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On periodic solutions for singular perturbation problems

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Synopsis

We apply a version of the Nash–Moser method to prove existence of periodic solutions for nonlinear elliptic equations and systems, involving singular perturbations. We allow nonlinearities depending on derivatives of order two more than that of the linear part, thus extending the previous results. Our result is new even in the case of one equation in one spatial dimension.

1. Introduction

We study the existence of periodic solutions for singularly perturbed elliptic equations and symmetric systems. This question was initiated by P. Rabinowitz [5, 6], who considered the problem of finding a function $u(x) = u(x_1, \ldots, x_n)$, which is 2π periodic in all variables and solves

$$-\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_j})_{x_i} + u = \varepsilon f(x, u, Du, D^2u, D^3u). \tag{1.1}$$

Here the function f depends on u and its derivatives up to order three, and is also 2π periodic in $x_1, \ldots, x_n$. The operator on the left is assumed to be uniformly elliptic with coefficients 2π periodic in $x_1, \ldots, x_n$. If one attempts to solve (1.1) using Picard's iterations, one has a loss of one derivative at each step. P. Rabinowitz [5] uses J. Moser's version of the "Nash-Moser" method to prove solvability of (1.1) for ε sufficiently small.

Another approach to this problem was subsequently found by T. Kato [1], by adopting some of his general techniques developed for evolution equations.

We consider nonlinearities of order two more than that of the linear part, i.e. problems of the type

$$-\Delta u + u = \varepsilon f(x, u, Du, D^2u, D^3u, D^4u). \tag{1.2}$$

We allow f to depend on the fourth-order derivatives of the type $u_{x_ix_ix_jx_j}$, provided a certain positivity condition is satisfied. We prove solvability of (1.2) for small ε by using a slight modification of J. Schwartz's [7] version of the Nash-Moser technique, which is rooted in the work of J. Nash. Similar perturbation results hold for elliptic equations of arbitrary order, and for symmetric elliptic systems of arbitrary order and size. For perturbations of order greater than two our technique does not apply, and we do not know of any results in that direction. To simplify the presentation we restrict ourselves to elliptic operators of the type $(-1)^{m_0}\Delta^{m_0}u + u$, and to second-order elliptic systems with two equations.

Next we discuss the notation and list some preliminary results. By T^n we denote the n-torus, $T^n = [0, 2\pi]^n$. We abbreviate $\int f = \int_{T^n} f(x)\,dx$. We write $u_{x_k} = u_k = D_k u$ for partial derivatives; $a_{ij,k} = D_k a_{ij}$; $D^\alpha u$ is the derivative corresponding to a multi-index α; D^{m_0} denotes the set of all partial derivatives of order m_0. The following notation will be used repeatedly:

$$D^\alpha(fg) = f^\alpha g + f^{\alpha-1}g^1 + \dots + f^1 g^{\alpha-1} + fg^\alpha,$$

where we denote $f^{\alpha-k}g^k = \sum_{|\gamma|=k} c_\gamma D^{\alpha-\gamma} f D^\gamma g$, with c_γ the coefficients from the Leibnitz rule. We shall write $\|\cdot\|_m$ for the norm on the Sobolev space $H^m(T^n)$, $|\cdot|_m$ for the one on $C^m(T^n)$. All positive constants independent of the unknown functions we denote by c.

We need the following standard lemmas, see [2] for proofs and references.

LEMMA 1.1. *For any integer $m \geq 0$ and any $\varepsilon > 0$, one can find a constant $c(\varepsilon)$ so that*

$$\|v\|_m \leq \varepsilon \|v\|_{m+1} + c(\varepsilon)\|v\|_0.$$

LEMMA 1.2. *Suppose $f_1, f_2 \in C^r(T^n)$, $r \geq 0$ is an integer. Then*

$$\|f_1 f_2\|_r \leq c(|f_1|_0 \|f_2\|_r + |f_2|_0 \|f_1\|_r).$$

LEMMA 1.3. *Suppose $w_1, \dots, w_s \in C^r(T^n)$. Suppose that $\phi = \phi(x, w_1, \dots, w_s)$ possesses continuous derivatives up to order $r \geq 1$ bounded by c for $x \in T^n$ and $\max_i |w_i| < 1$. Then*

$$\|\phi(x, w_1, \dots, w_s)\|_r \leq c(\max_i \|w_i\|_r + 1).$$

If in addition we assume that $\phi(x, 0, \dots, 0) \equiv 0$, $r \geq [n/2] + 1$, then

$$\|\phi(x, w_1, \dots, w_s)\|_r \leq \delta(\max_i \|w_i\|_r) \quad \textit{where} \quad \delta(t) \to 0 \textit{ as } t \to 0.$$

LEMMA 1.4. *Let l, k, m be non-negative integers, $k < m$. Then*

$$\|u\|_{k+l} \leq c\|u\|_{m+l}^{k/m} \|u\|_l^{1-(k/m)}.$$

The following theorem is a slight modification of J. Schwartz's form of J. Nash's implicit function theorem, see [3] for proof. We denote $B^m = H^m(T^n) \times \dots \times H^m(T^n) = H^m(T^n)^s$. If $u = (u_1, \dots, u_s) \in B^m$, then $\|u\|_m \equiv \sum_{k=1}^s \|u_k\|_m$.

