Global solution branches and exact multiplicity of solutions for two point boundary value problems

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Dedicated to the memory of my father Lev Korman

1 Introduction

We consider solutions of the two point boundary value problems

$$(1.1) \qquad u''(x) + \lambda f(x, u(x)) = 0, \;\; \text{for } a < x < b, \;\; u(a) = u(b) = 0,$$

depending on a parameter λ. We wish to know how many *exactly* solutions does the problem (1.1) have, and how these solutions change with λ. What is the role of the parameter λ? Of course, it could be absorbed into the nonlinearity f. However, as is often the case, it is helpful to have something "extra" in the statement of the problem. Consider for example the problem

$$(1.2) \qquad u''(x) + 4e^{\frac{5u(x)}{5+u(x)}} = 0, \;\; \text{for } 0 < x < 1, \;\; u(0) = u(1) = 0.$$

Problems of this type come up in combustion theory, referred to as "perturbed Gelfand problem", see e.g. J. Bebernes and D. Eberly [7]. It will follow from a result we present below that this problem has *exactly* three positive solutions. It appears next to impossible to establish this result directly. We introduce a parameter λ, and consider

$$(1.3) \qquad u''(x) + \lambda e^{\frac{5u(x)}{5+u(x)}} = 0, \;\; \text{for } 0 < x < 1, \;\; u(0) = u(1) = 0.$$

We now study *curves of solutions*, $u = u(x, \lambda)$. The advantage of this approach is that some parts of the solution curve are easy to understand, and it also becomes clear what are the tougher parts of the solution curve that we need to study - the *turning points*. For example, it is easy to understand the “small” solutions of (1.3), by applying the implicit function theorem (in Banach spaces) in the neighborhood of the trivial solution $\lambda = 0$, $u = 0$. We then continue this curve of solutions for increasing $\lambda > 0$ until a critical solution is reached, i.e. the implicit function theorem is no longer applicable. We show that at the critical solution the Crandall-Rabinowitz Theorem 1.2 (see below) applies. It implies that either the solution curve continues forward in λ through the critical solution, or it just bends back (no secondary bifurcations or other excentric behaviour is possible). We then show that the global solution curve makes exactly two turns, and the value of $\lambda = 4$ from (1.2) lies between the turns, thus establishing the existence of three solutions. The bifurcation approach, just described, has been developed in the recent years by Y. Li, T. Ouyang, J. Shi and the present author. It applies also to the semilinear elliptic problems for balls in R^n, however in the present paper we restrict to the ODE case (1.1).

The most detailed results are obtained when one considers positive solutions of autonomous problems, i.e. when $f = f(u)$. Since in that case both the length and the position of the interval (a, b) are irrelevant, and since positive solutions are symmetric with respect to the midpoint of the interval, it is convenient to pose the problem on the interval $(-1, 1)$, i.e. we consider

$$(1.4) \qquad u''(x) + \lambda f(u(x)) = 0, \ \text{ for } -1 < x < 1, \ \ u(-1) = u(1) = 0.$$

It turns out that convexity properties of $f(u)$ are important for determining the direction of the turn for solution curves. Accordingly, in the simplest case $f''(u) > 0$ and $f(u) > 0$, for $u > 0$, we can give an exhaustive analysis of the problem. (In case $f''(u) < 0$ and $f(u) > 0$, for $u > 0$, it is easy to prove uniqueness of solutions.) The next case in order of complexity is when $f(u)$ changes concavity exactly once. The prominent case is when $f(u)$ is modelled on a cubic with simple roots:

$$(1.5) \quad u'' + \lambda(u-a)(u-b)(c-u) = 0, \ \text{ for } -1 < x < 1, \ \ u(-1) = u(1) = 0.$$

We assume that $0 \le a < b < c$, since the analysis is easier if some root(s) is negative. We wish to describe how many *exactly* positive solutions does the problem (1.5) have for various λ.

This problem was studied in a 1981 paper by J. Smoller and A. Wasserman [58]. They succeeded in solving the problem for $a = 0$, while their proof for $a > 0$ case contained an error. This error was discovered by S.-H.

Wang [60], who was able to solve the problem under some restriction on $a > 0$. Both papers used the phase-plane analysis. P. Korman, Y. Li and T. Ouyang [30] used bifurcation theory to attack the problem, but again some restrictions were necessary (all of the above mentioned papers covered more general $f(u)$, behaving like cubic). Very recently, P. Korman, Y. Li and T. Ouyang [33], building on their previous work, have given a computer assisted proof for general cubic. It turns out that the set of all positive solutions consists of two curves, with the lower curve monotone in λ, and the upper curve having exactly one turn. The computations in P. Korman, Y. Li and T. Ouyang [30] also showed that the approach in J. Smoller and A. Wasserman [58] could not possibly cover the general cubic. (That approach required some integral to be positive, in order to derive a differential inequality for a time map. However, that integral changes sign for some cubics.) In the next section we state the optimal result, and describe the approach taken in [30] and [33].

Another prominent class of problems where $f(u)$ changes concavity exactly once is

$$u'' + \lambda e^{\frac{au}{u+a}} = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0, \tag{1.6}$$

from combustion theory. Here a is a second parameter. In case $a = 5$, we have the problem (1.3), discussed above. If $a \leq 4$, the problem is easy. In that case the solution curve is monotone, and it continues for all $\lambda > 0$ without any turns. Following some earlier results of K.J. Brown, M.M.A. Ibrahim and R. Shivaji [9] and others (see [61] and [29] for the earlier references), S.-H. Wang [61] has proved existence of a constant a_0, so that for $a > a_0$ the solution curve of (2.21) is exactly S-shaped, i.e. it starts at $\lambda = 0$, $u = 0$, it makes exactly two turns, and then it continues for all $\lambda > 0$ without any more turns. S.-H. Wang [61] gave an approximation of the constant $a_0 \simeq 4.4967$. That paper, as well as all previous ones, used a time map approach. P. Korman and Y. Li [29] have applied the bifurcation approach to the problem. Since bifurcation approach is more general, this opened a way to do other problems. In fact, Y. Du and Y. Lou [14] have used a similar approach, with several additional tricks of their own, to prove that for a ball in two dimensions a similar result holds for sufficiently large a.

