A Computer-Assisted Study of Periodic Parabolic Systems

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ABSTRACT

Both theoretical and computational studies of long-term behavior for a general class of parabolic systems with asymptotically time-periodic coefficients are presented. The results are applied to the Lotka-Volterra systems, describing interaction of two competing species.

1. INTRODUCTION

We study asymptotic behavior of quasimonotone asymptotically time-periodic parabolic systems; that is, roughly speaking, parabolic systems with off-diagonal elements increasing. We prove some general results on existence and uniqueness of periodic solutions, which we then use to get bounds on the ω-limit sets. We apply our results to asymptotically time-periodic Lotka–Volterra competition systems with diffusion and Dirichlet boundary conditions, for which our conclusions complement the recent work of P. Hess and A. C. Lazer [5].

Most of our results were suggested by numerical computations. In addition, we present a number of computations in cases where known results do not apply, suggesting possibilities for development. In particular, all the results seem to extend numerically for time-periodic and nonhomogeneous boundary conditions and to the initial data not satisfying compatibility conditions. For Lotka–Volterra competition systems our computations show that in some cases a superior on the average species may become extinct.

Sections 4 and 5 are numerical in nature. Section 4 deals with approximation of the principal eigenvalue and eigenfunction for a periodic-parabolic operator. We show the problem can be reduced to finding the principal eigenpair of a positive definite matrix with all entries positive, for

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which we use the power method with surprisingly fast convergence. Section 5 contains some computations and remarks on P. de Mottoni and A. Schiaffino's [10] example of a periodic Lotka-Volterra system with multiple coexistence states.

2. PRELIMINARY RESULTS ON PERIODIC PARABOLIC EQUATIONS

Let Ω be a smooth domain in R^n. For $(x, t) \in \Omega \times R$ we denote

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{ij} + \sum_{i=1}^{n} a_i(x,t)u_i + a_0(x,t)u, \qquad (2.1)$$

with a_{ij}, a_i, $a_0 \in C^{\alpha,\alpha/2}(\overline{\Omega} \times R)$ for some $\alpha > 0$ and p-periodic in t, $p > 0$. Assume further that for all x and t: $a_{ij} = a_{ji}$, $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i \xi_j > \mu_0 |\xi|^2$ for all $\xi \in R^n$ and some $\mu_0 > 0$, and that $a_0(x, t) \leqslant 0$. We denote by $u(x, t; u_0)$ (and sometimes by $u(x, t; u_0, f)$) the solution of

$$u_t - Lu = f(x,t) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad t > 0 \qquad (2.2)$$

$$u(x,0) = u_0(x). \qquad (2.3)$$

The following result is proved in H. Amann [1].

LEMMA 2.1. *For any p-periodic in t function $f(x,t) \in C^{\alpha,\alpha/2}(\overline{\Omega} \times R)$ the problem* (2.2) *has a unique p-periodic solution, denoted u_f, and $u_f > 0$ if $f > 0$, $u_f \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times R)$.*

We add the following stability assertion. It could also be derived from the abstract results of P. Takač [11]; however, we present a direct proof for completeness.

LEMMA 2.2. *For any continuous data $u_0(x)$ such that the problem* (2.2–3) *is classically solvable for $t > 0$ (e.g., $u_0 \in C_0^2(\Omega)$, see* [1, *p.* 20]) *we have $u(x, t; u_0) \to u_f(x,t)$ as $t \to \infty$ uniformly in $\overline{\Omega}$.*

PROOF. Note that Lemma 2.1 implies that $u_{\lambda f} = \lambda u_f$ for any constant λ. If $|f(x,t)| \leqslant M$ for all x and t, we can choose $\lambda \geqslant 1$ such that $u_{-\lambda M} <$

$u_0(x) < u_{\lambda M}$ for all $x \in \Omega$ (Hopf's lemma implies that $\partial u_{\lambda M}/\partial n < 0$ on $\partial\Omega$, and recall also that $u_M > 0$). Set $v_0(x, t) = u_{\lambda M}$. Define $v_1(x, t)$ as the solution of

$$\begin{gathered} v_{1t} - Lv_1 = f(x,t) \quad \text{in } \Omega, \qquad v_1 = 0 \quad \text{on } \partial\Omega \\ v_1(x,0) = v_0(x, p)(= v_0(x,0)). \end{gathered} \tag{2.4}$$

Since $v_0(x, p) = u_{\lambda M}(x, p)$ is smooth in $\overline{\Omega}$, it satisfies Equation (2.1) on $\partial\Omega$, thus providing the compatibility condition of Theorem 7 in [4, pp. 65, 75]. Hence (2.4) is classically solvable and $v_1(x, t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times R)$. By the maximum principle it follows that

$$v_1(x,t) \leqslant v_0(x,t) \quad \text{for all } x \in \Omega, \qquad t > 0. \tag{2.5}$$

Next we define $v_2(x, t)$ as the solution of

$$v_{2t} - Lv_2 = f(x,t) \quad \text{in } \Omega, \qquad v_2 = 0 \quad \text{on } \partial\Omega, \qquad v_2(x,0) = v_1(x, p). \tag{2.6}$$

Its existence follows as shown previously. By (2.5) and the maximum principle it follows that $v_2(x, t) \leqslant v_1(x, t)$ for all $x \in \Omega$ and $t > 0$. Continuing this way, we define $v_n(x, t) = v(x, t; v_{n-1}(x, p))$, $n \geqslant 2$ and obtain a decreasing sequence of iterates. Similarly, starting with $u_0(x, t) = u_{-\lambda M}(x, t)$, we obtain an increasing sequence of iterates $u_n(x, t)$. Applying the maximum principle again, we see that $u_i \leqslant v_i$ for all i, and so for all x and t,

$$u_0 \leqslant u_1 \leqslant u_2 \leqslant \cdots \leqslant v_2 \leqslant v_1 \leqslant v_0.$$

Call $u(x, t) = \lim_{n\to\infty} u_n(x, t)$, $v(x, t) = \lim_{n\to\infty} v_n(x, t)$. Clearly, $u = v = 0$ on $\partial\Omega$. By the interior Schauder's estimates [4, p. 64], $u_n(x, t)$ are uniformly bounded in $C^{2+\alpha,1+\alpha/2}(K \times [t_1, t_2])$ for any $K \subset\subset \Omega$ and $0 < t_1 < t_2 < T$. Hence $u_n(x, t)$ converge to $u(x, t)$ in $C^{2,1}(K \times [t_1, t_2])$, i.e., $u(x, t)$ is a solution of (2.2).