THEOREM 1.5. *Let $F[u]$: $B^m \to B^{m-\alpha}$ $(0 \leq \alpha \leq m)$ be a (non-linear) operator with the domain $D(F) = \{u \in B^m, \|u\|_m < \delta, \delta > 0\}$. Suppose that*

(i) *$F[u]$ has two continuous Frechet derivatives both bounded by c;*

(ii) *there exists a map $L(u)$ with domain $D(L) = D(F)$ and range in the space $B(B^{m-\alpha}, B^{m-\alpha})$ of bounded linear operators on $B^{m-\alpha}$ to itself, such that*

(iia) *$F'[u]L(u)h = h$, $h \in B^{m-\alpha}$, $u \in D(F)$,*

(iib) *$\|L(u)h\|_{m-\alpha} \leq c\,\|h\|_{m-\alpha}$, $h \in B^{m-\alpha}$, $u \in D(F)$,*

(iic) *$\|L(u)F[u]\|_{m+8\alpha} \leq c(1 + \|u\|_{m+9\alpha})$, $u \in B_1^{m+9\alpha} \cap D(F)$.*

Then if $\|F[0]\|_{m-\alpha}$ is small enough (compared with c), $F[D(F)]$ contains the origin.

2. *A priori* estimates and existence for the linear problem

LEMMA 2.1. *Consider the problem* ($m_0 =$ *integer* > 0)

$$u + (-1)^{m_0}\Delta^{m_0}u + Lu = f(x), \tag{2.1}$$

where

$$Lu \equiv \sum_{|\alpha|\le 2m_0} a_\alpha(x)D^\alpha u + \sum_{|\beta|=2m_0+1} b_\beta(x)D^\beta u + \sum_{|\gamma|=m_0+1} c_\gamma(x)D^{2\gamma}u.$$

All the functions a_α, b_β, c_γ, u *and* f *are assumed to be* 2π*-periodic in each variable* x_i, *i.e.* $x \in T^n$. *Assume that*

$$(-1)^{m_0+1}\sum_{|\gamma|=m_0+1} c_\gamma(x)\xi_\gamma^2 \ge 0 \quad \text{for all } x \in T^n, \tag{2.2}$$

and for any collection of real numbers ξ_γ, *indexed by the multi-index* γ. *For integer* $k \ge 0$, *denote*

$$a_k = \max_{|\alpha|\le 2m_0} |a|_k,\ b_k = \max_\beta |b_\beta|_k,\ c_k = \max_{|\gamma|=m_0+1} |c_\gamma|_k;$$

$$\rho_k = a_k + b_k + c_k;$$

$$p_{m_0+3} = \rho_{m_0+3},\ p_{m_0+4} = \rho_{m_0+3}p_{m_0+3} + \rho_{m_0+4}, \ldots;$$

$$p_l = \rho_{m_0+3}p_{l-1} + \rho_{m_0+4}p_{l-2} + \ldots + \rho_{l-1}p_{m_0+3} + \rho_l,\ l \ge m_0 + 5.$$

Then if $\rho_{m_0+2} < \varepsilon_1$ *with* ε_1 *sufficiently small, one has the following a priori estimates* (m*-integer*):

$$\|u\|_{m+m_0} + \|u\|_0 \le \begin{cases} c\|f\|_m, & \text{for } 0 \le m \le m_0+2,\\ c(\|f\|_m + p_{m_0+3}\|f\|_{m-1} + p_{m_0+4}\|f\|_{m-2} \\ \quad + \ldots + p_m\|f\|_{m_0+2}), & \text{for } m > m_0+2. \end{cases}$$