P. Korman and Y. Li [29] had also improved the value of the constant to $a_0 \simeq 4.35$, i.e. for $a > a_0$ the solution sets are S-shaped curves. But what about $4 < a < a_0$? S.-H. Wang [61] has conjectured existence of a critical number $\bar{a}$, so that for $a \leq \bar{a}$ the solution curve is monotone, while for $a > \bar{a}$

the solution curve is exactly S-shaped (the number a_0, mentioned above, is just an upper bound for $\bar{a}$). Recently, P. Korman, Y. Li and T. Ouyang [33] has given a computer assisted proof of the S.-H. Wang's conjecture. Numerical calculations show that $\bar{a} \simeq 4.07$.

Other topics we discuss using the bifurcation approach involve pitchfork bifurcation and symmetry breaking, sign changing solutions, and the Neumann problem. We also present a recent formula from P. Korman, Y. Li and T. Ouyang [33], which allows one to compute all possible values of $\alpha = u(0)$, at which solution of (1.4), with the maximal value equal to α, is singular.

The case when $f = f(x,u)$ is much harder than the autonomous case. In particular, the time map method does not apply. Bifurcation approach works, but it becomes much more complicated. For example, solutions of the corresponding linearized problem need not be of one sign (an implicit example of that is provided by the Theorem 1.10 in W.-M. Ni and R.D. Nussbaum [46]). In the papers P. Korman and T. Ouyang a class of $f(x,u)$ has been identified, for which the theory of positive solutions is very similar to that for the autonomous case, see e.g. [34], [35] and [36]. Further results in this direction have been given in P. Korman, Y. Li and T. Ouyang [30], and P. Korman and J. Shi [40]. Namely, assume that $f \in C^2$ satisfies

$$f(-x,u) = f(x,u) \quad \text{for all } -1 < x < 1, \text{ and } u > 0,$$

$$f_x(x,u) \le 0 \quad \text{for all } 0 < x < 1, \text{ and } u > 0.$$

Under the above conditions any positive solution of (1.1) is an even function, with $u'(x) < 0$ for all $x \in (0,1]$, see B. Gidas, W.-M. Ni and L. Nirenberg [15]. We show that any solution of the corresponding linearized problem is of one sign, and then outline a number of exact multiplicity results.

Without symmetry assumption on $f(x,u)$ things are even more hard. In Section 4 we present extensions of the previous results in P. Korman and T. Ouyang [38]. The notion of Schwarzian derivative from Complex Analysis turns out to play a role here.

The bifurcation approach is effective for other problems, in addition to the two point problems that we discuss in the present paper. Most notably, similar results were developed for PDE's on a ball or annulus in R^n, see e.g. P. Korman, Y. Li and T. Ouyang [31] or T. Ouyang and J. Shi [50]. It was also used for systems of equations in P. Korman [22], for fourth order equations in P. Korman [26], and for periodic problems in P. Korman and T. Ouyang [37].

In Section 5 we give a brief review of time map method. Let $u = u(t)$ be solution of the initial value problem,

$$u'' + f(u) = 0, \quad u(0) = 0, \; u'(0) = p.$$

Using ballistic analogy, we can interpret this as "shooting" from the ground level, at an angle $p > 0$. Let $T/2$ denote the time it takes for the projectile to reach its maximum amplitude. By symmetry of positive solutions, $T = T(p)$ is then the time when the projectile falls back to the ground, the *time map*. The function $u(t)$ then satisfies the two point Dirichlet problem

$$u'' + f(u) = 0, \; \text{ for } 0 < t < T, \; u(0) = u(T) = 0,$$

which by rescaling is equivalent to (1.4). There are two completely different formulas for the same time map $T = T(p)$. The first one is obtained by direct integration, see e.g. W.S. Loud [43] for an early reference, while the second one was derived by R. Schaaf [53] through a change of variables, which converts the problem into a harmonic oscillator. Both formulas for the time map are nontrivial to use. The first one involves improper integrals, while the second one is highly implicit. ("Name your poison", so to say.) However, both approaches are well developed by now, see the book by R. Schaaf [53], and the papers of S.-H. Wang and his coworkers, of I. Addou, and many other papers, including J. Cheng [10], [11], and K.J. Brown et al [9]. We give an exposition of the second approach, and connect it to the notion of *generalized averages* from P. Korman and Y. Li [28].

In the final Section 6 we discuss numerical computation of solutions of (1.4). Again, the autonomous case is much easier. We describe two efficient ways to compute the solutions, and explain why finite differences (or finite elements) are not appropriate for autonomous problems.

The basic tool for continuation of solutions is the implicit function theorem in Banach spaces. We present it here in the formulation of M.G. Crandall and P.H. Rabinowitz [12], see also L. Nirenberg [45].

