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## Pohozaev's Identity and Non-existence of Solutions for Elliptic Systems

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### Abstract

We extend the classical Pohozaev's identity to semilinear elliptic systems of Hamiltonian type, providing an alternative and simpler approach to the results of E. Mitidieri [8], R.C.A.M. Van der Vorst [15], and Y. Bozhkov and E. Mitidieri [2].

**Key words:** Pohozaev's identity, Non-existence of solutions.

**AMS Subject Classification:** 35J57

## 1 Introduction

Any solution  $u(x)$  of semilinear Dirichlet problem on a bounded domain  $\Omega \subset R^n$

$$\Delta u + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \quad (1.1)$$

satisfies the well known Pohozaev's identity

$$\int_{\Omega} [2nF(u) + (2-n)uf(u)] \, dx = \int_{\partial\Omega} (x \cdot \nu) |\nabla u|^2 \, dS. \quad (1.2)$$

Here  $F(u) = \int_0^u f(t) \, dt$ , and  $\nu$  is the unit normal vector on  $\partial\Omega$ , pointing outside. (From the equation (1.1),  $\int_{\Omega} uf(u) \, dx = \int_{\Omega} |\nabla u|^2 \, dx$ , which gives an alternative form of the Pohozaev's identity.) A standard use of this identity is to conclude that if  $\Omega$  is a star-shaped domain with respect to the origin, i.e.  $x \cdot \nu \geq 0$  for all  $x \in \partial\Omega$ , and  $f(u) = u|u|^{p-1}$ , for some constant  $p$ , then

the problem (1.1) has no non-trivial solution in the super-critical case, when  $p > \frac{n+2}{n-2}$ . In this note we present a proof of Pohozaev's identity, which appears a little more straightforward than the usual one, see e.g. L. Evans [3], and then use a similar idea for systems, generalizing the well-known results of E. Mitidieri [8]. After completing this work, we found out that this result appeared previously in Y. Bozhkov and E. Mitidieri [2]. However, our proof is different, and it appears to be much simpler. Similarly, we derive Pohozaev's identity for a version of  $p$ -Laplace equation.

Let  $z = x \cdot \nabla u = \sum_{i=1}^n x_i u_{x_i}$ . It is easy to verify that  $z$  satisfies

$$\Delta z + f'(u)z = -2f(u). \quad (1.3)$$

We multiply the equation (1.1) by  $z$ , and subtract from that the equation (1.3) multiplied by  $u$ , obtaining

$$\sum_{i=1}^n \left[ (zu_{x_i} - uz_{x_i})_{x_i} + x_i \frac{\partial}{\partial x_i} (2F(u) - uf(u)) \right] = 2f(u)u. \quad (1.4)$$

Clearly,

$$\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (2F - uf) = \sum_{i=1}^n \frac{\partial}{\partial x_i} [x_i (2F - uf)] - n(2F - uf).$$

We then rewrite (1.4)

$$\sum_{i=1}^n [(zu_{x_i} - uz_{x_i}) + x_i (2F(u) - uf(u))]_{x_i} = 2nF(u) + (2 - n)uf(u). \quad (1.5)$$

Integrating over  $\Omega$ , we conclude the Pohozaev's identity (1.2). (The only non-zero boundary term is  $\sum_{i=1}^n \int_{\partial\Omega} zu_{x_i} \nu_i dS$ . Since  $\partial\Omega$  is a level set of  $u$ ,  $\nu = \pm \frac{\nabla u}{|\nabla u|}$ , i.e.  $u_{x_i} = \pm |\nabla u| \nu_i$ . Then  $z = \pm (x \cdot \nu) |\nabla u|$ , and  $\sum_{i=1}^n u_{x_i} \nu_i = \pm |\nabla u|$ .)

It appears natural to refer to (1.5) as a *differential form* of Pohozaev's identity. For radial solutions on a ball, the corresponding version of (1.5) played a crucial role in the study of exact multiplicity of solutions, see T. Ouyang and J. Shi [9], and also P. Korman [6].

## 2 Non-existence of solutions for a class of systems

The following class of systems has attracted considerable attention recently

$$\begin{aligned} \Delta u + H_v(u, v) &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \\ \Delta v + H_u(u, v) &= 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \end{aligned} \quad (2.1)$$

where  $H(u, v)$  is a given differentiable function, see e.g. the following surveys: D.G. de Figueiredo [4], P. Quittner and P. Souplet [13], B. Ruf [14], see also P. Korman [5]. This system is of *Hamiltonian* type, which implies that it has some of the properties of scalar equations.