Next we show that $u(x, t)$ is periodic in t. Write

$$\begin{aligned} |u(x,t+p) - u(x,t)| \leqslant{} & |u_{n-1}(x,t+p) - u_n(x,t)| \\ & + |u(x,t+p) - u_{n-1}(x,t+p)| \\ & + |u_n(x,t) - u(x,t)|. \end{aligned}$$

The first term on the right is zero by periodicity of Equation (2.2). The other two are less than any $\varepsilon > 0$ for every fixed x and t, provided n is large, which proves periodicity. By the uniqueness claim of Lemma 2.1, $u(x,t) = v(x,t) = u_f$.

The functions $u_n(x,t) \in C(\overline{\Omega} \times [0,\ p])$ converge monotonously to a continuous u_f. By Vitali's theorem the convergence is uniform.

Turning to the stability, notice that by the maximum principle,

$$u_1(x,t) \leqslant u(x,t;u_0) \leqslant v_1(x,t) \quad \text{for all } x \in \overline{\Omega} \quad \text{and} \quad t > 0.$$

For $np \leqslant t \leqslant (n+1)p$, we have $u_1(x,t) = u_{n+1}(x, t-np)$ and $v_1(x,t) = v_{n+1}(x, t-np)$, which are uniformly close.

Remark. In our numerical experiments with (2.1) in case $L = \Delta$ and $\Omega = (0, 2)$, we were obtaining good approximations of 1-periodic solutions by $t = 2$ for most of $f(x,t)$ and $u_0(x)$ that we tried. Moreover, we obtained similar results when $u_0(x) \neq 0$ on $\partial\Omega$, and for time-periodic Dirichlet data, which suggests that Lemmas 2.1 and 2.2 might admit considerable generalizations.

Lemma 2.3. *The periodic solution u_f depends continuously on $f(x,t)$, i.e., if $g(x,t) \in C^{\alpha,\alpha/2}(\overline{\Omega} \times R)$ is another p-periodic in t function, then for any $\varepsilon > 0$, $|u_f - u_g|_{C^0(\overline{\Omega}\times[0,p])} < \varepsilon$, provided $|f - g|_{C^0(\overline{\Omega}\times[0,p])} < \delta$, $\delta = \delta(\varepsilon)$. Moveover, u_f is monotone in f, i.e., if $g(x,t) \geqslant f(x,t)$ for all x and t, then $u_g \geqslant u_f$.*

Proof. Let $u(x,t)$ be solution of

$$u_t - Lu = g(x,t) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$
$$u(x,0) = u_f(x,0).$$

By the maximum principle $u_f(x,t) \leqslant u(x,t)$, and by Lemma 2.2, $u(x,t)$ tends uniformly to $u_g(x,t)$. Hence, $u_g \geqslant u_f$.

In view of the monotonicity, it suffices to prove the continuity claim for $g = f + \varepsilon$. If the claim was false one would find a sequence $\varepsilon_j \to 0$ such that for some $c_0 > 0$ and all j,

$$\left| u_f - f_{f+\varepsilon_j} \right|_{C^0(\overline{\Omega}\times[0,p])} \geqslant c_0. \tag{2.7}$$

The sequence $\{u_{\varepsilon_i}\}$ is uniformly bounded in $C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0, T])$ for any $T > 0$ (by the Schauder's estimates). Hence a subsequence will converge in $C^{2,1}(\overline{\Omega} \times [0, T])$ to a periodic solution of (2.2), which by uniqueness is u_f. But that contradicts (2.7).

LEMMA 2.4. *Let $a(t)$ and $b(t)$ be continuous p-periodic functions, and let $u(t)$ be a solution of*

$$\dot{u} + a(t)u = b(t). \tag{2.8}$$

Assume that either one of the following two conditions is satisfied,

(*i*) $A \equiv \int_0^p a(\tau)\,dt > 0$, *and* $b(t) > 0$ *for all* t;
(*ii*) $A < 0$, *and* $b(t) < 0$ *for all* t.

Then (2.8) *has a positive p-periodic solution (which in the first case attracts all other solutions as $t \to +\infty$, and in the second case repels).*

PROOF. Denote $h(t) = \exp(\int_0^t a(\tau)\,d\tau)$. Then by direct integration it follows that (2.8) has a p-periodic solution given by

$$u(t) = \frac{1}{h(t)}\left(c_0 + \int_0^t b(\tau)h(\tau)\,d\tau\right), \qquad c_0 = \frac{\int_0^p b(\tau)h(\tau)\,d\tau}{e^A - 1}.$$

In case (i) its positivity is obvious, and in case (ii)

$$c_0 + \int_0^t b(\tau)h(\tau)\,d\tau \geqslant c_0 + \int_0^p b(\tau)h(\tau)\,d\tau > 0.$$

Let λ_1 and $\varphi_1(x) > 0$ in Ω satisfy $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$.