Theorem 1.1 *Let X, Λ and Z be Banach spaces, and $f(x, \lambda)$ a continuous mapping of an open set $U \subset X \times \Lambda \to Z$. Assume that f has a Frechet derivative with respect to x, $f_x(x, \lambda)$ which is continuous on U. Assume that*

$$f(x_0, \lambda_0) = 0 \quad \textit{for some } (x_0, \lambda_0) \in U.$$

If $f_x(x_0, \lambda_0)$ is an isomorphism (i.e. $1:1$ and onto) of X onto Z, then there is a ball $B_r(\lambda_0) = \{\lambda : ||\lambda - \lambda_0|| < r\}$ and a unique continuous map

$x(\lambda) : B_r(\lambda_0) \to X$, such that

$$f(x(\lambda), \lambda) \equiv 0, \quad x(\lambda_0) = x_0.$$

If f is of class C^p, so is $x(\lambda)$, $p \geq 1$.

In the conditions of the above theorem, we refer to (x_0, λ_0) as a regular solution, otherwise we call a solution singular. What happens at a singular solution? (I.e. when $f_x(x_0, \lambda_0)$ is not an isomorphism.) In general, practically anything imaginable may happen, as one can see even for functions of two variables. However, in a lucky case solution will continue through a critical point, either by making a simple turn there, or maybe it even continues forward in λ (the critical point is then like a point of inflection). M.G. Crandall and P.H. Rabinowitz [13] have given conditions for that to occur. The following result is one of our principal tools.

Theorem 1.2 *[13] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) =$ span x_0 be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span x_0 in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + s x_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

Except for a brief discussion on p-Laplace equations, we consider only the classical solutions throughout this paper. We shall denote the derivatives of $u(x)$ by either $u'(x)$ or u_x, and mix both notations sometimes to make our discussion more transparent.

Most of the results in the present paper are based on our joint papers with Y. Li, T. Ouyang and J. Shi. Working with these talented colleagues has been a wonderful experience for me, and I wish to thank them for this opportunity. I also wish to thank Professors A. Canada, P. Drabek and A. Fonda for inviting me to write this review paper.

2 Bifurcation theory approach

2.1 Some general properties of solutions of autonomous problems

We will consider positive, negative and sign-changing solutions of the Dirichlet problem (for $u = u(x)$)

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0, \tag{2.1}$$

depending on a parameter λ. We assume throughout this section that $f(u) \in C^2(\bar{R}_+)$. We choose to consider the problem on the interval $(-1,1)$ for convenience (which is related to the symmetry of solutions). By shifting and scaling, we can replace the interval $(-1,1)$ by any other interval (a,b).

Lemma 2.1 *Let $\xi \in (-1,1)$ be any critical point of $u(x)$, i.e. $u'(\xi) = 0$. Then $u(x)$ is symmetric with respect to ξ.*

Proof: Let $v(x) \equiv u(2\xi - x)$. Then $v(x)$ satisfies the same equation (2.1), and moreover $v(\xi) = u(\xi)$ and $v'(\xi) = u'(\xi) = 0$. By uniqueness of initial value problems, $u(x) \equiv v(x)$, and the proof follows. ♢

Lemma 2.2 *Solution of (2.1) cannot have points of positive minimum, and of negative maximum.*

Proof: Let us rule out the case of positive minimums, with the other case being similar. Assume on the contrary that there are points of positive minimums, and let ξ be the largest such point. Since $u(\xi) > 0$ and $u(1) = 0$, we can find a point $\eta \in (\xi, 1)$, so that $u(\xi) = u(\eta)$. Observe that $u'(\eta) < 0$. Indeed, if we had $u'(\eta) = 0$, then by the preceding lemma, η would have to be a point of minimum, contradicting the maximality of ξ. We know that the energy $E(x)) = \frac{1}{2}u'(x)^2 + \lambda F(u(x))$ is constant, but by above, $E(\eta) > E(\xi)$, a contradiction. ♢

We now consider positive solutions of (2.1). It follows from the lemmas above, that any positive solution is an even function, with $u'(x) > 0$ on $(-1,0)$, and $u'(x) < 0$ on $(0,1)$. (Of course, by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [15] this result holds for balls in R^n for any $n \geq 1$). Hence $\alpha \equiv u(0)$ is the maximal value of solution. We show next that it is impossible for two solutions of (2.1) to share the same α.

Lemma 2.3 *The value of $u(0) = \alpha$ uniquely identifies the solution pair $(\lambda, u(x))$ (i.e. there is at most one λ, with at most one solution $u(x)$, so that $u(0) = \alpha$).*

Proof: Assume on the contrary that we have two solution pairs $(\lambda, u(x))$ and $(\mu, v(x))$, with $u(0) = v(0) = \alpha$. Clearly, $\lambda \neq \mu$, since otherwise we have a contradiction with uniqueness of initial value problems. (Recall that $u'(0) = v'(0) = 0$.) Then $u(\frac{1}{\sqrt{\lambda}}x)$ and $v(\frac{1}{\sqrt{\mu}}x)$ are both solutions of the same initial value problem

$$u'' + f(u) = 0, \quad u(0) = \alpha, \;\; u'(0) = 0,$$

and hence $u(\frac{1}{\sqrt{\lambda}}x) = v(\frac{1}{\sqrt{\mu}}x)$, but that is impossible, since the first function vanishes at $x = \sqrt{\lambda}$, while the second one at $x = \sqrt{\mu}$. ♢

Bifurcation theory approach revolves around the study of the linearized equation for (2.1)

$$w'' + \lambda f'(u(x))w = 0 \quad \text{for } -1 < x < 1, \;\; w(-1) = w(1) = 0, \tag{2.2}$$

where $u(x)$ is a solution of (2.1). If this problem has a nontrivial solution, we call $u(x)$ a singular solution of (2.1). We say that the solution $u(x)$ is non-singular, if $w(x) \equiv 0$ is the only solution of (2.2). The following lemma is easy to prove in the autonomous case.