Proof. Multiply (2.1) by u and integrate over T^n,

$$\int u(u + (-1)^{m_0}\Delta^{m_0}u) + \int uLu = \int fu. \tag{2.3}$$

Denote $I = \sum_{|\beta|=2m_0+1} \int b_\beta u D^\beta u$. We integrate by parts successively, taking one derivative from $D^\beta u$ at each step. After $2m + 1$ steps, we obtain

$$I = -I + \ldots,$$

where all the terms not shown on the right-hand side have b_β differentiated exactly once. Solving for I, and using further repeated integration by parts, we estimate

$$I \le cb_{m_0+1}\|u\|_{m_0}^2 \le c\varepsilon_1\|u\|_{m_0}^2.$$

Similarly (using previously defined notation)

$$\sum_{|\gamma|=m_0+1}\int c_\gamma D^{2\gamma}u\,u=(-1)^{m_0+1}\Big[\sum_{|\gamma|=m_0+1}c_\gamma(D^\gamma u)^2$$
$$+\sum_{|\gamma|=m_0+1}\int c_\gamma^1 u^\gamma u^{\gamma-1}+\ldots+\sum_{|\gamma|=m_0+1}\int c_\gamma^{m_0-1}u^\gamma u^1$$
$$+\sum_{|\gamma|=m_0+1}\int D^\gamma c_\gamma u^\gamma u\Big].$$

As above, we see that all terms on the right hand side from the second one onwards are bounded by $c\,|c|_{m_0+2}\,\|u\|^2_{m_0+1}\le c\varepsilon_1\,\|u\|^2_{m_0+1}$. Using these remarks and our conditions, we easily estimate from (2.3)

$$\|u\|_{m_0}+\|u\|_0\le c\,\|f\|_0,$$

which implies (2.2) for $m=0$.

Higher-order estimates are obtained by differentiating the equation (2.1). Denote $D^\delta u=u^\delta$, $|\delta|\le m$. Differentiate (2.1) and multiply the resulting equation by u^δ,

$$\int(u^\delta)^2+(-1)^{m_0}\int u^\delta\Delta^{m_0}u^\delta$$
$$+\sum_{|\alpha|\le 2m_0}\int u^\delta(a_\alpha D^\alpha u^\delta+a_\alpha^1 D^\alpha u^{\delta-1}+\ldots+D^\delta a_\alpha D^\alpha u)$$
$$+\sum_{|\beta|=2m_0+1}\int u^\delta(b_\beta D^\beta u^\delta+b_\beta^1 D^\beta u^{\delta-1}+\ldots+D^\delta b_\beta D^\beta u)$$
$$+\sum_{|\gamma|=m_0+1}\int u^\delta(c_\gamma D^{2\gamma}u^\delta+c_\gamma^1 D^{2\gamma}u^{\delta-1}+\cdots+D^\delta c_\gamma D^{2\gamma}u)=\int f^\delta u^\delta. \quad (2.4)$$

For $m>m_0+2$, we estimate (using repeated integration by parts on the first group of terms and the Schwarz inequality on the second)

$$A\equiv\Big|\int u^\delta(a_\alpha D^\alpha u^\delta+a_\alpha^1 D^\alpha u^{\delta-1}+\ldots+a_\alpha^{m_0+2}D^\alpha u^{\delta-m_O-2}$$
$$+a_\alpha^{m_0+3}D^\alpha u^{\delta-m_0-3}+\ldots+D^\delta a_\alpha D^\alpha u)\Big|$$
$$\le ca_{m_0+2}\,\|u\|^2_{m+m_0}+\varepsilon\,\|u\|^2_m$$
$$+c(\varepsilon)(a^2_{m_0+3}\,\|u\|^2_{m+m_0-3}+\ldots+a^2_m\,\|u\|^2_{2m_0}),$$

while $A\le ca_{m_0+2}\,\|u\|^2_{m+m_0}\le c\varepsilon_1\,\|u\|^2_{m+m_0}$ for $m\le m_0+2$. Similarly (see the estimate of I above)

$$\Big|\sum_{|\beta|=2m_0+1}\int u^\delta(b_\beta D^\beta u^\delta+\ldots+D^\delta b_\beta D^\beta u)\Big|$$
$$\le\begin{cases} cb_{m_0+2}\,\|u\|^2_{m+m_0}+\varepsilon\,\|u\|^2_m+c(\varepsilon)(b^2_{m_0+3}\,\|u\|^2_{m+m_0-2}+\ldots+b^2_m\,\|u\|^2_{2m_0+1}),\\ \text{for}\quad m>m_0+2,\\ cb_{m_0+2}\,\|u\|^2_{m+m_0},\quad\text{for}\quad m\le m_0+2.\end{cases}$$