THEOREM 2.1. *Consider the problem*

$$u_t - k(t)\Delta u = u(a(x,t) - b(x,t)u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.10}$$

with the p-periodic functions $k(t) \in C^{\alpha}(R)$, $a(x,t)$ and $b(x,t) \in C^{\alpha,\alpha/2}(\overline{\Omega} \times R)$, and $k(t) \geqslant k_0 > 0$, $b(x,t) \geqslant b_0 > 0$ for all x and t. Assume also

$u(x,0) \in C_0^2(\Omega)$. *We consider two possibilities.*

(*i*) *Assume there exists a p-periodic function* $a_0(t) \in C^\alpha(R)$, *such that*

$$a(x,t) \geqslant a_0(t) \quad \text{and} \quad \int_0^p a_0(\tau)\, d\tau > \lambda_1 \int_0^p k(\tau)\, d\tau. \tag{2.11}$$

Then problem (2.10) *has a unique positive p-periodic in t solution, to which all other non-negative solutions of* (2.10) *tend uniformly in x as* $t \to +\infty$. *Moreover, this solution is continuous and monotone increasing in* $a(x, t)$.

(*ii*) *If, on the other hand, there is a p-periodic in t function* $a_1(t) \in C^\alpha(R)$ *such that*

$$a(x,t) \leqslant a_1(t) \quad \text{with} \int_0^p a_1(\tau)\, d\tau < \lambda_1 \int_0^p k(\tau)\, d\tau,$$

then any solution of (2.10) *with non-negative data tends to zero uniformly in x as* $t \to +\infty$.

PROOF.

(i) To apply Theorem 3.1 we need super- and subsolutions. Large constants are clearly supersolutions. Let $v(t)$ be the positive p-periodic solution of

$$v'(t) + (\lambda_1 k(t) - a_0(t))v(t) = -1,$$

which exists by Lemma 2.4. Then it is easy to check that $\psi(x, t) = \varepsilon\varphi_1(x)v(t)$ is a positive subsolution of (2.10), provided ε is sufficiently small. By Theorem 3.1, Problem (2.10) has a positive p-periodic in t solution that attracts all strictly positive data (i.e., $u(x,0) \geqslant \delta\varphi_1(x)$ for some $\delta > 0$). If the data is assumed to be only non-negative, then the strong parabolic maximum principle implies that $u(x, \tau)$ is strictly positive for any $\tau > 0$, and the stability claim follows. The monotonicity and continuity claims are proved as in Lemma 2.3, using the usual monotone iterations.

(ii) Let $w(t)$ be a positive p-periodic solution of

$$w' + (\lambda_1 k(t) - a_1(t))w = 1.$$

It is easy to check that $\varphi_\lambda(x, t) = \lambda\varphi_1(x)w(t)$ is a supersolution of (2.10) for any $\lambda > 0$. Take a subsolution $\psi = 0$. It follows by Theorem 3.1 that (2.10) has a maximal non-negative p-periodic solution $\bar{u}$. By the strong maximum principle $\bar{u}$ is either strictly positive or zero. In the first case for λ small we would have a supersolution φ_λ below $\bar{u}$, which is impossible by Proposition 3.1. Hence $\bar{u} = 0$ and the claim follows.

3. QUASIMONOTONE PERIODIC PARABOLIC SYSTEMS

We study the periodic and asymptotically periodic systems of the form

$$u_t - L(x,t,D)u = f(x,t,u) \quad \text{in } \Omega \times R, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

Here $u(x,t) = (u^1, \dots, u^m)$, $L = (L^1, \dots, L^m)$ with each L^i of the form (2.1) and satisfying the same conditions as listed there. The vector-valued function $f = (f^1, \dots, f^m)$: $\Omega \times R^{m+1} \to R^m$ is assumed to be of the class $C^{\alpha,\alpha/2}(\bar{\Omega} \times R)$ for any u belonging to some bounded set $D \subset R^m$, and locally Lipschitz continuous in u uniformly with respect to $(x, t) \in \bar{\Omega} \times R$.

DEFINITION. We say that $f(x, t, u)$ is quasimonotone increasing in u on D if

$$\frac{\partial f^i}{\partial u^j} \geqslant 0 \quad \text{for all } i \neq j, \qquad x \in \Omega, \qquad t \in R, \qquad u \in D. \tag{3.2}$$

DEFINITION. A p-periodic in t vector-valued function $(\psi(x, t) = (\psi^1, \dots, \psi^m))$ is called a subsolution of (3.1) if

$$\psi_t - L(x,t,D)\psi \leqslant f(x,t,\psi) \quad \text{in } \Omega \times R, \qquad \psi \leqslant 0 \quad \text{on } \partial\Omega. \tag{3.3}$$

A supersolution $\varphi(x, t)$ is defined by reversing the preceding inequalities.

THEOREM 3.1. *Assume that L and f are p-periodic in t, and there is a pair of p-periodic in t super- and subsolutions with $\psi \leqslant \varphi$ on $\Omega \times R$. Assume f is quasimonotone increasing in u for $\psi \leqslant u \leqslant \varphi$. Then* (3.1) *has minimal and maximal p-periodic in t solutions $\underline{u}(x,t)$ and $\bar{u}(x,t)$, respectively. The ω-limit set for any regular solution with data $u(x,0) \in C_0^2(\Omega)$*

in the order interval $[\psi(x,0),\ \varphi(x,0)]$ *is contained in* $[\underline{u}(x,t),\ \bar{u}(x,t)]$ *(throughout the paper the ω-limit set is understood with respect to uniform convergence in x and t).*

PROOF. Existence of u and $\bar{u}$ is of course standard; however we review the construction briefly, since we need it to prove stability. We construct a sequence of approximations $\{u_n(x,t)\}$ defined as p-periodic solutions of

$$u^i_{nt} - Lu^i_n + Mu^i_n = Mu^i_{n-1} + f^i(x,t,u_{n-1}) \quad \text{in } \Omega,$$

$$u^i_n = 0 \quad \text{on} \, \partial\Omega, \qquad i=1,\dots,m, \tag{3.4}$$

where the constant $M > 0$ on the right is chosen so large that the right-hand side is an increasing function of u^i_{n-1}. The iterates starting with $u_0 = \psi(x,t)$ will converge increasing in n to $\underline{u}$ (and the ones starting with $v_0 = \varphi(x,t)$ will produce another sequence $v_n(x,t)$ converging to $\bar{u}$).