Lemma 2.4 *Let $u(x)$ be a positive solution of (2.1), with*

$$u'(1) < 0. \tag{2.3}$$

If the problem (2.2) admits a nontrivial solution, then it does not change sign, i.e. we may assume that $w(x) > 0$ on $(-1, 1)$.

Proof: The function $u'(x)$ also satisfies the linear equation in (2.2). By the condition (2.3), $u'(x)$ is not a multiple of $w(x)$. Hence its roots are interlaced with those of $w(x)$. If $w(x)$ had a root ξ inside say $(-1, 0)$, then $u'(x)$ would have to vanish on $(-1, \xi)$, which is impossible by the remarks following Lemma 2.2. ♢

The condition (2.3) will hold for any positive solution, provided that

$$f(0) \geq 0, \tag{2.4}$$

see e.g. p. 107 in M. Renardi and R.C. Rogers [52]. If $f(0) < 0$ it is possible to have $u'(1) = 0$. We shall encounter such a situation later, in connection with symmetry-breaking bifurcation. What we see here is a manifestation of the "divide" between the problems when (2.4) holds, and the case of $f(0) < 0$.

Lemma 2.5 *If the problem (2.2) admits nontrivial solutions, then the solution set is one dimensional. If moreover $u(x)$ is a positive solution, satisfying (2.3), then $w(x)$ is an even function.*

Proof: By uniqueness of initial value problem the value of $w'(1)$ uniquely determines $w(x)$, and hence the null space is one dimensional. Turning to the second claim, if $u(x)$ is positive, then it is even. Hence $w(-x)$ also solves (2.2). Since the null space is one dimensional, $w(-x) = cw(x)$ for some constant c. Evaluating this relation at $x = 0$, we conclude that $c = 1$ (since $w(0) > 0$ by the previous lemma), which is the desired symmetry. $\diamondsuit$

The following lemma gives a simple condition for positive solutions of (2.1) to be non-singular.

Lemma 2.6 *Assume that either*

$$f'(u) > \frac{f(u)}{u} \quad \textit{for all } u > 0,$$

or the opposite inequality holds. Then the linearized problem (2.2) has only the trivial solution.

Proof: If we rewrite the equation in (2.1) in the form

$$u'' + \lambda \frac{f(u)}{u} u = 0,$$

and use the Sturm comparison theorem, we conclude that the positive solution $u(x)$ oscillates faster than $w(x)$, and hence it must vanish on $(-1, 1)$, which is impossible. $\diamondsuit$

Another very simple condition is the following.

Lemma 2.7 *Assume that*

$$f'(u) < 0 \;\; \textit{for all } u > 0.$$

Then the linearized problem (2.2) has only the trivial solution.

Proof: Multiplying the equation in (2.2) by w, and integrating, we conclude that the problem (2.2) can have only the trivial solution. $\diamondsuit$

We shall need the following lemma, which “connects” the solutions of (2.1) and (2.2). It will allow us to verify the crucial condition of the Crandall-Rabinowitz Theorem 1.1 for both positive and sign changing solutions.

Lemma 2.8 *If the problem (2.2) admits a nontrivial solution, then*

$$\int_{-1}^{1} f(u)w\,dx = \frac{1}{\lambda}u'(1)w'(1). \tag{2.5}$$

Proof: The quantity $u''(x)w(x) - u'(x)w'(x)$ is a constant, and hence

$$u''(x)w(x) - u'(x)w'(x) = -u'(1)w'(1).$$

Integrating over $(-1,1)$ (by parts), we conclude the lemma. ♢

Very often one is looking for positive solutions of (2.1). A possible reason for this emphasis, is that only positive solutions have a chance to be stable, a property significant for applications. Let us recall the notion of stability. For any solution $u(x)$ of (2.1) let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. $w(x) > 0$ satisfies

$$w'' + \lambda f'(u)w + \mu w = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0. \tag{2.6}$$

The solution $u(x)$ of (2.1) is called unstable if $\mu < 0$, otherwise it is stable. (This is so called linear stability. It means, roughly, that solutions of the corresponding heat equation, with the initial data near $u(x)$ will tend to $u(x)$, as $t \to \infty$, see the book by D. Henry [16]).

Proposition 1 *Let $u(x)$ be a solution of (2.1) that changes sign on $(-1,1)$. Then $u(x)$ is unstable.*

Proof: Let $(\mu, w(x))$ denote the principal eigenpair of (2.6). Assume that on the contrary $\mu \geq 0$. Since $u(x)$ changes sign, we can find $-1 < x_1 < x_2 < 1$, such that $u'(x_1) = u'(x_2) = 0$ and say $u'(x) < 0$ on (x_1, x_2) (the other case is similar). Observe that $u''(x_1) < 0$ and $u''(x_2) > 0$ ($u'(x)$ satisfies a linear equation, it cannot vanish together with its derivative). Denoting $p(x) = u''(x)w(x) - u'(x)w'(x)$, we have

$$p'(x) = \mu u'(x)w(x) \leq 0 \ \text{ for } x \in (x_1, x_2).$$

We see that $p(x)$ is nonincreasing on (x_1, x_2). But, $p(x_1) = u''(x_1)w(x_1) < 0$ and $p(x_2) = u''(x_2)w(x_2) > 0$, a contradiction. ♢

This result was also proved by R. Schaaf [53]. A similar result for balls in R^n can be found in C.S. Lin and W.-M. Ni [42].