Next,

$$\sum_{|\gamma|=m_0+1} \int c_\gamma u^\delta D^{2\gamma} u^\delta = (-1)^{m_0+1} \sum_{|\gamma|=m_0+1} \int c_\gamma (u^{\delta+\gamma})^2 + \ldots,$$

where the first term on the right-hand side is positive by our assumptions, and all the others are easily estimated by $cc_{m_0+2}\,\|u\|^2_{m+m_0}$. The remaining terms involving c_γ are estimated as before:

$$\left| \sum_{|\gamma|=m_0+1} \int u^\delta (c^1_\gamma D^{2\gamma} u^{\delta-1} + \ldots + D^\delta c_\gamma D^{2\gamma} u) \right|$$

$$\begin{cases} c\, c_{m_0+2}\,\|u\|^2_{m+m_0} + \varepsilon\,\|u\|^2_m + c(\varepsilon)(c^2_{m_0+3}\,\|u\|^2_{m+m_0-1} + \ldots + c^2_m\,\|u\|^2_{2m_0+2}), \\ \text{for} \quad m > m_0 + 2, \\ c\, c_{m_0+2}\,\|u\|^2_{m+m_0}, \quad \text{for} \quad m \le m_0 + 2. \end{cases}$$

Using all these estimates in (2.4), summing in δ and fixing ε and ε_1 sufficiently small, we easily get the estimates:

$$\|u\|_{m+m_0} + \|u\|_0 \le c(\|f\|_m + \rho_{m_0+3}\,\|u\|_{m+m_0-1} + \rho_{m_0+4}\,\|u\|_{m+m_0-2} + \ldots + \rho_m\,\|u\|_{2m_0+2}) \quad \text{for} \quad m > m_0 + 2,$$

from which the proof easily follows.

Lemma 2.2. *Assume all conditions of Lemma* 2.1, *and that* $\rho_m \le c$ *and* $f \in H^m(T^n)$, $m > m_0 + [n/2] + 3$. *Then, for* ρ_{m_0+2} *sufficiently small, the problem* (2.1) *has a unique solution of class* $H^{m+m_0}(T^n)$.

Proof. For $\sigma = \text{const} > 0$, $0 \le \varepsilon \le 1$, and $x \in T^n$, consider an auxiliary problem

$$u + (-1)^{m_0}\Delta^{m_0} u + \varepsilon L u + \sigma(-1)^{m_0+1}\Delta^{m_0+1} u = f. \tag{2.5}$$

This is a uniformly elliptic equation on T^n, so that its index as an operator from $H^{m+2m_0+2}(T^n)$ to $H^m(T^n)$ is defined and homotopy invariant. By letting $\varepsilon \to 0$, we get an equation

$$u + (-1)^{m_0}\Delta^{m_0} u + \sigma(-1)^{m_0+1}\Delta^{m_0+1} u = f,$$

whose index (and hence that of (2.5)) is zero, as can be seen by a simple Fourier analysis. One easily sees that the estimates of Lemma 2.1 also hold for (2.5) with c independent of σ. This implies that (2.5) can have at most one solution, and since its index is zero, it is solvable. Let u^σ be the solution of (2.5) corresponding to $\varepsilon = 1$. Since $\|u^\sigma\|_{m+m_0} \le c$ uniformly in $\sigma > 0$, it follows that as $\sigma \to 0$ along some sequence, $u^\sigma \to u$ in $H^{m+m_0-1}(T^n)$ along a subsequence, where u is a solution of (2.1). Applying Lemma 2.1 again, we conclude that $u \in H^{m+m_0}(T^n)$.

3. Existence for singular perturbation equations

The following is a perturbation result, providing existence of a "small" solution.

Theorem 3.1. *On the torus* T^n *consider the equation*

$$F[u] \equiv u + (-1)^{m_0}\Delta^{m_0} u + f(x, u, Du, \ldots, D^{2m_0+2} u) = 0. \tag{3.1}$$

Assume that $f=f_1(x,u,Du,\ldots,D^{2m_0+2}u)+\varepsilon f_2(x,u,Du,\ldots,D^{2m_0+2}u)$, *with* $f_1(x,0,0,\ldots,0)=\partial f_1/\partial u\,(x,0,0,\ldots,0)=\partial f_1/\partial D^\alpha u\,(x,0,0,\ldots,0)\equiv 0$, *where* $D^\alpha u$ *is any derivative present among the arguments of* f. *Denote* $a_\alpha=\partial f/\partial D^\alpha u$ *for* $|\alpha|\le 2m_0$, $b_\beta=\partial f/\partial D^\beta u$ *for* $|\beta|=2m_0+1$. *For* $|\alpha|=2m_0+2$, *assume that* $\partial f/\partial D^\alpha u\equiv 0$ *unless* $\alpha=2\gamma$ *for some* $|\gamma|=m_0+1$; *in such a case denote* $c_\gamma=\partial f/\partial D^{2\gamma}u$. *For* $x\in T^n$ *and all other variables of* f *being sufficiently small in absolute values, assume that* $(-1)^{m_0+1}\sum_{|\gamma|=m_0+1}c_\gamma\xi_\gamma^2\ge 0$ *for any collection of real numbers* ξ_γ, *and that* $f\in C^{\mu_0}$ *with* $\mu_0=19m_0+10[n/2]+31$. *Then for* ε *sufficiently small the problem* (2.1) *has a* 2π *periodic in each* x_i *solution of class* $C^{2m_0+2}(T^n)$.