Turning to the stability, notice that by the maximum principle for weakly coupled parabolic systems $\psi(x,t) \leqslant u(x,t) \leqslant \varphi(x,t)$ for all $x \in \Omega$ and $t > 0$. Define $U_1(x,t) = (U_1^1, \dots, U_1^m)$ as the solution of

$$U^i_{1t} - LU^i_1 + MU^i_1 = M\psi^i + f(x,t,\psi) \quad \text{in } \Omega,$$

$$U^i_1 = 0 \quad \text{on } \partial\Omega, \qquad i=1,\dots,m, \qquad U^i_1(x,0) = u^i(x,0).$$

By the same maximum principle $u(x,t) \geqslant U_1(x,t)$ for all x and t. Since by Lemma 2.1, $U_1(x,t) \to u_1(x,t)$ [as defined in (3.4)] uniformly in x, it follows that for any $\varepsilon_1 > 0$ we can find T_1, so that

$$u(x,t) \geqslant u_1(x,t) - \varepsilon_1 \quad \text{for } t > T_1 \quad \text{and} \quad x \in \Omega.$$

By our assumptions on f it follows that for all x and $t > T_1$

$$Mu^i + f^i(x,t,u) \geqslant Mu^i_1 + f(x,t,u_1) - \delta_1, \quad \text{with } \delta_1 \to 0 \quad \text{when } \varepsilon_1 \to 0.$$

Next for $t \geqslant T_1$ we define $U_2(x,t)$ as the solution of

$$U^i_{2t} - LU^i_2 + MU^i_2 = Mu^i_1 + f(x,t,u_1) - \delta_1 \quad \text{in } \Omega, \qquad U^i_2 = 0 \quad \text{on } \partial\Omega,$$

$$U^i_2(x,t_1) = u(x,t_1).$$

By the maximum principle $u(x, t) \geqslant U_2(x, t)$ for $t \geqslant T_1$ and all $x \in \Omega$. By Lemmas 2.2 and 2.3, $U_2(x, t)$ tends to a p-periodic solution of the preceding equation, which is close to the p-periodic solution of the same equation with $\delta_1 = 0$, that is, $u_2(x, t)$. Hence for $t > T_2$ and all x,

$$u(x,t) \geqslant u_2(x,t) - \varepsilon_2, \quad \text{where } \varepsilon_2 \to 0 \quad \text{when } \delta_1 \to 0,$$

and so on.

The following lemma asserts that the bounds on the ω-limit set of (3.1) are stable under small perturbations.

LEMMA 3.1. *Let $v(x,t) = (v^1, \dots, v^m)$ be the solution of the system (3.1) with f replaced by another p-periodic function $g(x,t,u)$, satisfying the same conditions, and assume all other conditions of the Theorem 3.1 are satisfied with the same ψ and φ, and suppose finally that*

$$|f(x,t,u) - g(x,t,u)| < \delta \quad \text{for all } x \in \Omega, \qquad t \geqslant 0 \quad \text{and} \quad \psi \leqslant u \leqslant \varphi.$$

If $\underline{v}$ and $\bar{v}$ are, respectively, the minimal and the maximal p-periodic solutions of the new system, then for all x and t, $|\underline{u} - \underline{v}| + |\bar{u} - \bar{v}| < \varepsilon$, and $\varepsilon \to 0$ when $\delta \to 0$. Moreover, if $g \geqslant f$ then $\underline{u} \leqslant \underline{v}$ and $\bar{u} \leqslant \bar{v}$.

PROOF. Since $\bar{u}$, $\underline{u}$, $\bar{v}$ and $\underline{v}$ are uniform limits of solutions of linear parabolic problems, the proof easily follows using Lemma 2.3.

Next we consider asymptotically periodic systems. In the following f, $\underline{u}$, $\bar{u}$, ψ, and φ are defined as in Theorem 3.1.

THEOREM 3.2. *Consider the quasimonotone system ($v \in R^m$)*

$$v_t - Lv = g(x,t,v) \quad \text{in } \Omega,\ v = 0 \quad \text{on } \partial\Omega,$$

$$v(x,0) = v_0(x) \in W_0^{2,p}(\Omega). \tag{3.5}$$

Assume that for any $\varepsilon > 0$ one can find $T_1 > 0$ and two p-periodic functions $f_1(x,t,v)$ and $f_2(x,t,v)$ such that $f_1 \leqslant g \leqslant f_2$ and $|f_i - f| < \varepsilon$, $i = 1, 2$, for all $t > T_1$, $x \in \Omega$ and $\psi \leqslant u \leqslant \varphi$. (This condition implies that g tends to f uniformly in x and v.) Assume the functions f_1, f_2, and g are quasimonotone increasing and satisfy the same smoothness conditions as f in Theorem

3.1, *and that for each of these functions* φ *and* ψ *are super- and subsolutions. Assume finally that* $\psi(x, t) \leqslant v(x, t) \leqslant \varphi(x, t)$ *for all* $t > T_2$ *and* $x \in \Omega$. *Then the* ω*-limit set of the solution of* (3.5) *is contained in the order interval* $[\underline{u}, \bar{u}]$.

PROOF. By the maximum principle for $t > \max(T_1, T_2)$,

$$v(x,t;v_0,f_1) \leqslant v(x,t;v_0,g) \leqslant v(x,t;v_0,f_2).$$

Both $v(x, t; v_0, f_i)$, $i = 1, 2$, tend by Theorem 3.1 to the order intervals $[\underline{v}_{f_i}, \bar{v}_{f_i}]$, which by Lemma 3.1 are uniformly close to $[\underline{u}, \bar{u}]$.