Lemma 2.9 *Any two positive solutions of (2.1) do not intersect inside $(-1,1)$ (i.e. they are strictly ordered on $(-1,1)$).*

Proof: Let $u(x)$ and $v(x)$ be two intersecting solutions. Since both of them are even functions, they intersect on the half-interval $(0,1)$ as well. Let $0 < \xi < \eta < 1$ be two consecutive intersection points. If $v(x) > u(x)$ on (ξ, η), then $|u'(\xi)| > |v'(\xi)|$, while $|u'(\eta)| < |v'(\eta)|$. The energy $E(x) = \frac{1}{2}u'(x)^2 + \lambda F(u(x))$ is constant for any solution $u(x)$. But at ξ, $u(x)$ has higher energy than $v(x)$, and at η the order is reversed, a contradiction. $\diamondsuit$

The following result from P. Korman [21] gives a detailed description of the solution shape for large λ. (In [21] we proved this result for balls in R^n.) If for some reason solutions cannot be of that shape, it follows that there are no positive solutions of (2.1) for large λ. Recall that root α of $f(u)$ is called *stable* if $f(\alpha) = 0$ and $f'(\alpha) < 0$.

Theorem 2.1 *Let $u(x, \lambda)$ be a positive solution of (2.1), that exists for all $\lambda > \bar{\lambda}$, for some $\bar{\lambda} > 0$. Assume that either $\lim_{u \to \infty} \frac{f(u)}{u} = \infty$, or there is $u_0 > 0$, so that $f(u) \le 0$ for $u \ge u_0$. Then the interval $(-1, 1)$ can be decomposed into a union of open intervals, whose total length $= 2$, so that on each such subinterval $u(x, \lambda)$ tends to a stable root of $f(u)$, as $\lambda \to \infty$.*

Example Assume that $f(u) < 0$ for $0 < u < \bar{u}$, with some $\bar{u} > 0$, $f(u) > 0$ for $u > \bar{u}$, and $\lim_{u \to \infty} \frac{f(u)}{u} = \infty$, e.g. $f(u) = u^p - 1$, with $p > 1$. Then the problem

$$u'' + \lambda\,(u^p - 1) = 0, \;\; -1 < x < 1, \;\; u(-1) = u(1) = 0$$

has no positive solution for λ large enough. Indeed, since $f(u)$ has no stable roots, solution cannot exhibit the behaviour described in the above theorem, and hence the solution cannot exist for all large λ.

The bifurcation approach applies also to the quasilinear problems of the type

$$(2.7) \qquad (\varphi(u'))' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0.$$

The prominent example is that of p-Laplacian $\varphi(t) = t|t|^{p-2}$, with $p \ge 2$. Motivated by this example, we assume that $\varphi(t) \in C^2(R \setminus \{0\})$ satisfies

$$(2.8) \qquad \varphi'(t) > 0, \qquad \text{for all } t \ne 0.$$

We consider weak solutions of (2.7), which are of class $C[-1,1] \cap C^1(-1,1) \cap C^2\left((-1,1) \setminus \{0\}\right)$. (Since the problem is degenerate elliptic, the value of $u''(0)$ might be infinite.)

Assuming the condition (2.8), we will show that the conclusions of the Lemmas 2.1, 2.2, 2.4 and 2.5 hold verbatim for the problem (2.7). In case $\varphi(t) = t|t|^{p-2}$, the Lemmas 2.3 and 2.8 hold too. The proofs are basically the same, but there are some difficulties due to degeneracy. For example, in the proof of Lemma 2.1 it is not apparent that $u(x)$ is symmetric with respect to ξ in case $\xi = 0$. (If $\xi \neq 0$, the proof is as before.) We therefore combine the first two lemmas to assert that the solution has the same shape as before.

Lemma 2.10 *Assuming the condition (2.8), any positive solution of (2.7) is an even function, with $u'(x) > 0$ on $(-1, 0)$, and $u'(x) < 0$ on $(0, 1)$.*

Proof: We need to adjust the definition of energy. Define $\Phi(z) = \int_0^z t\varphi'(t)\,dt$. Then the energy $\Phi(u'(x)) + \lambda F(u(x))$ is constant (in case of p-Laplacian, $\frac{p}{p-1}|u'(x)|^p + \lambda F(u(x)) = constant$). Using the energy, we conclude as before that $u(x)$ cannot have any points of minimum inside the interval $(-1, 1)$. Also, since the energy is constant, it follows that

$$u'(-1) = -u'(1). \tag{2.9}$$

To prove the symmetry, we consider $v(x) \equiv u(-x)$. The function $v(x)$ satisfies the same problem (2.7). By (2.9) it has the same initial conditions at $x = 1$ as $u(x)$. Hence $u(x) \equiv v(x)$. And finally, observe that an even function with no interior minimums has the desired shape. ♢

It is easy to see that the Lemma 2.3 holds in case of p-Laplacian. Since we need homogeneity for rescaling, we cannot assert it for the general problem (2.7). Next we consider the linearized problem for (2.7)

$$(\varphi'(u')w')' + \lambda f'(u)w = 0 \quad \text{for } -1 < x < 1, \;\; w(-1) = w(1) = 0. \tag{2.10}$$

Lemma 2.11 *Assume that the condition (2.3) holds. If the problem (2.10) admits a nontrivial solution, then it does not change sign, i.e. we may assume that $w(x) > 0$ on $(-1, 1)$. Moreover, in the case of p-Laplacian, the following generalization of the formula (2.5) holds:*

$$\int_{-1}^{1} f(u)w\,dx = \frac{2}{p\lambda}\varphi'(u'(1))u'(1)w'(1). \tag{2.11}$$

Proof: The proof of the first statement is exactly the same as before. Turning to the other one, we differentiate the equation (2.10)

$$(\varphi'(u')u'_x)' + \lambda f'(u)u_x = 0. \tag{2.12}$$

Combining the problems (2.10) and (2.12), we have

$$\varphi'(u'(x))(u''(x)w(x) - u'(x)w'(x)) = -\varphi'(u'(1))u'(1)w'(1). \tag{2.13}$$

