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POSITIVITY FOR THE LINEARIZED PROBLEM FOR SEMILINEAR EQUATIONS

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Abstract. Using recent results of M. Tang [10], we provide a simple approach to proving positivity for the linearized problem of semilinear equations, which is crucial for establishment of exact multiplicity results, and for symmetry breaking.

1. Introduction

When one studies multiplicity of positive solutions of the Dirichlet problem

$$\Delta u + f(u) = 0, \quad \text{for } |x| < 1, u = 0 \text{ when } |x| = 1 \tag{1.1}$$

crucial role is usually played by the corresponding linearized problem

$$\Delta w + f'(u)w = 0, \quad \text{for } |x| < 1,\ w = 0 \text{ when } |x| = 1, \tag{1.2}$$

see e.g. P. Korman, Y. Li and T. Ouyang [4] or T. Ouyang and J. Shi [7]. Recall that by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [2] positive solutions are radially symmetric, and hence they satisfy the equation (2.1) below, with (2.2) giving the corresponding linearized problem, in view C. S. Lin and W.-M. Ni [6]. In the studies of symmetry-breaking bifurcation, i.e. when there are non-radial solutions bifurcating off the curve of radial ones, central role was

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again played by the linearized problem, see e.g. J. Smoller and A. Wasserman [9] and P. Korman [3]. In both cases the interest was in the nodal properties of the solution of the linearized problem (2.2). For exact multiplicity results it was necessary to prove that $w(r)$ cannot change sign, while for symmetry breaking one needed in [3] to prove that $w(r)$ cannot change sign more than once. In both cases this was one of the most difficult steps. In this work we observe that a certain zeta function, introduced recently by M. Tang [10] provides a considerable simplification in the analysis of $w(r)$. In particular, we obtain a short and self-contained proof of positivity of $w(r)$ in case of cubic $f(u) = u(u-a)(c-u)$. The previous proof in T. Ouyang and J. Shi [7] was using some deep results of M. K. Kwong and L. Zhang [5].

In case of symmetry breaking we obtain a result extending the one in [3], with a considerable relaxation of conditions in the region where $f(u) < 0$, especially in the case of $n = 2$.

2. Positivity for the linearized problem

We assume that $f \in C^1(\overline{R}_+)$ satisfies $f(0) \le 0$, and that $f(u) < 0$ for $u \in (0,b)$ and $f(u) > 0$ for $u \in (b,c)$ for some $0 < b < c \le \infty$. In view of the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [2] positive solutions of the Dirichlet problem (1.1) are radially symmetric, with $u'(r) < 0$, and hence they satisfy

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \quad u'(0) = 0, \quad u(1) = 0. \tag{2.1}$$

By the result of C. S. Lin and W.-M. Ni any solution of the linearized problem (1.2) is also radially symmetric, and hence it satisfies

$$w'' + \frac{n-1}{r}w' + f'(u)w = 0, \quad w'(0) = 0, \quad w(1) = 0. \tag{2.2}$$

We shall give conditions under which $w(r)$ is of one sign, which are crucial for exact multiplicity results, as well as conditions under which $w(r)$ cannot have more than one interior root, with applications to symmetry breaking. Following M. Tang [10], we consider the functions $\xi(r) = r^{n-1}(u'w - uw')$ and $\zeta(r) = r^n[u'w' + f(u)w] + (n-2)r^{n-1}u'w$. While $\xi(r)$ can be regarded as the Wronskian of u and v, the function $\zeta(r)$ is a useful invention of M. Tang. As was observed in M. Tang [10], these functions have the following derivatives

$$\xi'(r) = r^{n-1}[uf'(u) - f(u)]w, \tag{2.3}$$

$$\zeta'(r) = 2f(u)wr^{n-1}. \tag{2.4}$$

Clearly, $u(0) > b$ for any positive solution (otherwise multiply by u and integrate, to obtain a contradiction). Denote by η the point where $u(\eta) = b$, i.e. $f(u(r)) > 0$ on $[0,\eta)$ and $f(u(r)) < 0$ on $(\eta,1)$. It is well-known that $w(r)$ cannot vanish on

$(\eta, 1)$. To underscore the usefulness of $\zeta(r)$, observe that this assertion follows immediately by integrating (2.4) between the (hypothetical) largest root of $w(r)$ on $(\eta, 1)$ and 1. We shall also need the following function from M. Tang [10]:

$$Q(r) = r^n[u'^2 + f(u)u] + (n-2)r^{n-1}u'u.$$

THEOREM 2.1.