Proof. Consider $F[u]$ as a map $F\colon B^\mu(T^n)\to H^{\mu-\alpha}(T^n)$, where $B^\mu(T^n)=\{u\in H^\mu(T^n)\colon \|u\|_\mu\le\delta\}$, with constant $\delta>0$ and positive integers $\mu\ge\alpha$ to be specified. We shall solve (3.1) by applying Theorem 1.5. Notice that

$$F'[u]v=v+(-1)^{m_0}\Delta^{m_0}v+Lv\quad (L\text{ as defined in (2.1)}).$$

It is straightforward to show that $F'[u]$, $F''[u]$ are continuous and bounded operators provided $\mu-\alpha>[n/2]$, $\alpha\ge 2m_0+2$ (see [2, 3] for similar arguments).

Conditions (ii*a*) and (ii*b*) of Theorem 1.5 follow directly from Lemmas 2.1 and 2.2. We need to require that $\mu-\alpha\ge m_0+[n/2]+4$ for Lemma 2.2. Assuming further that $\alpha\ge 2m_0+[n/2]+3$, we estimate, using Lemma 1.3,

$$\rho_{m-\alpha}\le c(\|u\|_{\mu-\alpha+[n/2]+1+2m_0+2}+1)\le c(\delta+1)\le c,$$
$$\rho_{m_0+2}\le o(\|u\|_{m_0+2+[n/2]+1+2m_0+2})=o(\delta)\quad\text{as}\quad \delta\to 0,$$

which makes both Lemmas 2.1 and 2.2 applicable. To verify condition (ii*c*), we apply Lemma 2.1 again:

$$\|L(u)F[u]\|_{\mu+8\alpha}\le c(\|F[u]\|_{\mu+8\alpha}+p_{m_0+3}\|F[u]\|_{\mu+8\alpha-1}+\ldots+p_{\mu+8\alpha}\|F[u]\|_{m_0+2}).\quad (3.2)$$

If we denote $\tau=\|u\|_{\mu+9\alpha}^{1/(\mu+9\alpha-m_0-[n/2]-5)}$, then by Lemma 1.4 (since $\mu\ge m_0+[n/2]+5$)

$$\|u\|_k\le c\,\|u\|_{\mu+9\alpha}^{(k-m_0-[n/2]-5)/(\mu+9\alpha-m_0-[n/2]-5)}\times\|u\|_{m_0+[n/2]+5}^{1-[(k-m_0-[n/2]-5)/(\mu+9\alpha-m_0-[n/2]-5)]}\le c\tau^{k-m_0-[n/2]-5},\quad (3.3)$$

for $k=m_0+[n/2]+6,\ldots,\mu+9\alpha-1$. Then by (3.3),

$$\rho_k\le c(\|u\|_{k+[n/2]+2m_0+3}+1)\le c(\tau^{k+m_0-2}+1),\quad k=m_0+3,\ldots,\mu+8\alpha;$$
$$p_k\le c(\tau^{k+m_0-2}+1),\quad k=m_0+3,\ldots,\mu+8\alpha;$$
$$\|F[u]\|_k\le c(\|u\|_{k+2m_0+2}+1)\le c(\tau^{k+m_0-[n/2]-3}+1),\quad k=\mu-2m_0-1,\ldots,\mu+8\alpha;$$
$$\|F[u]\|_k\le c,\quad k=m_0+2,m_0+3,\ldots,\mu-2m_0-2.$$

Using these estimates in (3.2), we estimate

$$\|L(u)F[u]\|_{\mu+8\alpha}\le c(\tau^{\mu+8\alpha+m_0-2}+1)\le c(\|u\|_{m+9\alpha}+1),$$

provided that $\mu+8\alpha+m_0-2\le\mu+9\alpha-m_0-[n/2]-5$. By fixing $\alpha=2m_0+[n/2]+3$, $\mu=3m_0+2[n/2]+7$, $\mu_0=\mu+8\alpha$, and δ sufficiently small we satisfy all of the above requirements, and conclude the proof. □

Remark 3.2. It is clear from the proof that if we assume f to be of class C^{ν} with $\nu > \mu_0$, then the solution is of class $C^{2m_0+2+\nu-\mu_0}$, and if $f \in C^{\infty}$ so does the solution.

Example 3.3. Let a, ϕ be 2π periodic in each x_i and C^{∞} in all arguments; $a(x) \geq 0$ for $x \in T^n$. The equation

$$\Delta u - u = a(x)u^3_{x_1x_1x_1x_1} + \varepsilon\phi(x, u, Du, D^2u)$$

has a solution $u \in C^{\infty}(T^n)$ for ε sufficiently small.

