A remark on Pinney's equation

Philip Korman
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025

Abstract

We show that Pinney's equation [2] with a constant coefficient can be reduced to its linear part by a simple change of variables. Also, Pinney's original solution is simplified slightly.

Key words: Pinney's equation, general solution.

AMS subject classification: 34A05.

In 1950 Edmund Pinney published a very influential paper [2], which was less than half a page long. That paper provided a general solution of the nonlinear differential equation

$$y'' + a(x)y + \frac{c}{y^3} = 0\,, \quad y(x_0) = q \neq 0\,,\; y'(x_0) = p\,, \tag{1}$$

with a given function $a(x)$ and a constant $c \neq 0$. Namely, the solution is

$$y(x) = \pm\sqrt{u^2(x) - cv^2(x)}\,, \tag{2}$$

where $u(x)$ and $v(x)$ are the solutions of the linear equation

$$y'' + a(x)y = 0\,, \tag{3}$$

for which $u(x_0) = q$, $u'(x_0) = p$, and $v(x_0) = 0$, $v'(x_0) = \frac{1}{q}$. One takes "plus" in (2) if $q > 0$, and "minus" if $q < 0$. Clearly, $u(x)$ and $v(x)$ form a fundamental set of (3), and by Liouville's formula their Wronskian at any x is the same as at x_0, i.e.,

$$u'(x)v(x) - u(x)v'(x) = 1\,, \quad \text{for all } x\,.$$

A substitution of $y = \sqrt{u^2(x) - cv^2(x)}$ into (1) gives

$$y'' + a(x)y + \frac{c}{y^3} = -c\,\frac{\left[u'(x)v(x) - u(x)v'(x)\right]^2 - 1}{\left[u^2(x) - cv^2(x)\right]^{\frac{3}{2}}} = 0\,.$$

If $c < 0$, the solution is valid for all x, while for $c > 0$ some singular points are possible.

The nonlinear equation equation (1) possessing a *general solution* is very special, and it attracted a lot of attention (there are currently 92 MathSciNet and 543 Google Scholar citations). It turns out that this equation was considered back in 1880 by Ermakov [1].

Our remark is that in case of constant $a(x) = a_0$, Pinney's equation becomes linear for $z(x) = y^2(x)$. Indeed, we multiply the equation

$$y'' + a_0 y + \frac{c}{y^3} = 0 \tag{4}$$

by y', and integrate to get

$$y'^2 + a_0 y^2 - cy^{-2} = p^2 + a_0 q^2 - c\frac{1}{q^2}\,. \tag{5}$$

Now multiply the same equation by y:

$$yy'' + a_0 y^2 + cy^{-2} = 0\,, \tag{6}$$

and set $z = y^2$. Since $yy'' = \frac{1}{2}z'' - y'^2$, by using (5), one transforms (6) to

$$z'' + 4a_0 z = 2\left(p^2 + a_0 q^2 - c\frac{1}{q^2}\right)\,, \quad z(0) = q^2\,,\ z'(0) = 2pq\,.$$

References

[1] V.P. Ermakov, Second order differential equations, Kiev University Izvestia, **9**, 125 (1880) (Russian).

[2] E. Pinney, The nonlinear differential equation $y'' + p(x)y + cy^{-3} = 0$, *Proc. Amer. Math. Soc.* **1**, 681 (1950).