EXAMPLE. In the conditions of Theorem 2.1 (i), let $d(x, t) \to a(x, t)$ as $t \to \infty$ uniformly in x. Then any non-negative solution of

$$z_t - k(t)\Delta z = z(d(x,t) - b(x,t)z) \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega$$

converges as $t \to \infty$ to the unique p-periodic solution of (2.10). (Indeed we can take $f_{1,2} = z(a \pm \varepsilon - bz)$, $0 < \varepsilon \leqslant \varepsilon_0$, $\varphi = M$, $\psi = \delta\varphi_1 v$ with $M > 0$ large and ε_0, $\delta > 0$ small.)

Next we shall give some simple conditions for uniqueness that are similar to the ones given in [9] for the autonomous systems; see also P. Takač [11].

DEFINITION. We say that a vector function $\mathbf{f}(x, t, u)$: $\Omega \times R \times R^m_+ \to R^m$ is sublinear in u if for any constant $0 < \gamma < 1$

$$\mathbf{f}(x,t,\gamma u) \geqslant \gamma \mathbf{f}(x,t,u) \quad \text{for } x \in \Omega, \qquad t > 0 \quad \text{and} \quad u \geqslant 0. \quad (3.6)$$

We say that $\mathbf{f}$ is strictly sublinear if in addition for any fixed $u > 0$, $0 < \gamma < 1$ and $1 \leqslant i \leqslant m$ the functions $\mathbf{f}^i(x, t, \gamma u)$ and $\gamma f^i(x, t, u)$ are not identically equal on $\Omega \times R$.

PROPOSITION 3.1. *If the problem* (3.1) *with* f *as in Theorem* 3.1 *and moreover strictly sublinear, has a positive p-periodic solution* u, *then it cannot have a positive p-periodic supersolution* $\varphi \leqslant u$, $\varphi \neq u$. (*In particular, this implies uniqueness of a positive p-periodic solution. The last conclusion* *also follows from* [11].)

PROOF. Assume such φ exists. Let $0 < \gamma < 1$ be the maximal number such that $\varphi \geqslant \gamma u$. Set $z = \varphi - \gamma u$. Notice that by strict sublinearity of **f**, no component of z can be identically zero. Then for $M > 0$ sufficiently large and any $1 \leqslant i \leqslant m$, it follows by (3.6),

$$z_t^i - Lz^i + Mz^i \geqslant M\varphi^i + f^i(x,t,\varphi) - M\gamma u^i - f^i(x,t,\gamma u) \geqslant 0,$$
$$z^i(x,0) \geqslant 0,$$

with both inequalities being strict on some open sets. By the maximum principle $z > 0$ on $\Omega \times R_+$, which contradicts the maximality of γ.

In the following we denote by $\mu_1(a)$ the principal eigenvalue of the periodic parabolic problem

$$u_t - Lu - a(x,t)u = \mu u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.7}$$

and by $\psi_1(a) > 0$ the corresponding p-periodic eigenfunction. Existence of such eigenpairs is known; see e.g., [5]. We abbreviate $\mu_1 = \mu_1(0)$, $\psi_1 = \psi_1(0)$.

PROPOSITION 3.2. *Consider Problem* (3.1) *with p-periodic L and* **f** *as in Theorem* 3.1. *Assume that for all* $1 \leqslant i \leqslant m$, *any* $z \in R_+^m$, $h \in R_+$ *we have*

$$f^i(x,t,z+he) \leqslant f^i(x,t,z) + \mu_1 h, \quad \text{with } e = (1,1,\ldots,1) \in R^m, \tag{3.8}$$

and both sides of (3.8) *are not identical. Then the conclusion of Proposition* 3.1 *holds.*

PROOF. This time we let $\gamma > 0$ to be the minimal number such that $\varphi \geqslant u - \gamma\psi_1$. By our assumptions, the last inequality is strict on some set in each component. Setting $z = \varphi - u + \gamma\psi_1$, we obtain for $M > 0$ large and any $1 \leqslant i \leqslant m$,

$$\begin{aligned} z_t^i - Lz^i + Mz^i &\geqslant M(\varphi^i + \gamma\psi_1) + f^i(x,t,\varphi) - Mu^i - f^i(x,t,u) \\ &\quad + \gamma\mu_1\psi_1 \geqslant M(\varphi^i + \gamma\psi_1) + f^i(x,t,\varphi + \gamma\psi_1) \\ &\quad - Mu^i - f^i(x,t,u) \geqslant 0, \end{aligned}$$

with the inequality strict on some open set. This contradicts the minimality of γ.

Next we study periodic competition systems of the type

$$u_t = k_1(t)\Delta u + u(a(x,t) - b(x,t)u - c(x,t)v) \text{ in } \Omega, \qquad u=0 \text{ on } \partial\Omega$$
$$v_t = k_2(t)\Delta v + v(d(x,t) - e(x,t)u - f(x,t)v) \text{ in } \Omega, \qquad v=0 \text{ on } \partial\Omega, \tag{3.9}$$

where $k_1(t)$ and $k_2(t)$ are positive p-periodic functions of class $C^{\alpha}(R)$ and $a,\ldots,f$ are either p-periodic or asymptotically p-periodic functions of class $C^{\alpha,\alpha/2}(\bar{\Omega} \times R)$ and furthermore b, c, e, and f are strictly positive.