We now integrate over $(-1,1)$. In the case of p-Laplacian, $\varphi(t) = t|t|^{p-2}$, $\varphi'(t) = (p-1)|t|^{p-2}$, and $t\varphi'(t) = (p-1)\varphi(t)$. Hence the second term on the left is equal to $-(p-1)\int_{-1}^{1}\varphi(u'(x))w'(x)\,dx$. Integrating this term by parts, we can combine it with the first term of the resulting equation. Finally, we observe that $\varphi'(u'(x))u''(x) = -\lambda f(u(x))$. $\diamondsuit$

2.2 Convex nonlinearities

For convex nonlinearities one can give an exhaustive description of the bifurcation diagrams for the problem (2.1), since we are able to show that the solution curve cannot turn more than once. Namely, we assume that $f(u) \in C^2(\bar{R}_+)$ satisfies

$$f(0) > 0, \text{ and } f(u) > 0 \text{ for } u > 0, \tag{2.14}$$

$$f''(u) > 0, \quad \text{for } u > 0, \tag{2.15}$$

$$f(u) \geq au - b, \text{ for } u > 0, \text{ and some constants } a > 0 \text{ and } b \geq 0. \tag{2.16}$$

Theorem 2.2 *The problem (2.1), under the above conditions, has at most two positive solutions for any λ. Moreover, all positive solutions lie on a unique curve in the $(\lambda, u(0))$ plane. This curve begins at the point $(\lambda = 0, u(0) = 0)$, and either it tends to infinity at some $\lambda_0 > 0$, or else it bends back at some $\lambda_0 > 0$, and then continues without any more turns, and tends to infinity at some $\bar{\lambda}$, $0 \leq \bar{\lambda} < \lambda_0$.*

Proof: When $\lambda = 0$ we have a trivial solution $u = 0$. It follows by the implicit function theorem that for small $\lambda > 0$ there is a continuous in λ curve of solutions, passing through $(0,0)$. We claim that this solution curve cannot be continued indefinitely for all $\lambda > 0$. Assume on the contrary that solutions can be continued as $\lambda \to \infty$. Write the problem (2.1) in the corresponding integral form,

$$u(x) = \lambda \int_{-1}^{1} G(x,\xi) f(u(\xi))\,d\xi, \tag{2.17}$$

where $G(x,\xi)$ is the corresponding Green's function. It is well-known that $G(x,\xi) > 0$ for all $0 < x, \xi < 1$. Since by our assumptions $f(u)$ is bounded from below by a positive constant, it follows that $u(x)$ will become uniformly large, as $\lambda \to \infty$. Let $\phi_1(x)$ be the principal eigenvalue of $-u''$ on the interval $(-1,1)$ subject to zero boundary conditions, and λ_1 the corresponding principal eigenvalue (here $\phi_1(x) = \cos \frac{\pi}{2} x$, and $\lambda_1 = \frac{\pi^2}{4}$). Multiplying the equation (2.1) by $\phi_1(x)$, and integrating, we have

$$\lambda_1 \int_{-1}^{1} u\phi_1 \, dx = -\int_{-1}^{1} u''\phi_1 \, dx = \lambda \int_{-1}^{1} f(u)\phi_1 \, dx$$
$$\geq \lambda a \int_{-1}^{1} u\phi_1 \, dx - \lambda b \int_{-1}^{1} \phi_1 \, dx.$$

But this leads to a contradiction, as $\lambda \to \infty$, since $\int_{-1}^{1} u\phi_1 \, dx \to \infty$. ($u(x)$ is a convex function, tending to infinity.)

Let λ_0 denote the supremum of λ, for which the solution curve continues to the right. It is possible that solutions become unbounded as $\lambda \to \lambda_0$ (this is one of the possibilities discussed in the statement of the theorem). So assume that the solutions stay bounded, as $\lambda \to \lambda_0$. Passing to the limit in the integral form of the equation, see (2.17), we conclude the existence of a bounded solution $u_0(x)$, which our solution curve enters at $\lambda = \lambda_0$. Clearly the pair $(\lambda_0, u_0(x))$ is a singular solution of (2.1) (since it cannot be continued to the right in λ). We show next that the Crandall-Rabinowitz Theorem 1.2 applies at $(\lambda_0, u_0(x))$.

We begin by recasting the equation in operator form $F(\lambda, u) = 0$, where the map $F(\lambda, u) : C^2(-1,1) \times R_+ \to C(-1,1)$ is defined by $F(\lambda, u) = u''(x) + \lambda f(u(x))$. Observe that $F_u(\lambda, u)w$ is given by the left hand side of the linearized equation (2.2). Since the point (λ_0, u_0) is singular, it follows that the linearized equation (2.2) has a non-trivial solution $w(x)$, which is positive by Lemma 2.4. By Lemma 2.5 it follows that the null-space $N\left(F_u(\lambda_0, u_0)\right) = span\{w(x)\}$ is one dimesional, and then $codim R\left(F_u(\lambda_0, u_0)\right) = 1$, since $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero. To apply the Crandall-Rabinowitz Theorem 1.2, it remains to show that $F_\lambda(\lambda_0, u_0) \notin R\left(F_u(\lambda_0, u_0)\right)$. Assuming the contrary would imply existence of a non-trivial $v(x)$, a solution of

$$v'' + \lambda_0 f_u(\lambda_0, u_0)v = f(\lambda_0, u_0), \quad \text{for } x \in (-1,1), \quad v(-1) = v(1) = 0.$$

By the Fredholm alternative (or just multiplying this equation by w, the equation (2.2) by v, subtracting and integrating)

$$\int_{-1}^{1} f(\lambda_0, u_0))w(x) \, dx = 0,$$

which contradicts Lemma 2.8. (Since $f(0) > 0$, we have $u'(1) < 0$, and $w'(1) < 0$ by uniqueness of initial value problems. Hence by Lemma 2.8, the above integral is positive.) Hence the Crandall-Rabinowitz Theorem 1.2 applies at $(\lambda_0, u_0(x))$.