4. A symmetric singularly perturbed system

We show that the results of the preceding sections extend to symmetric systems. To simplify the presentation, we consider two equations of second order, but our results easily generalise to an arbitrary order and number of equations. As before we start with *a priori* estimates and existence for the linear case.

LEMMA 4.1. *On T^n consider the system*

$$\begin{aligned} u(x) - a_{ij}(x)u_{ij} - b_{ij}(x)v_{ij} - a_i(x)u_i - b_i(x)v_i - a_0(x)u - b_0(x)v &= f(x), \\ v(x) - b_{ij}(x)u_{ij} - c_{ij}(x)v_{ij} - b_i(x)u_i - c_i(x)v_i - d_0(x)u - c_0(x)v &= g(x), \end{aligned} \tag{4.1}$$

where all the functions involved are 2π periodic in each x_i, $i = 1, \ldots, n$; a_{ij}, b_{ij}, c_{ij} are symmetric matrices, and the summation convention is used throughout this section. Assume that

(i) $a_{ij}(x)\xi_i\xi_j + 2b_{ij}(x)\xi_i\eta_j + c_{ij}(x)\eta_i\eta_j \geq 0$ *for all* $x \in T^n$, $\xi\eta \in R^n$;

(ii) *if* $n > 2$ *then* $a_{ij} = b_{ij} = c_{ij} \equiv 0$ *for* $i \neq j$.

For integer $k \geq 0$, denote

$$\rho_k = \max_{i,j} (|a_{ij}|_k, |b_{ij}|_k, |c_{ij}|_k, |a_i|_k, |b_i|_k, |c_i|_k, |a_0|_k, |b_0|_k, |c_0|_k, |d_0|_k),$$

$$p_3 = \rho_3, \; p_l = \rho_3 p_{l-1} + \rho_4 p_{l-2} + \ldots + \rho_{l-1}p_3 + \rho_l \; \textit{for } l \geq 4.$$

Introduce the vectors $U = \binom{u}{v}$ and $F = \binom{f}{g}$ with the norms $\|U\|_m = \|u\|_m + \|v\|_m$, $\|F\|_m = \|f\|_m + \|g\|_m$.

Then for ρ_2 sufficiently small, the following estimates hold (m = integer)

$$\begin{aligned} &\|U\|_m \leq c\,\|F\|_m \quad \textit{for} \quad m = 0, 1, 2, \\ &\|U\|_m \leq c(\|F\|_m + p_3\|F\|_{m-1} + p_4\,\|F\|_{m-2} + \ldots + p_m\,\|F\|_2) \quad \textit{for} \quad m \geq 3. \end{aligned} \tag{4.2}$$

Proof. To simplify the presenation, assume that $a_i(x) = b_i(x) = c_i(x) = a_0(x) = b_0(x) = c_0(x) = d_0(x) \equiv 0$ for all i. Multiply the first equation in (4.1) by u, the second one by v, integrate both equations over T^n and add:

$$\begin{aligned} \int (u^2 + v^2) + \int [a_{ij}u_iu_j + b_{ij}(u_iv_j + v_iu_j) + c_{ij}v_iv_j] - \frac{1}{2}\int a_{ij,ij}u^2 \\ - \frac{1}{2}\int c_{ij,ij}v^2 + \int b_{ij,i}(uv_j + u_jv) = \int fu + \int gv. \end{aligned} \tag{4.3}$$

The second term on the left-hand side in (4.3) is nonnegative, while the last one is bounded by $\rho_2 \int (u^2 + v^2)$, and then the estimate (4.2) for $m = 0$ easily follows.

Next we differentiate both equations in (4.1), and denote $D^\delta u = u^\delta$, $D^\delta v = v^\delta$, $|\delta| \le m$. Multiply the first equation by u^δ, the second one by v^δ, integrate over T^n and add:

$$\int (u^\delta)^2 + \int (v^\delta) - \int a_{ij} u^\delta u_{ij}^\delta - \int a_{ij}^1 u^\delta u_{ij}^{\delta-1} - \int a_{ij}^2 u^\delta u_{ij}^{\delta-2} - \dots - \int a_{ij}^\delta u^\delta u_{ij}$$

$$- \int b_{ij} u^\delta v_{ij}^\delta - \int b_{ij}^1 u^\delta v_{ij}^{\delta-1} - \int b_{ij}^2 u^\delta v_{ij}^{\delta-2} - \dots - \int b_{ij}^\delta u^\delta v_{ij} - \int b_{ij} u_{ij}^\delta v^\delta$$

$$- \int b_{ij}^1 u_{ij}^{\delta-1} v^\delta - \int b_{ij}^2 u_{ij}^{\delta-2} v^\delta - \dots - \int b_{ij}^\delta u_{ij} v^\delta - \int c_{ij} v_{ij}^\delta v^\delta$$

$$- \int c_{ij}^1 v_{ij}^{\delta-1} v^\delta - \int c_{ij}^2 v_{ij}^{\delta-2} v^\delta - \dots - \int c_{ij}^\delta v_{ij} v^\delta = \int (f^\delta u^\delta + g^\delta v^\delta).$$