The problems of this type as well as their autonomous versions have been studied by a number of people; see, e.g., [5, 7, 9] and the references therein. It turns out that the asymptotic behavior of (3.9) is governed largely by the stability of the semitrivial solutions $(u^*, 0)$ and $(0, v^*)$ where u^* is the solution of the first equation in (3.9) when v is set equal to zero, and v^* is defined similarly. P. Hess and A. C. Lazer [5] have studied abstract competition models and found that if both semitrivial solutions are unstable then "compression" occurs, i.e., a situation similar to our Theorem 3.1. Conditions for instability of semitrivial solutions can be easily given in terms of $\mu_1(a - cv^*)$ and $\mu_1(d - eu^*)$. More explicit sufficient conditions were given by P. Hess and A. C. Lazer in case of (3.9) with periodic coefficients and Neumann boundary conditions [5, p. 28]. Their result can be easily modified to cover the Robin boundary conditions, but not the Dirichlet case, since the positivity of ψ_1 on $\bar{\Omega}$ was used. The following result fills the gap and moreover allows asymptotically periodic coefficients. Define

$$\mathfrak{f}_M = \sup_{(-\infty,\infty)} \mathfrak{f}(t).$$

THEOREM 3.3. *For Problem* (3.9) *with periodic coefficients, assume there exist p-periodic functions* $a_1(t)$, $a_2(t) \in C^{\alpha}(R)$ *such that*

$$a(x,t) - c(x,t)\left(\frac{d}{f}\right)_M > a_1(t), \qquad d(x,t) - e\left(\frac{a}{b}\right)_M > a_2(t),$$

$$\int_0^p a_i\,dt > \lambda_1 \int_0^p k_i\,dt, \qquad i=1,2.$$

Then (3.9) *has positive p-periodic solutions* $(\underline{u}, \bar{v})$ *and* $(\bar{u}, \underline{v})$, *such that any other positive p-periodic solution of* (3.9) *satisfies* $\underline{u} \leqslant u \leqslant \bar{u}$, $\underline{v} \leqslant v \leqslant \bar{v}$, *and for any nonzero data* $u(x,0) \geqslant 0$ *and* $v(x,0) \geqslant 0$ *the* ω*-limit set of the corresponding solution of* (3.9) *is contained in the order rectangle* $Q \equiv [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$. *Moveover, if one perturbs the system* (3.9) *replacing* $a(x,t)$ *by* $a(x,t) + a_1(x,t), \ldots, f(x,t)$ *by* $f(x,t) + f_1(x,t)$ *with* $a_1, \ldots, f_1$ *tending to zero uniformly in* x *as* $t \to \infty$, *then the* ω*-limit set of any non-negative solution is still in* Q.

PROOF. By letting $\bar{v} = -v$ we transform (3.9) into a new system

$$\begin{aligned} u_t &= k_1 \Delta u + u(a - bu + c\bar{v}) \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega \\ \bar{v}_t &= k_2 \Delta \bar{v} + \bar{v}(d - eu + f\bar{v}) \quad \text{in } \Omega, \qquad \bar{v}=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.10}$$

which is quasimonotone increasing in $u > 0$ and $\bar{v} < 0$. Let u_1 be a positive p-periodic solution of

$$u_t = k_1(t)\Delta u + u\left(a - bu - c\left(\frac{d}{f}\right)_M - (c+1)\varepsilon_0\right) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

and similarly v_1 a positive p-periodic solution of

$$v_t = k_2(t)\Delta v + v\left(d - e\left(\frac{a}{b}\right)_M - fv - (e+1)\varepsilon_0\right) \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega,$$

whose existence for sufficiently small ε_0 follows by Theorem 2.1. Then $\varphi = ((a/b)_M + \varepsilon_0, -v_1)$ and $\psi = (u_1, -(d/f)_M - \varepsilon_0)$ are super- and subsolutions of (3.10), and Theorem 3.1 applies (the inequality $\psi < \varphi$ follows easily by the maximum principle). To apply Theorem 3.2, define $f_1 = f_-$, $f_2 = f_+$ and

$$f_\pm = \begin{pmatrix} u(a \pm \varepsilon - bu + c\bar{v}) \\ \bar{v}(d \mp \varepsilon - eu + f\bar{v}) \end{pmatrix},$$

with $0 < \varepsilon \leqslant \varepsilon_1$. One checks that for ε_1 sufficiently small, $\varepsilon_1 = \varepsilon_1(\varepsilon_0)$, φ and ψ are super- and subsolutions for both f_1 and f_2. Using comparison arguments and Theorem 2.1, one shows that solutions of (3.10) lie eventually between ψ and φ, and the proof follows.

REMARKS.

1. On the basis of the second part of Theorem 2.1, one can easily write down conditions for stability of one or both semitrivial solutions, and then get conditions for extinction of one of the species.

2. Using Proposition 3.2, we can translate the uniqueness results of McKenna-Walter [9] to our situation. However, a fundamental problem remains open: does instability of semitrivial solutions imply uniqueness of a p-periodic solution?

3. The trick of converting (3.9) to a quasimonotone increasing form by letting $\bar{v} = -v$ is known, see [9]. In Korman-Leung [7] it was shown that a similar trick works for larger systems iff all variables can be divided into two groups, such that any two variables from different groups compete and any two variables from the same group cooperate. All our results extend to such systems.

For the cases when one or two of the semitrivial solutions are stable there are some general results in [5]. Here we illustrate these cases numerically. The phenomena we observed in our numerical examples are not for the most part covered by the results of [5].

(i) Both semitrivial solutions are stable. We computed solutions of the following system $[\Omega = (0, 2)]$:

$$\begin{aligned} u_t &= u_{xx} + u(a + p\sin 2\pi t - u - cv), \qquad u(0) = u(2) = 0 \\ v_t &= v_{xx} + v(d - eu - v), \qquad v(0) = v(2) = 0, \end{aligned} \tag{3.11}$$

with $a = 6$, $d = 5$, $c = e = 2$, $p = 3$. We verified the stability of semitrivial solutions computationally: by setting $c = 0$ in the first equation (then $u \to u^*$) and observing that the v component of the solution coverges to zero, we conclude stability of $(u^*, 0)$ (alternatively we could compute $\mu_1(d - eu^*)$ by using the results of the next section). Similarly we verified that $(0, v^*)$ is stable.