(a) *Assume that*

$$K(u) \equiv \frac{uf'(u)}{f(u)} \quad \textit{is decreasing for } u \in (b, c) \textit{ and } \lim_{u \to c} K(u) > 1, \tag{2.5}$$

$$Q(r) > 0 \quad \textit{for } r \in (0, \eta). \tag{2.6}$$

Then $w(r)$ cannot have more than one interior root on $(0, \eta)$.

(b) *If, in addition,*

$$K(u) < 1 \quad \textit{for } u \in (0, b), \tag{2.7}$$

then $w(r)$ has no interior roots on $(0, 1)$, i.e. we may assume that $w(r) > 0$ on $[0, 1)$.

PROOF. (a) Assume on the contrary that $w(r)$ has two interior roots $0 < \tau < \overline{\tau} < \eta$, so that $w(r) > 0$ on $[0, \tau)$ and $w(r) < 0$ on $(\tau, \overline{\tau})$. (Observe that $\overline{\tau} < \eta$, since w has no roots on $(\eta, 1)$, i.e. we have $f(u) > 0$ on $[0, \overline{\tau}]$.) We consider a function $O(r) = 2\xi(r)/\gamma' - \zeta(r)$, with a positive constant γ' to be specified. Using (2.3) and (2.4), we compute

$$O'(r) = \frac{2}{\gamma'} f(u) r^{n-1} (K(u) - \gamma) w, \tag{2.8}$$

with $\gamma = \gamma' + 1$. Observe that by our conditions $K(u(r))$ is an increasing function of r, with $K(u(0)) > 1$. We now fix γ (and hence γ') by setting $\gamma = K(u(\tau))$. Then we see from (2.8) that the function $O(r)$ is decreasing on $(0, \overline{\tau})$. Since $O(0) = 0$, it follows that $O(r) < 0$ for $r \in (0, \overline{\tau})$. Since $\xi(\tau) > 0$, while $\xi(\overline{\tau}) < 0$, there exists a $t \in (\tau, \overline{\tau})$ so that $\xi(t) = 0$. It follows that

$$\zeta(t) = -O(t) > 0. \tag{2.9}$$

On the other hand, since $u(t)/w(t) = u'(t)/w'(t)$,

$$\zeta(t) = \left[t^n\left(u'w'\frac{u}{w} + f(u)u\right) + (n-2)t^{n-1}u'u\right]\frac{w}{u} = Q(t)\frac{w(t)}{u(t)} < 0,$$

contradicting (2.9).

(b) We now exclude the possibility that $w(r)$ has exactly one interior root τ, so that $w(r) > 0$ on $[0, \tau)$ and $w(r) < 0$ on $(\tau, 1)$. The argument is similar. Clearly, $\tau \in (0, \eta)$. Again, we consider the function $O(r)$, and set $\gamma = K(u(\tau)) > 1$. Thanks to the assumption (2.7), we see from (2.8) that $O(r)$ is decreasing on

$(0,1)$. (Observe that at $r=\tau$ both $K(u(r))-\gamma$ and $w(r)$ change sign, while at $r=\eta$ both $K(u(r))-\gamma$ and $f(u(r))$ change sign.) It follows that $O(1)<0$, i.e. $\zeta(1)=-O(1)>0$. But $\zeta(1)=u'(1)w'(1)<0$, a contradiction. $\square$

Remarks 2.2. (a) Assume that we know that $w(r)$ cannot vanish in the region where $u(r)>\rho$, for some $\rho\in(b,c)$. Then we may replace the condition $\lim_{u\to c}K(u)>1$ by requiring that $K(\rho)>1$ and $K(u)<K(\rho)$ for $u\in(\rho,c)$.