More generally, we assume that  $H(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m)$ , with integer  $m \geq 1$ , and consider the Hamiltonian system of  $2m$  equations

$$\begin{aligned} \Delta u_k + H_{v_k} &= 0 \text{ in } \Omega, \quad u_k = 0 \text{ on } \partial\Omega, \quad k = 1, 2, \dots, m \\ \Delta v_k + H_{u_k} &= 0 \text{ in } \Omega, \quad v_k = 0 \text{ on } \partial\Omega, \quad k = 1, 2, \dots, m. \end{aligned} \quad (2.2)$$

We call solution of (2.2) to be positive, if  $u_k(x) > 0$  and  $v_k(x) > 0$  for all  $x \in \Omega$ , and all  $k$ . We consider only the classical solutions, with  $u_k$  and  $v_k$  of class  $C^2(\Omega) \cap C^1(\bar{\Omega})$ . We have the following generalization of Pohozaev's identity, see also [2].

**Theorem 2.1** Assume that  $H(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m) \in C^2(R_+^m \times R_+^m) \cap C(\bar{R}_+^m \times \bar{R}_+^m)$ . For any positive solution of (2.2), and any real numbers  $a_1, \dots, a_m$ , one has

$$\begin{aligned} \int_{\Omega} [2nH(u, v) + (2-n)\sum_{k=1}^m (a_k u_k H_{u_k} + (2-a_k)v_k H_{v_k})] dx \\ = 2\sum_{k=1}^m \int_{\partial\Omega} (x \cdot \nu) |\nabla u_k| |\nabla v_k| dS. \end{aligned} \quad (2.3)$$

**Proof.** Define  $p_k = x \cdot \nabla u_k = \sum_{i=1}^n x_i u_{kx_i}$ , and  $q_k = x \cdot \nabla v = \sum_{i=1}^n x_i v_{kx_i}$ ,  $k = 1, 2, \dots, m$ . These functions satisfy the system

$$\begin{aligned} \Delta p_k + \sum_{j=1}^m H_{v_k u_j} p_j + \sum_{j=1}^m H_{v_k v_j} q_j &= -2H_{v_k}, \quad k = 1, 2, \dots, m \\ \Delta q_k + \sum_{j=1}^m H_{u_k u_j} p_j + \sum_{j=1}^m H_{u_k v_j} q_j &= -2H_{u_k}, \quad k = 1, 2, \dots, m. \end{aligned} \quad (2.4)$$

We multiply the first equation in (2.2) by  $q_k$ , and subtract from that the first equation in (2.4) multiplied by  $v_k$ . The result can be written as

$$\begin{aligned} \sum_{i=1}^n [(u_{kx_i} q_k - p_{kx_i} v_k)_{x_i} + (-u_{kx_i} q_{kx_i} + v_{kx_i} p_{kx_i})] \\ + H_{v_k} q_k - \sum_{j=1}^m H_{v_k u_j} p_j v_k - \sum_{j=1}^m H_{v_k v_j} q_j v_k = 2v_k H_{v_k}. \end{aligned} \quad (2.5)$$

Similarly, we multiply the second equation in (2.2) by  $p_k$ , and subtract from that the second equation in (2.4) multiplied by  $u_k$ , and write the result as

$$\begin{aligned} \sum_{i=1}^n [(v_{kx_i} p_k - q_{kx_i} u_k)_{x_i} + (-v_{kx_i} p_{kx_i} + u_{kx_i} q_{kx_i})] \\ + H_{u_k} p_k - \sum_{j=1}^m H_{u_k u_j} p_j u_k - \sum_{j=1}^m H_{u_k v_j} q_j u_k = 2u_k H_{u_k}. \end{aligned} \quad (2.6)$$

Adding the equations (2.5) and (2.6), we get

$$\begin{aligned} \sum_{i=1}^n [u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k]_{x_i} + H_{u_k} p_k + H_{v_k} q_k - \sum_{j=1}^m H_{u_k u_j} p_j u_k \\ - \sum_{j=1}^m H_{u_k v_j} q_j u_k - \sum_{j=1}^m H_{v_k u_j} p_j v_k - \sum_{j=1}^m H_{v_k v_j} q_j v_k = 2u_k H_{u_k} + 2v_k H_{v_k}. \end{aligned}$$