We started with $u(x, 0) = v(x, 0) = 1$. Not surprisingly, $v \to 0$, since u is a superior on the average species. Next we considered $a = 6$, $c = e = 2$, $d = 5$, $p = 3$, $u(x, 0) = 1$ but $v(x, 0) = 5$. This time $u \to 0$, i.e., the superior on the average species went extinct, as a result of being outnumbered initially. When $a = 6$, $c = e = 2$, $d = 5$, $u(x, 0) = v(x, 0) = 1$, but $p = -12$, then again the superior species u went extinct as a result of initially declining carrying capacity. Finally we tried $a = 6$, $d = 5$, $p = 3$, $u(x, 0) = v(x, 0) = 1$, $e = 2$ but $c = 2 + 2/\ln(2 + t)$. Here again the superior in the long run species u went extinct because of the stiff initial competition.

(ii) One semitrivial solution is stable, another unstable. We consider again (3.11) with $a = 6$, $d = 5$, $c = 0.5$, $e = 1$, and $p = 3$. Similarly to the preceding we had verified computationally that $(u^*, 0)$ is stable and $(0, v^*)$ is unstable. We had computed solutions for various choices of $u(x,0)$ and $v(x,0)$, and in all cases we had $v(x, t) \to 0$, even for $u(x,0)$ "small" and $v(x,0)$ "large." These computations suggest that either (3.11) has no positive p-periodic solution, or else such a solution has a small domain of attraction.

4. COMPUTATION OF PRINCIPAL EIGENPAIRS FOR A CLASS OF PERIODIC PARABOLIC PROBLEMS

We are interested in computing the principal eigenvalue and eigenfunction of the following problem: the function $u(x, t)$ is a nontrivial p-periodic in t solution of

$$u_t - k(t)\Delta u - a(x,t)u = \mu u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

with p-periodic in t coefficients $k(t) \in C^{\alpha}(R)$ and $a(x,t) \in C^{\alpha,\alpha/2}(\overline{\Omega} \times R)$, $k > 0$. It is known (see Lazer [8]) that the smallest eigenvalue μ_1 of (4.1) is simple and the corresponding eigenfunction ψ_1 can be taken positive. We had mentioned in the previous sections the importance of μ_1 for bifurcation of positive solutions.

We replace the domain Ω by the uniform square mesh Ω_h of step size h, denoting $k = (k_1, \dots, k_n)$, $x_k = (k_1 h, \dots, k_n h)$ and $u_k = u(x_k)$. We replace the Laplacian by its finite difference version

$$\sum_{i=1}^{n} \frac{u_{k+e_i} - 2u_k + u_{k-e_i}}{h^2},$$

and then we number the grid points in an arbitrary way $x^1, \dots, x^N$, which produces the ordering of corresponding values of $u(x)$: $u_1, \dots, u_N$, $u_i = u(x^i)$. The problem (4.1) is then replaced by the following: find the nontrivial p-periodic vector function $\mathbf{u}(t) = (u_1(t), \dots, u_N(t))$, such that

$$\dot{\mathbf{u}}(t) - A(t)u(t) = \mu u(t), \tag{4.2}$$

with symmetric matrix $A(t)$, such that all of its off-diagonal elements are either strictly positive or identically zero. We assume the ordering of the grid points is such that $(A + \gamma I)^n$ has all its entries positive for large enough $\gamma > 0$ and n (e.g., order A by rows).

Proposition 4.1. *The eigenvalues of* (4.2) *are* $\mu_i = -1/p \ln \rho_i$, *where* ρ_i *are the Floquet multipliers of* $A(t)$.

Proof. Setting $\mathbf{v}(t) = e^{-\mu t}\mathbf{u}(t)$, we rewrite (4.2) as

$$\dot{\mathbf{v}}(t) - A(t)\mathbf{v} = 0. \tag{4.3}$$

According to the Floquet theory, the solution of (4.3) can be written as $\mathbf{v}(t) = F(t)e^{\Lambda t}\mathbf{v}(0)$, where $F(t)$ is a p-periodic matrix with $F(0) = I$ – identity matrix, and Λ is a constant matrix. Then the solution of (4.2) with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ is

$$\mathbf{u}(t) = F(t)e^{(\Lambda + \mu I)t}\boldsymbol{u}_0. \tag{4.4}$$

For $\mathbf{u}(t)$ to be p-periodic, the matrix $\Lambda + \mu I$ must have a zero eigenvalue and $\boldsymbol{u}_0$ be its eigenvector. Recalling that $\Lambda = 1/p \ln X(p)$, where $X(t)$ is the fundamental solution matrix of (4.3), the proof follows.

Proposition 4.2. *Let* $X(p)$ *be the Floquet matrix of the matrix* $A(t)$ *as previously described. Then* $X(p)$ *is positive definite, and all its entries are positive.*

Proof. The fundamental solution matrix $X(t)$ satisfies

$$\dot{X} = A(t)X, \qquad X(0) = I. \tag{4.5}$$

Consider an $n \times n$ matrix $Z(t) = e^{\gamma t}X(t)$. Then from (4.5),

$$\dot{Z} = (A(t) + \gamma I)Z, \qquad Z(0) = I. \tag{4.6}$$

Fix γ large so that all diagonal elements of $A + \gamma I$ are positive. On finite time intervals, solution of (4.6) can be obtained as a uniform limit of the sequence of iterates defined as follows: $Z_0 = I$ and

$$Z_n = I + \int_0^t (A(s) + \gamma I)Z_{n-1}(s)\,ds, \qquad n > 1; \tag{4.7}$$

See [2, p. 168]. Iterating in (4.7) with $\bar{A} = A + \gamma I$,

$$Z_n = I + \int_0^t \bar{A}(s)\,ds + \cdots + \frac{1}{n!}\left(\int_0^t \bar{A}(s)\,ds\right)^n. \tag{4.8}$$

It is clear that for sufficiently large n the matrix $(\int_0^t \bar{A}(s)\,ds)^n$ has all entries positive, from which it follows that the same is true for $X(t)$. From (4.8) we also conclude that $X(t)$ is symmetric. By Liouville's formula det $X(t) > 0$ for all $t \geqslant 0$, so that the eigenvalues of $X(t)$ are nonzero for all t. Since the eigenvalues vary continuously with t, positive definiteness of $X(t)$ follows.