(b) The first part of the theorem was also proved in T. Ouyang and J. Shi [8] by a different and more involved method. The second part could be also proved by using a certain integral identity of M. K. Kwong and L. Zhang [5], see [4] or [8] for details. We see that considering the function $O(r)$, as above, provides a unified and simplified approach.

(c) We needed positivity of $Q(r)$ only between the roots of $w(r)$. So if, for example, $w(r)$ cannot vanish in the region where $u(r)$ is greater than some constant, then we do not need $Q(r)$ to be positive in that region. In particular, in case $n=2$ the condition (2.6) can be dropped.

(d) We show that are many $f(u)$ for which $K(u)$ is decreasing. To emphasize the dependence on $f(u)$, we denote $K_f(u)=uf'(u)/f(u)$. Then we have for any two functions f and g

$$K_f=K_g+K_{f/g}. \tag{2.10}$$

So if K_f is decreasing, the same is true for K_{u^pf}, for any $p>0$. Also, we see from (2.10) that

$$K_{f^2}=K_f+F_{f^2/f}=2K_f,$$

and in general $K_{f^m}=mK_f$. Hence K_{f^m} is decreasing, provided K_f is decreasing.

We now give some conditions on $f(u)$ under which $Q(r)>0$, i.e. the condition (2.6) holds. Following M. Tang [10], we let

$$P(r)=r^n({u'}^2+2F(u))+(n-2)r^{n-1}u'u,$$

where as usual $F(u)=\int_0^u f(t)\,dt$. Then

$$Q(r)-P(r)=r^n(uf(u)-2F(u)),$$

and

$$P'(r)=r^{n-1}[2nF(u)-(n-2)uf(u)]\equiv r^{n-1}I(u). \tag{2.11}$$

To show that $Q(r)$ is positive, suffices to show that both $P(r)$ and $uf(u)-2F(u)$ are positive.

Lemma 2.3. *Assume that* $uf(u)-2F(u)>0$ *for* $u>b$, (*i.e. where* $f(u)>0$), *and one of the following three conditions holds:*

(a) $I(u)=2nF(u)-(n-2)uf(u)>0$ *for* $u>b$,
(b) $I(u)=2nF(u)-(n-2)uf(u)<0$ *for* $u>0$,
(c) $I(u)<0$ *for* $u\in(0,c)$, $I(u)>0$ *for* $u>c$, *for some* $c>0$.

We claim that $P(r)>0$ *for* $r\in(0,\eta)$ (*i.e. where* $f(u)>0$).

Proof. Since $P(0)=0$ and $P(1)>0$, it follows by integrating (2.11) that $P(r)>0$ for $r\in(0,\eta)$. (In case (a) $P(r)$ is increasing on $(0,\eta)$, in case (b) is decreasing for all r, and in the last case $P(r)$ is positive for all r.) Since $Q(r)>P(r)$ over $(0,\eta)$, the proof follows. $\square$

Example 2.4. Consider positive solutions of the cubic problem

$$u''+\frac{n-1}{r}u'+u(u-b)(c-u)=0,\quad u'(0)=0,\quad u(1)=0, \tag{2.12}$$

with constants $c>2b>0$. Then any non-trivial solution of the corresponding linearized problem (2.2) is of one sign. We verify that the Theorem 2.1 applies. The function $f(u)=u(u-b)(c-u)$ is convex for $u\in(0,(b+c)/3)$ and concave when $u>(b+c)/3$. It is easy to see, see below or [4] and [7], that $w(r)$ cannot vanish in the region where $f(u)$ is concave, so that we only need to verify that $Q(r)>0$ in the region where $u\in J\equiv(0,\frac{b+c}{3})$. Compute

$$uf(u)-2F(u)=u^3\left[-\frac{1}{2}u+\frac{1}{3}(b+c)\right]>0\quad\text{on } J.$$