We now sum in  $k$ , then switch the orders of summation in  $i$  and  $k$  in the second group of terms on the left (the ones involving  $H$ ), putting the result into the form

$$\begin{aligned} \sum_{k=1}^m \sum_{i=1}^n [u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k]_{x_i} \\ + \sum_{i=1}^n x_i (2H - \sum_{k=1}^m u_k H_{u_k} - \sum_{k=1}^m v_k H_{v_k})_{x_i} = 2\sum_{k=1}^m u_k H_{u_k} + 2\sum_{k=1}^m v_k H_{v_k}. \end{aligned}$$

Writing,

$$\begin{aligned} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (2H - \sum_{k=1}^m u_k H_{u_k} - \sum_{k=1}^m v_k H_{v_k}) &= \sum_{i=1}^n \frac{\partial}{\partial x_i} [x_i (2H - \sum_{k=1}^m u_k H_{u_k} - \sum_{k=1}^m v_k H_{v_k})] \\ &\quad - n(2H - \sum_{k=1}^m u_k H_{u_k} - \sum_{k=1}^m v_k H_{v_k}), \end{aligned}$$

we obtain the differential form of Pohozaev's identity

$$\begin{aligned} \sum_{k=1}^m \sum_{i=1}^n [u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k + x_i (2H - \sum_{k=1}^m u_k H_{u_k} - \sum_{k=1}^m v_k H_{v_k})]_{x_i} \\ = 2nH + (2-n) (\sum_{k=1}^m u_k H_{u_k} + \sum_{k=1}^m v_k H_{v_k}). \end{aligned}$$

Integrating, we obtain as before

$$\begin{aligned} \int_{\Omega} [2nH(u, v) + (2 - n) (\sum_{k=1}^m u_k H_{u_k} + \sum_{k=1}^m v_k H_{v_k})] dx \\ = 2 \sum_{k=1}^m \int_{\partial\Omega} (x \cdot \nu) |\nabla u_k| |\nabla v_k| dS. \end{aligned} \quad (2.7)$$

(Since we consider positive solutions, and  $\partial\Omega$  is a level set for both  $u_k$  and  $v_k$ , we have  $\nu = -\frac{\nabla u_k}{|\nabla u_k|} = -\frac{\nabla v_k}{|\nabla v_k|}$ , i.e.,  $u_{ki} = -|\nabla u_k| \nu_i$  and  $v_{ki} = |\nabla v_k| \nu_i$  on the boundary  $\partial\Omega$ .) From the first equation in (2.1),  $\int_{\Omega} v_k H_{v_k} dx = \int_{\Omega} \nabla u_k \cdot \nabla v_k dx$ , while from the second equation  $\int_{\Omega} u_k H_{u_k} dx = \int_{\Omega} \nabla u_k \cdot \nabla v_k dx$ , i.e., for each  $k$

$$\int_{\Omega} v_k H_{v_k} dx = \int_{\Omega} u_k H_{u_k} dx.$$

Using this in (2.7), we conclude the proof.  $\diamond$

**Remark** Here and later on, we consider only the classical solutions. Observe that by our conditions and elliptic regularity, classical solutions are in fact of class  $C^3(\Omega)$ , so that all quantities in the above proof are well defined. Also, it suffices to assume that  $\Omega$  is star-shaped with respect to any one of its points (which we then take to be the origin).

As a consequence, we have the following non-existence result.

**Proposition 1** *Assume that  $\Omega$  is a star-shaped domain with respect to the origin, and for some real constants  $\alpha_1, \dots, \alpha_m$ , and all  $u_k > 0$ ,  $v_k > 0$ , we have*

$$nH + (2 - n) \sum_{k=1}^m (\alpha_k u_k H_{u_k} + (1 - \alpha_k) v_k H_{v_k}) < 0. \quad (2.8)$$

*Then the problem (2.2) has no positive solutions.*

**Proof.** We use the identity (2.3), with  $a_k/2 = \alpha_k$ . Then, assuming existence of positive solution, the left hand side of (2.3) is negative, while the right hand side is non-negative, a contradiction.  $\diamond$

**Example** Assume that  $m = 2$ , and consider  $H(u_1, v_1, u_2, v_2) = \frac{1}{p}(v_1^p + v_2^p) + u_1^r u_2^s$ , with  $p > \frac{2n}{n-2}$ , and  $r + s > \frac{2n}{n-2}$ . Then the inequality (2.8) holds, with  $\alpha_1 = \alpha_2 = \frac{1}{2}$ . It follows that the system

$$\begin{aligned} \Delta u_1 + v_1^{p-1} &= 0 \quad \text{in } \Omega, & u_1 &= 0 \quad \text{on } \partial\Omega, \\ \Delta v_1 + r u_1^{r-1} u_2^s &= 0 \quad \text{in } \Omega, & v_1 &= 0 \quad \text{on } \partial\Omega, \\ \Delta u_2 + v_2^{p-1} &= 0 \quad \text{in } \Omega, & u_2 &= 0 \quad \text{on } \partial\Omega, \\ \Delta v_2 + s u_1^r u_2^{s-1} &= 0 \quad \text{in } \Omega, & v_2 &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has no positive solutions.