Notice that

$$-\int a_{ij} u^\delta u_{ij}^\delta - \int b_{ij} u^\delta v_{ij}^\delta - \int b_{ij} u_{ij}^\delta v^\delta - \int c_{ij} v_{ij}^\delta v^\delta$$

$$= \int (a_{ij} u_i^\delta u_j^\delta + b_{ij} u_i^\delta v_j^\delta + b_{ij} u_j^\delta v_i^\delta + c_{ij} v_i^\delta v_j^\delta) - \frac{1}{2} \int a_{ij,ij} (u^\delta)^2$$

$$- \frac{1}{2} \int c_{ij,ij} (v^\delta)^2 + \int b_{ij,i} (u^\delta v_j^\delta + u_j^\delta v^\delta).$$

Here the first integral on the right-hand side is positive by (i), while all others are bounded by $c\rho_2(\|u\|_m^2 + \|v\|_m^2)$. Next,

$$-\int b_{ij}^1 u^\delta v_{ij}^{\delta-1} - \int b_{ij}^1 u_{ij}^{\delta-1} v^\delta = \int b_{ij,i}^1 (u^\delta v_j^{\delta-1} + u_j^{\delta-1} v^\delta) + \int b_{ij}^1 (u_i^\delta v_j^{\delta-1} + u_j^{\delta-1} v_i^\delta).$$

The first term on the right-hand side is bounded by $c\rho_2(\|u\|_m^2 + \|v\|_m^2)$. For the second one, we consider two cases:

Case (i) $n = 2$. Then we may assume that $u_j^{\delta-1} = u^\delta$, $v_j^{\delta-1} = v^\delta$ (otherwise interchange i and j in case they are different, or refer to the next case if they are equal). Then

$$\left| \int b_{ij}^1 (u_i^\delta v_j^{\delta-1} + u_j^{\delta-1} v_i^\delta) \right| = \left| \int b_{ij,i}^1 u^\delta v^\delta \right| \le c\rho_2(\|u\|_m^2 + \|v\|_m^2).$$

Case (ii) $n > 2$. Then we may assume by the assumption (ii) that $i = j$ (the other terms are zero). Then a typical member of the second term is estimated as follows:

$$\left| \int b_{ij,p} (u_j^\delta v_j^{\delta-e_p} + u_j^{\delta-e_p} v_j^\delta) \right| = \left| \int b_{jj,pp} u_j^{\delta-e_p} v_j^{\delta-e_p} \right| \le c\rho_2(\|u\|_m^2 + \|v\|_m^2).$$

The terms $\int a_{ij}^1 u^\delta u_{ij}^{\delta-1}$ and $\int c_{ij}^1 v_{ij}^{\delta-1} v^\delta$ are similarly bounded by $c\rho_2(\|u\|_m^2 + \|v\|_m^2)$. The remaining terms in (4.4) are estimated in a uniform manner, which we illustrate on one of the terms,

$$\int a_{ij}^3 u^\delta u_{ij}^{\delta-3} \leq \varepsilon \, \|u\|_m^2 + c(\varepsilon)\rho_3^2 \, \|u\|_{m-1}^2.$$

Using all these considerations in (4.4), and summing in all $|\delta| \leq m$, we obtain (choosing ε sufficiently small)

$$\|U\|_m \leq c\,(\|F\|_m + \rho_3 \,\|U\|_{m-1} + \rho_4 \,\|U\|_{m-2} + \ldots + \rho_m \,\|U\|_2),$$

and the proof follows. □

LEMMA 4.2. *For the problem* (4.1), *assume that all conditions of Lemma* 4.1 *are satisfied*; f, $g \in H^m(T^n)$, $\rho_m \leq c$, $m \geq [n/2] + 4$. *Then for* ρ_2 *sufficiently small the problem* (4.1) *has a unique solution with* u, $v \in H^m(T^n)$.