Next we compute the direction of bifurcation at the point $(\lambda_0, u_0(x))$. According to the Crandall-Rabinowitz Theorem 1.2, the solution set near the point $(\lambda_0, u_0(x))$ is a curve $\lambda = \lambda(s)$, $u = u(s)$, with $\lambda(0) = \lambda_0$ and $u(0) = u_0(x)$. Observe that $\lambda'(0) = 0$, and $u_s(0) = w(x)$, according to the Crandall-Rabinowitz theorem. If we can show that $\lambda''(0) < 0$, it would follow that the solution curve turns to the left at $(\lambda_0, u_0(x))$, since the function $\lambda(s)$ has a maximum at $s = 0$. To express $\lambda''(s)$, we differentiate the equation (2.1) twice in s, obtaining

$$u''_{ss} + \lambda f_u u_{ss} + \lambda f_{uu} u_s^2 + 2\lambda' f_u u_s + \lambda'' f = 0, \quad u_{ss}(-1) = u_{ss}(1) = 0.$$

Letting $s = 0$, we have by the above remarks

$$u''_{ss} + \lambda_0 f_u u_{ss} + \lambda_0 f_{uu} w^2 + \lambda''(0) f = 0, \quad u_{ss}(-1) = u_{ss}(1) = 0. \tag{2.18}$$

Multiplying this equation by w, the equation (2.2) by u_{ss}, subtracting and integrating

$$\lambda''(0) = -\lambda_0 \frac{\int_{-1}^{1} f_{uu}(\lambda_0, u_0(x)) w^3(x)\, dx}{\int_{-1}^{1} f(\lambda_0, u_0)) w(x)\, dx} < 0, \tag{2.19}$$

with the last inequality due to convexity of $f(u)$ and Lemma 2.8.

The above analysis is valid not only at the point $(\lambda_0, u_0(x))$, but also at any other critical point. Hence, locally near any critical point, the solution set consists of a parabola-like curve, facing to the left in the $(\lambda, u(0))$ plane. Hence, after bending back at the point $(\lambda_0, u_0(x))$, our solution curve continues for decreasing λ, without ever encountering critical points. (At any critical point, we could not possibly have a parabola-like curve, described above, since our curve has arrived from the right.) Hence, the solution curve continues globally, without any turns, and it has to go to infinity at some $\bar{\lambda} \geq 0$. We then have one of the solution curves, described in the theorem, and the maximum value of solutions on this curve, $u(0)$ varies from zero to infinity. Hence all possible maximum values are "taken", and so by Lemma 2.3 this curve exhausts the solution set. $\diamondsuit$

Remarks

1. If, moreover, we have

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty, \tag{2.20}$$

Figure 1: Three types of solution curves for convex $f(u)$

then the curve cannot go to infinity at a finite λ. Hence, it will bend back at some λ_0, and go to infinity at $\lambda = 0$. Indeed, since $f(u) > 0$, the solutions $u(x, \lambda)$ are concave in x. So that if $u(x, \lambda)$ gets large near some $\lambda = \lambda_1$, it would have to get uniformly large on some interval, say on $(-1/2, 1/2)$. Writing our equation in the form $u'' + \lambda \frac{f(u)}{u} u = 0$, and using the Sturm comparison theorem, we conclude that the positive solution $u(x)$ has to vanish on $(-1/2, 1/2)$, which is impossible.

2. One can show that the solutions on the lower branch are increasing in λ, for all $x \in (-1, 1)$, see P. Korman and T. Ouyang, [34], [35]. (On the upper branch this is no longer true, but the maximal value $u(0)$ is increasing as we trace the branch, i.e. it is decreasing in λ.)

3. Our assumptions did not require for $f(u)$ to be increasing.

4. All three possibilities, mentioned in the theorem, can actually occur, see Figure 1 for the results of numerical computations. Observe that in the second and third cases $f(u)$ is asymptotically linear, and bifurcation from infinity happens.

2.3 S-shaped solution curves

We saw in the previous section that solution curves are relatively simple for convex $f(u)$. If $f(u)$ changes concavity, then the solution curve may admit more than one turn. One prominent nonlinearity, with change in concavity, is connected to combustion theory, see the nice book of J. Bebernes and D. Eberly [7]. Namely, we consider the problem

$$(2.21) \qquad u'' + \lambda e^{\frac{au}{u+a}} = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0,$$

where a is a constant (this problem is referred to as "the perturbed Gelfand problem" in [7]). In case $a \le 4$, the problem is easy. In that case $uf'(u) - f(u) < 0$ for all $u > 0$, and hence all positive solutions are non-singular. This means that the solution curve is monotone, i.e. it continues for all $\lambda > 0$ without any turns. S.-H. Wang [60] has proved existence of a constant a_0, so that for $a > a_0$ the solution curve of (2.21) is exactly S-shaped, i.e. it starts at $\lambda = 0$, $u = 0$, it makes exactly two turns, and then it continues for all $\lambda > 0$ without any more turns. S.-H. Wang [60] gave an approximation of the constant $a_0 \simeq 4.4967$. That paper, as well as all previous ones, used a time map approach. P. Korman and Y. Li [28] have applied the bifurcation approach to the problem to show the exactness of the S-shaped curves, and they also improved the value of the constant to $a_0 \simeq 4.35$, i.e. for $a > a_0$ the solution sets are S-shaped curves. S.-H. Wang [60] has conjectured existence of critical number $\bar{a}$, so that for $a \le \bar{a}$ the solution curve is monotone, while for $a > \bar{a}$ the solution curve is exactly S-shaped. Recently, P. Korman, Y. Li and T. Ouyang [33] has given a computer assisted proof of the S.-H. Wang's conjecture. Numerical calculations show that $\bar{a} \simeq 4.07$.