Turning to $P(r)$, observe that $P(0)=0$, $P(1)>0$, and

$$P'(r)=r^{n-1}I(u)=r^{n-1}u^2\left[\frac{n-4}{2}u^2+\frac{-n+6}{3}(b+c)u-2bc\right]. \tag{2.13}$$

If $n\geq 4$, the quadratic polynomial in the square brackets in (2.13) is negative when $u=0$, and changes sign at most once for $u>0$. We see that $P'(r)$ is either negative or changes sign exactly once on $(0,1)$, with negative values near $r=1$. In both cases $P(r)$ is positive. When $n=3$ we have a quadratic $-u^2/2+(b+c)u-2bc$ which is positive at $u=c$, and since any positive solution of (2.12) satisfies $u(0)<c$, we see that again $P'(r)$ changes sign at most once, and we proceed similarly. In case $n=2$, we see directly from the definition of $Q(r)$ that it is positive between any two roots of $w(r)$, since $f(u)$ is positive there.

Observing that $K(u)=u(\ln f(u))'$, one easily verifies that

$$K'(u)=-\frac{b}{(u-b)^2}-\frac{c}{(c-u)^2}<0$$

for all $u\in(0,c)\setminus\{b\}$. Since $K(0)=1$, the condition (2.7) holds. Following [8], we define $\alpha=(b+c)/3$ (i.e. $f''(\alpha)=0$), and $\rho=\alpha-f(\alpha)/f'(\alpha)$. It was shown in [8] that $w(r)$ cannot vanish in the region where $u(r) > \rho$ (a sharper statement than $w(r)$ not vanishing where $u > \alpha$, which was used above). We recall the proof for completeness. One begins by observing that

$$f'(u) < \frac{f(u)}{u-\rho} \quad \text{for all } u \in (\rho, c). \tag{2.14}$$

Indeed, if we define $g(u) = f(u) - f'(u)(u-\rho)$, then $g(\alpha) = 0$ and $g'(u) = -f''(u)(u-\rho)$, so that $g(u)$ has a minimum at α and (2.14) follows. If we now consider a test function $z(r) = u(r) - \rho$, then $z(r) > 0$, while in view of (2.14),

$$z'' + \frac{n-1}{r} z' + f'(u(r))z = -f(u) + f'(u)(u-\rho) < 0,$$

and hence, by a standard comparison argument, $w(r)$ cannot vanish where $u(r) > \rho$. We now observe that since $K(u)$ is decreasing,

$$K(\rho) > K(\alpha) = 1 + \frac{\rho f'(\alpha)}{f(\alpha)} > 1.$$

This completes verification of the condition (2.5) (see also Remark 2.2(a)).

3. Application to symmetry breaking

We show that the results of the preceeding section can be used to prove that symmetry breaking occurs for semilinear equations on balls. We now recall the basic set up, and some results from P. Korman [3].

According to the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [2] any positive solution of the problem (depending on a positive parameter λ)

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{for } |x| = 1. \tag{3.1}$$

is radially symmetric, i.e. $u = u(r)$, with $r = |x|$, and $u(r)$ satisfies (2.1). Moreover, they proved that $u'(r) < 0$ for all $r \in (0,1)$. If $f(0) \geq 0$ then as we vary λ solutions of (3.1) stay positive, and hence radially symmetric. Indeed, there are two ways in which positive solutions may eventually become sign-changing: either they develop an interior root, or a zero slope at the boundary $r = 1$. The first possibility cannot happen since positive solutions are decreasing functions, and the second one is ruled out by the Hopf's boundary lemma. Hence for the symmetry breaking it is necessary that

$$f(0) < 0.$$

It was shown in P. Korman [3] that symmetry breaking actually occurs, provided the linearized equation (2.2) cannot have more than one interior zero. Our Theorem 2.1 provides conditions for that to happen. In fact, we have the following result.