Proof. To simplify the presentation, we shall again assume that $a_i = b_i = c_i = a_0 = b_0 = c_0 = d_0 \equiv 0$. For $\sigma = \text{const} > 0$ and $0 \leq t \leq 1$, consider a new system on T^n:

$$\begin{aligned} u - \sigma\Delta u - t a_{ij} u_{ij} - t b_{ij} v_{ij} &= f(x), \\ v - \sigma\Delta v - t b_{ij} u_{ij} - t c_{ij} v_{ij} &= g(x). \end{aligned} \tag{4.5}$$

Examining the proof of Lemma 4.1, one verifies the following estimates for (4.5):

$$\|u\|_{m+2} + \|v\|_{m+2} \leq c(\|f\|_m + \|g\|_m) \quad \text{with} \quad c = c(\sigma), \tag{4.6}$$

$$\|u\|_m + \|v\|_m \leq c(\|f\|_m + \|g\|_m) \leq c \text{ with } c \text{ independent of } \sigma. \tag{4.7}$$

Let S denote the set of $t \in [0, 1]$ such that the system (4.5) has a unique solution with u, $v \in H^{m+2}(T^n)$. Obviously $0 \in S$.

One easily shows that S is open in $[0, 1]$. (If (4.5) is solvable for t_0, then for $|t - t_0|$ small one sets up a contractive mapping on a ball of sufficiently large radius around the origin in $H^{m+2}(T^n)$, using the estimate (4.6).) To see that S is closed in $[0, 1]$, we assume there is a sequence of $t_n \to t_0$ with corresponding solution $(u_n, v_n) \in H^{m+2}(T^n) \times H^{m+2}(T^n)$ of (4.5). This implies existence of some $(u, v) \in H^{m+1}(T^n)^2$ so that $u_n \to u$ and $v_n \to v$ in $H^{m+1}(T^n)$ along a subsequence. passing to the limit in (4.5) along this subsequence, we see that (u, v) is a solution of (4.5) corresponding to $t = t_0$. Applying (4.6) we conclude that $(u, v) \in H^{m+2}(T^n) \times H^{m+2}(T^n)$. We see that (4.5) is solvable for all t in $[0, 1]$. Denote by (u^σ, v^σ) its solution corresponding to $t = 1$.

Now let $\sigma \to 0$ along some sequence. In view of the estimate (4.7), there exists some $(u, v) \in H^{m-1}(T^n) \times H^{m-1}(T^n)$ so that $u^\sigma \to u$ and $v^\sigma \to v$ in $H^{m-1}(T^n)$ along a subsequence. Passing to the limit in (4.5) along this subsequence, we see that (u, v) is a solution of (4.1). Using (4.7) again, we conclude that $(u, v) \in H^m(T^n) \times H^m(T^n)$.

Next we state the main existence result of this section. Its proof is similar to that of Theorem 3.1, and is therefore omitted.

THEOREM 4.3. *On the torus* T^n, *consider the system*

$$\begin{aligned} u &= f_1(x, u, v, Du, Dv, D^2u, D^2v), \\ v &= f_2(x, u, v, Du, Dv, D^2u, D^2v). \end{aligned} \tag{4.8}$$

Assume that $f_p = f_p^1(x, u, v, Du, Dv, D^2u, D^2v) + \varepsilon f_p^2(x, u, v, Du, Dv, D^2u, D^2v)$, *with* $f_p^1(x, 0, \ldots, 0) = f_{pu}^1(x, 0, \ldots, 0) = f_{pu_i}^1(x, 0, \ldots, 0) = f_{pu_{kl}}^1(x, 0, \ldots, 0) \equiv 0$ *for* $p = 1, 2$ *and* $i, k, l = 1, \ldots, n$. *For* $x \in T^n$ *and all other variables sufficiently small in absolute values, assume that* $\dfrac{\partial f_1}{\partial v_i} = \dfrac{\partial f_2}{\partial u_i}$, $\dfrac{\partial f_1}{\partial v_{ij}} = \dfrac{\partial f_2}{\partial u_{ij}}$ *for all* i *and* j, *and that if we define*

$$a_{ij} = \frac{\partial f_1}{\partial u_{ij}}, \quad a_i = \frac{\partial f_1}{\partial u_i}, \quad a_0 = \frac{\partial f_1}{\partial u}, \quad b_{ij} = \frac{\partial f_1}{\partial v_{ij}}, \quad b_i = \frac{\partial f_1}{\partial v_i},$$

$$b_0 = \frac{\partial f_1}{\partial v}, \quad c_{ij} = \frac{\partial f_2}{\partial v_{ij}}, \quad c_i = \frac{\partial f_2}{\partial v_i}, \quad c_0 = \frac{\partial f_2}{\partial v}, \quad d_0 = \frac{\partial f_2}{\partial u},$$

then these functions satisfy conditions (i) *and* (ii) *of Lemma* 4.1. *Assume that* $f \in C^{m_0}$ *in all arguments with* $m_0 = 10[n/2] + 31$. *Then for* ε *sufficiently small the problem* (4.8) *has a* 2π *periodic in each* x_i *solution of class* $C^2(T^n) \times C^2(T^n)$.

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