In case  $m = 1$ , we recover the following result of E. Mitidieri [8]. We provide some details, in order to point out that some restrictions in [8] can be relaxed.

**Proposition 2** *Assume that  $\Omega$  is a star-shaped domain with respect to the origin, and for some real constant  $\alpha$ , and all  $u > 0$ ,  $v > 0$  we have*

$$\alpha u H_u(u, v) + (1 - \alpha) v H_v(u, v) > \frac{n}{n-2} H(u, v). \quad (2.9)$$

Then the problem (2.1) has no positive solution.

Comparing this result to E. Mitidieri[8], observe that we do not require that  $H_u(0, 0) = H_v(0, 0) = 0$ .

An important subclass of (2.1) is

$$\begin{aligned} \Delta u + f(v) &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \\ \Delta v + g(u) &= 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \end{aligned} \quad (2.10)$$

which corresponds to  $H(u, v) = F(v) + G(u)$ , where as before,  $F(v) = \int_0^v f(t) dt$ ,  $G(u) = \int_0^u g(t) dt$ . Unlike [8], we do not require that  $f(0) = g(0) = 0$ . The Theorem 2.1 now reads as follows.

**Theorem 2.2** *Let  $f, g \in C(\bar{R}_+)$ . For any positive solution of (2.10), and any real number  $a$ , one has*

$$\begin{aligned} \int_{\Omega} [2n(F(v) + G(u)) + (2 - n)(avf(v) + (2 - a)ug(u))] dx \\ = 2 \int_{\partial\Omega} (x \cdot \nu) |\nabla u| |\nabla v| dS. \end{aligned} \quad (2.11)$$

We now consider a particular system

$$\begin{aligned} \Delta u + v^p &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \\ \Delta v + g(u) &= 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \end{aligned} \quad (2.12)$$

with  $g(u) \in C(\bar{R}_+)$ , and a constant  $p > 0$ .

**Theorem 2.3** *Assume that  $\Omega$  is a star-shaped domain with respect to the origin, and*

$$nG(u) + (2 - n) \left( 1 - \frac{n}{(n-2)(p+1)} \right) ug(u) < 0, \quad \text{for all } u > 0. \quad (2.13)$$

*Then the problem (2.12) has no positive solution.*

**Proof.** We use Pohozaev's identity (2.11), with  $f(v) = v^p$ . We select the constant  $a$ , so that

$$2nF(v) + (2 - n)avf(v) = 0,$$

i.e.,  $a = \frac{2n}{(n-2)(p+1)}$ . Then, assuming existence of a positive solution, the left hand side of (2.11) is negative, while the right hand side is non-negative, a contradiction.  $\diamond$

Observe that in case  $p = 1$ , the Theorem 2.3 provides a non-existence result for a biharmonic problem with Navier boundary conditions (in E. Mitidieri [8], a separate identity was used to cover the biharmonic case)

$$\Delta^2 u = g(u) \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega. \quad (2.14)$$

**Proposition 3** *Assume that  $\Omega$  is a star-shaped domain with respect to the origin, and the condition (2.13) holds. Then the problem (2.14) has no positive solution.*

Finally, we consider the system

$$\begin{aligned} \Delta u + v^p &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \\ \Delta v + u^q &= 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \end{aligned} \quad (2.15)$$

The curve  $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}$  is called a *critical hyperbola*. We recover the following well known result of E. Mitidieri [8], see also R.C.A.M. Van der Vorst [15]. (Observe that we relax the restriction  $p, q > 1$  from [8].)

**Proposition 4** *Assume that  $p, q > 0$ , and*

$$\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n}. \quad (2.16)$$

*Then the problem (2.15) has no positive solution.*

**Proof.** Condition (2.16) is equivalent to (2.13), and the Theorem 2.3 applies.  $\diamond$

In case  $p = 1$ , we have the following known result, see E. Mitidieri [8].

