do not differ much from one another, relatively speaking. Curve fitting, using Nicolas's values of $f(n)$ from $n = 4$ to $n = 95$, yields

(10)
$$a'_0 = -0.179\,690, \quad a'_1 = 0.427\,217, \quad a'_2 = -8.735\,45, \quad a'_3 = 9.065\,95,$$
$$b'_1 = -0.055\,965\,2, \quad b'_2 = 5.176\,99, \quad b'_3 = 5.370\,44, \quad b'_4 = -0.479\,231.$$

Again, similar tests were carried out with the same degree of success. Curve fitting using all but the last of Nicolas's results gives, besides an unmodified a'_0, coefficients differing from those in (10) in the fourth or third decimal, and by extrapolation $(f(n) - \ln n)_{n=95} = -0.176\,191\,961\,7$ compared to $-0.176\,191\,988\,2$ according to Nicolas. In all tests, the values of a_0 and a'_0 have appeared to be the least sensitive for modification, which is important within the present context.

This enables us to conclude that curve fitting, using all known $(f(n) - \ln n)$ values, by means of the asymptotic approximation (9) leads to

$$\limsup_{n \to +\infty} \{f(n) - \ln n\} = -0.179\,690\ldots \tag{11}$$

with reliable precision. This answers part (a) of Erdős's problem.

Note that Nicolas started his partial solution by proving that

$$\gamma - 1 < f(n) - \ln n < \gamma, \qquad n \geq 2$$

and that our result (11) is in agreement with this enclosure of the values of $f(n) - \ln n$, our value in (11) lying closer to $\gamma - 1$ than to γ as $f(95) - \ln 95 = -0.176\ldots$ made it possible to expect. This result also proves that the answer to part (b) is not equidistance of the grid points for the absolute maximum of (1). Qualitatively, the $(x_{k+1} - x_k)$-intervals are somewhere minimal and increase from there onward in both directions of the x-axis.

REFERENCE

[1] J.-L. NICOLAS, *Partial solution of Problem* 92-9* *(An extremal problem)*, SIAM Rev., 36 (1994), pp. 493–495.

Periodic Solutions of a Differential Equation

Problem 95-9, *by* PHILIP KORMAN (University of Cincinnati).

Consider the equation

$$\dot{x} = x(x - a(t))(b(t) - x) \tag{1}$$

with continuous positive p-periodic functions $a(t)$ and $b(t)$ such that

$$\max_t a(t) < \min_t b(t).$$

Show that (1) has exactly two positive p-periodic solutions.

Addendum to the solution by HONGWEI CHEN (Christopher Newport University).

David Ross recently pointed out that there was a gap in the proof of the exact multiplicity result in [1]. The argument can be modified as follows. First, we establish an exact multiplicity result for the general equation

$$x' = f(t, x) \quad \text{with} \quad f(t + p, x) = f(t, x). \tag{2}$$

If $f_{xx} > 0$ for all t and all $x \in I$ (where I is any subinterval of the real axis), then (2) has at most two p-periodic solutions with values in I. In fact, let

$$0 < x_1(t) < x_2(t) < x_3(t)$$

be three positive p-periodic solutions of (2). Define

$$y_1 = x_2(t) - x_1(t), \qquad y_2 = x_3(t) - x_2(t).$$

Then y_1 and y_2 are positive p-periodic functions and satisfy

$$y_1' = c_1(t)y_1 \tag{3}$$

and

$$y_2' = c_2(t)y_2, \tag{4}$$

respectively, where

$$c_1(t) = \int_0^1 f_x(t, \tau x_2 + (1-\tau)x_1)\, d\tau,$$
$$c_2(t) = \int_0^1 f_x(t, \tau x_3 + (1-\tau)x_2)\, d\tau.$$

Equations (3) and (4) give

$$\int_0^p c_i(t)\, dt = \int_0^p y_i'/y_i\, dt = 0$$

for $i = 1, 2$. This contradicts the fact that $c_2 > c_1$.

The exact multiplicity result of (1) is concluded as follows. Letting $z = 1/x$ in (1) gives

$$z' = \frac{1}{z} + abz - (a+b). \tag{5}$$

Denoting the right-hand side of (5) by $f(t, z)$,

$$f_{zz} = \frac{2}{z^3} > 0.$$

Therefore, (5) has at most two positive p-periodic solutions, as does (1).

REFERENCE

[1] H. CHEN, *Solution of Problem* 95-9 *(Periodic solution of a differential equation)*, SIAM Rev., 38 (1996), pp. 326–328.

An Integral from Electron Gas Theory

Problem 96-6 *by* M. L. GLASSER (Clarkson University).

The following integral arose in calculating the exchange-correlation energy of an electron gas in a strong magnetic field [1, p. 99]:

$$f(z) = \int_0^\infty \exp(-x^2/z^2) \tan^{-1}(1/x)\, dx.$$

Find computationally effective expansions for $f(z)$.