From the preceding propositions and the Perron-Frobenius theorem it follows that $X(p)$ has eigenvalues $0 < \rho_1 \leqslant \rho_2 \leqslant \cdots < \rho_n$ with ρ_n simple corresponding to a positive eigenvector ξ, and then Problem (4.2) has eigenvalues $\mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_n$ with μ_1 simple and corresponding eigenvector $\mathbf{u}_1 = \xi > 0$. (The principal eigenfunction $\psi_1(x_i, t)$ is then obtained by solving (4.2) with $\mu = \mu_1$ and $u(0) = \xi$).

To calculate ρ_n and ξ we used the power method. Starting with an initial guess ξ_0, $\|\xi_0\| = 1$, we compute $\xi_k = X(p)\xi_{k-1}/\|X(p)\xi_{k-1}\|$, and $\lambda_k = (X(p)\xi_{k-1}, \xi_{k-1})$. Since ρ_n is simple it follows that $\lambda_k \to \rho_n$ and $\xi_k \to \xi$.

Example. Let $\Omega = (0, 2)$, $p = 1$ and consider the eigenvalue problem

$$u_t - u_{xx} - c(x+1)(1 + \sin 2\pi t)u = \mu u \quad \text{in } \Omega, \qquad u(0) = u(2) = 0, \tag{4.9}$$

with $c = 4$ and $2\pi \simeq 6.283$. We subdivided the interval (0, 2) into 20 pieces, obtaining a 19×19 system, which was then solved using Euler's method with time step 1/2000, obtaining $X(1)$. We then applied the power method starting with $\xi_0 = 1/\sqrt{19}\ (1, 1, \ldots, 1)$. To our great surprise the power method converged in just one step, giving $\mu_1 = -1.888$ and the corresponding ξ. (The computation took only several minutes using VAX/PASCAL.) To test our program, we set $c = 0$ in (4.9), obtaining $\mu_1 = 2.464 \simeq \pi^2/4$ as expected ($\psi_1 = \sin \pi x/2\ x$).

Remark. The power method converged in just one step in all our experiments with constant $k(t)$. To understand the reason behind this, let us examine the Floquet matrix for the matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},$$

corresponding to two-point discretization of the Laplacian. It is clear that $X(1) = e^A$ has $\xi_0 = 1/\sqrt{2}(1, 1)^T$ as an eigenvector.

5. A PERIODIC COMPETITION MODEL WITH TWO POSITIVE SOLUTIONS

In [10] P. de Mottoni and A. Schiaffino have presented the following example of a periodic competition system of ODE's

$$\begin{aligned} \dot{x} &= x(a(t) - b(t)x - c(t)y) \\ \dot{y} &= y(d(t) - e(t)x - f(t)y), \end{aligned} \tag{5.1}$$

with $a = b = 1 + \varepsilon + \varepsilon \sin t$, $c = a(t)/(1 + \varepsilon \sin t)$, $d = f = a + \varepsilon \cos t (1 + \varepsilon \sin t)^{-1}$, $e = 2a - d(1 + \varepsilon \sin t)$, and $\varepsilon \in (0, 1)$. The system (5.1) has an explicit positive 2π-periodic solution $x_0 = \frac{1}{2}$, $y_0 = \frac{1}{2}(1 + \varepsilon \sin t)$ and semitrivial solutions: $(1, 0)$, which is unstable, and $(0, 1)$ which is stable. For $\varepsilon = 0.66$ Mottoni and Schiaffino report on calculations suggesting asymptotic stability of (x_0, y_0) and existence on another positive 2π-periodic solution. In this short section we pursue this example further. As was pointed out in [10] its importance lies in the fact that the corresponding averaged system has no positive solutions, which means that the principle of competitive exclusion may be violated in the periodic case.

It follows from the results of [10] that the period map T of the system (5.1) has an invariant curve Γ, which is a graph of continuous nonincreasing function joining $(0, 1)$ and $(1, 0)$, and that all fixed points of T lie on Γ. Hence stability of (x_0, y_0) will imply existence of another positive fixed point of T, i.e., another positive 2π-periodic solution of (5.1). In other words, if λ_1 and λ_2 are eigenvalues of the Floquet matrix, we need to show that $|\lambda_1|$, $|\lambda_2| < 1$.

We had linearized the system (5.1) near (x_0, y_0) with $\varepsilon = 0.33$ and computed the fundamental solution matrix $X(2\pi)$ using Euler's method, subdividing the interval [0, 6.283153] into 32000 equal pieces, obtaining

$$X(2\pi) = \begin{pmatrix} 0.423339 & -0.423126 \\ -0.419396 & 0.419653 \end{pmatrix},$$

$$\lambda_1 = 0.842757, \qquad \lambda_2 = 0.000235.$$

We performed standard error analysis, taking into account the following: Euler's method errors, round-off errors, errors in function evaluations, and the error in representation of 2π. Our results indicate that the error in the eigenvalues is considerably smaller than 0.15, which is small enough to justify stability of (x_0, y_0). (We also computed with $\varepsilon = 0$, both to test the program and to get an empirical error estimate, which turned out to be less than 0.001 in each entry of $X(2\pi)$). For $\varepsilon = 0.66$ we obtained $\lambda_1 = 0.50441$ and $\lambda_2 = 0.00003$ and the same conclusion.

Finally, we remark that although it is very plausible that (5.1) has only two positive p-periodic solutions, this remains to be proved.

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