0 . \tag{3.5}$$

Assume that

$$f''(u) < 0 \ \text{ for } u \in (0, b) .$$

Assume, finally, that $f(u)$ *satisfies the condition* (2.2)*. Also, assume for definiteness that* $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$*. Then the set of positive solutions of* (3.1) *consists of two solution curves. The lower curve is parabola-like, opening to the right, and the upper one is without any turns, and it joins* $(0, \infty)$ *to* (∞, γ_1) *in the* $(\lambda, u(0))$ *plane. The lower branch of the lower curve terminates at some finite* $\bar{\lambda}$ *(at* $\bar{\lambda}$, $u'(1) = 0$, *and the solutions are sign-changing for* $\lambda > \bar{\lambda}$*).*



Bifurcation diagram for Theorem 3.4

Proof. Observe that (3.5) implies that (2.3) holds. Theorem 3.1 applies, and provides us with the upper curve. For $u(0) \in (a, b)$, only turns to the right are possible on the solution curve, see [3], and since $f(0) < 0$, arguing as above, we see that the lower branch terminates at some finite $\bar{\lambda}$. ☐

Let us compare this result with Theorem 2.2 in K.-C. Hung [2]. We do not impose any concavity assumptions on $f(u)$ for $u \in (b, \infty)$, and we dropped several technical assumptions. However, we added an extra condition $\int_a^\gamma f(u)\, du < 0$, which K.-C. Hung [2] does not have. For the cubic $f(u) = (u - a)(u - b)(u - c)$, $0 < a < b < c$, K.-C. Hung's theorem produces the optimal result (i.e., the bification diagram is the same as for Theorem 3.4), requiring only that $\int_0^b f(u)\, du > 0$, while our result requires that in addition, $\int_a^c f(u)\, du < 0$. On the other hand, our result allows many changes in convexity on (b, ∞).

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