**Proposition 5** *Assume that  $\Omega$  is a star-shaped domain with respect to the origin, and  $q > \frac{n+4}{n-4}$ . Then the problem*

$$\Delta^2 u = u^q \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega$$

*has no positive solutions.*

### 3 Pohozaev's identity for a version of $p$ -Laplace equation

We consider the following version of  $p$ -Laplace equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \varphi(u_{x_i}) + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (3.1)$$

Here  $\varphi(t) = t|t|^{p-2}$ , with a constant  $p > 1$ . This is a variational equation for the functional  $\int_{\Omega} \left[ \frac{1}{p} (|u_{x_1}|^p + \dots + |u_{x_n}|^p) - F(u) \right] dx$ . This equation is known to the experts, see P. Lindqvist [7], but it has not been studied much.

Observe that  $\varphi(at) = a^{p-1}\varphi(t)$ , for any constant  $a > 0$ . Also,  $\varphi'(t) = (p-1)|t|^{p-2}$ , i.e.,

$$t\varphi'(t) = (p-1)\varphi(t). \quad (3.2)$$

Letting, as before,  $z = x \cdot \nabla u = \sum_{i=1}^n x_i u_{x_i}$ , we see that  $z$  satisfies

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} [\varphi'(u_{x_i}) z_{x_i}] + f'(u)z = -pf(u). \quad (3.3)$$

To derive (3.3), we consider  $u^s(x) \equiv u(sx)$ , which satisfies

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \varphi\left(\frac{\partial}{\partial x_i} u^s\right) = -s^p f(u^s). \quad (3.4)$$

(To see that, it is convenient to write (3.1) as  $\sum_{i=1}^n \varphi'(\frac{\partial}{\partial x_i} u) \frac{\partial^2}{\partial x_i^2} u + f(u) = 0$ .) Then differentiating (3.4) with respect to  $s$ , and setting  $s = 1$ , we obtain (3.3). (Alternatively, to derive (3.3), one could differentiate (3.1) in  $x_j$ , then multiply by  $x_j$ , and sum in  $j$ .)

**Proposition 6** *Any solution of (3.1) satisfies*

$$\int_{\Omega} [pnF(u) + (p-n)uf(u)] dx = (p-1) \int_{\partial\Omega} (x \cdot \nu) |\nabla u| \sum_{i=1}^n \varphi(|\nabla u| \nu_i) \nu_i dS, \quad (3.5)$$

where  $\nu_i$  is the  $i$ -th component of  $\nu$ , the unit normal vector on  $\partial\Omega$ , pointing outside.

**Proof.** Multiply the equation (3.1) by  $z$ , and write the result as

$$(p-1) \sum_{i=1}^n \frac{\partial}{\partial x_i} [z\varphi(u_{x_i})] - (p-1) \sum_{i=1}^n \varphi(u_{x_i}) z_{x_i} + (p-1)f(u)z = 0. \quad (3.6)$$

Multiply the equation (3.3) by  $u$ , and write the result as

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} [u\varphi'(u_{x_i}) z_{x_i}] - \sum_{i=1}^n u_{x_i} \varphi'(u_{x_i}) z_{x_i} + f'(u)uz = -puf(u). \quad (3.7)$$

We now subtract (3.7) from (3.6). In view of (3.2), we have a cancellation, and so we obtain

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} [(p-1)z\varphi(u_{x_i}) - u\varphi'(u_{x_i})z_{x_i}] + [(p-1)f(u) - uf'(u)]z = puf(u).$$

As before,

$$\begin{aligned} [(p-1)f(u) - uf'(u)]z &= \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (pF(u) - uf(u)) \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} [x_i(pF(u) - uf(u))] - n(pF(u) - uf(u)). \end{aligned} \quad (3.8)$$

This gives us a differential form of Pohozaev's identity

$$\begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial x_i} [(p-1)z\varphi(u_{x_i}) - u\varphi'(u_{x_i})z_{x_i} + x_i(pF(u) - uf(u))] \\ = pnF(u) + (p-n)uf(u). \end{aligned} \quad (3.9)$$

Integrating, and using the divergence theorem, we conclude the proof.  $\diamond$

For star-shaped domains, the right hand side of (3.5) is non-negative, so if

$$pnF(u) + (p-n)uf(u) < 0 \text{ for all } u,$$

then the problem (3.1) has no non-trivial solutions.

**Example** For star-shaped domains, the problem

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \varphi(u_{x_i}) + u|u|^{r-1} = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has no non-trivial solutions, provided the constant  $r$  satisfies  $r > \frac{np-n+p}{n-p}$ .

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