Theorem 3.1. *Assume that* $f(u) \in C^2(\overline{R}_+)$, *and there is some* $b > 0$ *so that*

$$f(u) < 0 \quad \text{for } u \in [0, b), \quad f(u) > 0 \quad \text{for } u > b,$$

and that

$$\lim_{u \to \infty} \frac{f(u)}{u^l} = m > 0, \quad \text{where } l < \frac{n+2}{n-2} \text{ if } n \geq 3,\ l < \infty \text{ for } n = 1, 2.$$

Assume also that the conditions (2.5) *and* (2.6) *hold. Then there is a* $\lambda_0 > 0$, *so that for* $\lambda \in (0, \lambda_0)$ *there is a curve of positive* (*and hence radial*) *solutions of* (3.1). *At* $\lambda = \lambda_0$ *there is a* n*-dimensional family of symmetry breaking solutions bifurcating off this curve.*

Proof. Existence of positive solutions for small λ follows from A. Ambrosetti, D. Arcoya and B. Buffoni [1]. By the Theorem 2.1 in [3] the curve of positive solutions cannot be continued for all λ (the function $f(u)$ has no stable roots). Hence after some λ_0 solutions on the curve must cease to be positive. As was mentioned above, we have $u'(1) = 0$ at λ_0. By the Theorem 2.1 above the solution of the linearized equation (2.2) cannot have more than one interior root on $(0, 1)$. Hence the Theorem 4.1 in P. Korman [3] applies, implying that symmetry breaking occurs at λ_0. $\square$

Example 3.2. $f(u) = u^p - a$, with constants $1 < p < (n+2)/(n-2)$ for $n > 2$ and $1 < p < \infty$ for $n = 2$, and $a > 0$. Here the second part of the Theorem 2.1 applies, implying that $w > 0$ (which was already observed in [3] in case $p < n/(n-2)$), and hence the Theorem 3.1 applies, implying that symmetry breaking occurs. Indeed, here $K(u) = pu^p/(u^p - 1)$, and $K'(u) = -ap^2u^{p-1}/(u^p - a)^2$. We see that $\lim_{u \to \infty} K(u) = p > 1$, and $K(u)$ is decreasing for all $u > 0$. Since also $K(0) = 0$, we see that the conditions (2.5) and (2.7) hold. Turning to the condition (2.6), compute $uf(u) - 2F(u) = au + (1 - 2/(p+1))u^{p+1} > 0$. We also observe that here

$$I(u) = -a(n+2)u + \left[\frac{2n}{p+1} - n + 2\right] u^{p+1}. \tag{3.2}$$

The quantity in the square bracket is positive, and hence either the second or the third case of the Lemma 2.3 holds.

Example 3.3. Let us consider a modification of the above example, beginning with the case of $n = 2$, and $1 < p < \infty$. The function $f(u)$ is negative for $u \in [0, \sqrt[p]{a})$ and positive for $u > \sqrt[p]{a}$. We now consider an *arbitrary* modification of $f(u)$ on the interval $[0, \sqrt[p]{a})$, which keeps $f(u)$ negative there, and $f(\sqrt[p]{a}) = 0$. The first part of the Theorem 2.1 applies, implying that w cannot change sign more than once. Again, the Theorem 3.1 applies, implying that the symmetry breaking occurs.

In the case $n > 2$ and $1 < p < (n+2)/(n-2)$, we can allow any modification of $f(u)$ on the interval $[0, \sqrt[p]{a})$, which keeps $f(u)$ negative there, and $f(\sqrt[p]{a}) = 0$, provided that it keeps $I(u) < 0$ on the interval $(0, \sqrt[p]{a})$. Indeed, from (3.2) we see that $I(\sqrt[p]{a}) < 0$, and hence either the second or the third case of the Lemma 2.3 continues to hold for the modified function.

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