We are going to discuss the S-shaped curves, mostly following P. Korman and Y. Li [29]. However, in that paper time maps were still used at one point. Subsequently, in P. Korman and J. Shi [40] an argument not using time maps was given. Next we present this result, dealing with instability of solutions (it also turned out to be of independent interest), after we recall the notion of stability. For any solution $u(x)$ of (2.1) let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. $w(x) > 0$ satisfies

$$w'' + \lambda f'(u)w + \mu w = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0. \tag{2.22}$$

The solution $u(x)$ of (2.1) is called unstable if $\mu < 0$, otherwise it is stable.

Let $F(u) = \int_0^u f(t)\,dt$, $h(u) = 2F(u) - uf(u)$. The instability result from P. Korman and J. Shi [40] is

Theorem 2.3 *Assume that $f \in C^1[0,\infty)$, $f(0) > 0$, and for some $\alpha > \beta > 0$ we have:*

$$h'(u) \ge 0 \quad \text{for} \quad 0 < u < \beta, \qquad h'(u) \le 0 \quad \text{for} \quad \beta < u < \alpha, \tag{2.23}$$

$$h(\alpha) \le 0. \tag{2.24}$$

Then the solution of (2.1) with $u(0) = \alpha$ is unstable, if it exists.

Proof: We have $h(0) = 0$, $h'(u) = f(u) - uf'(u)$, $h'(0) = f(0) > 0$. It follows from our conditions that $h(u)$ is unimodular on $[0, \alpha]$, and it takes its positive maximum at $u = \beta$. Define $x_0 \in (0,1)$ by $u(x_0) = \beta$. We then conclude

$$f(u(x)) - u(x)f'(u(x)) \le 0 \quad \text{on } (0, x_0),$$
$$f(u(x)) - u(x)f'(u(x)) \ge 0 \quad \text{on } (x_0, 1). \tag{2.25}$$

We also remark that by the condition (2.24),

$$\int_0^1 [f(u) - uf'(u)]\, u'(x)\, dx = \int_0^1 \frac{d}{dx} h(u(x))\, dx = -h(\alpha) \ge 0. \tag{2.26}$$

Assume now that $u(x)$ is stable, i.e. $\mu \ge 0$ in (2.22). Without loss of generality, we assume that $w > 0$ in $(-1, 1)$. By the maximum principle, $u'(1) < 0$, so near $x = 1$ we have $-u'(x) > w(x)$. Since $-u'(0) = 0$, while $w(0) > 0$, the functions $w(x)$ and $-u'(x)$ change their order at least once on $(0, 1)$. We claim that the functions $w(x)$ and $-u'(x)$ change their order exactly once on $(0, 1)$. (We ignore the points where these functions merely "touch".) Observe that $-u'(x)$ satisfies

$$(-u')'' + \lambda f'(u)(-u') = 0 \quad \text{on } (0, 1), \tag{2.27}$$

while $w(x)$ (and any of its positive multiples) is a supersolution of the same equation. Let $x_3 \in (0, 1)$ be the largest point where $w(x)$ and $-u'(x)$ change the order. Assuming the claim to be false, let x_2, with $0 < x_2 < x_3$, be the next point where the order changes. We have $w > -u'$ on (x_2, x_3), and the opposite inequality to the left of x_2. Since $w(0) > -u'(0)$, there is another point $x_1 < x_2$, where the order is changed. We can now find a constant $\gamma > 1$, and a point $x_0 \in (x_1, x_2)$ so that a $\gamma w(x)$, a supersolution of (2.27), touches at x_0 from above a solution $-u'(x)$ of the same equation, a contradiction.

Since the point of changing of order is unique, by scaling of $w(x)$ we can achieve

$$-u'(x) \le w(x) \quad \text{on } (0, x_0),$$
$$-u'(x) \ge w(x) \quad \text{on } (x_0, 1). \tag{2.28}$$

Using (2.25), (2.28), and also (2.26), we have

$$\int_0^1 [f(u) - uf'(u)]\, w(x)\, dx < \int_0^1 [f(u) - uf'(u)]\, (-u'(x))\, dx \le 0, \tag{2.29}$$

since the integrand on the left is pointwise smaller than the one on the right. On the other hand, multiplying the equation (2.22) by u, the equation (2.1) by w, subtracting and integrating over $(0,1)$, we have

$$\int_0^1 [f(u) - uf'(u)]\, w(x)\, dx = \frac{\mu}{\lambda} \int_0^1 uw\, dx \geq 0,$$

which contradicts (2.29). So $\mu < 0$. $\diamondsuit$

We will consider a class of nonlinearities, including $f(u) = e^{\frac{au}{u+a}}$, so let us list our assumptions. We assume that $f(u) \in C^2[0, \bar{u}]$ for some $0 < \bar{u} \leq \infty$, and that it satisfies

$$f(u) > 0 \quad \text{for all } 0 \leq u < \bar{u}. \tag{2.30}$$

We assume $f(u)$ to be convex-concave, i.e. there an $\alpha \in (0, \bar{u})$, such that

$$f''(u) > 0 \ \text{ for } u \in (0, \alpha), \quad f''(u) < 0 \ \text{ for } u \in (\alpha, \bar{u}). \tag{2.31}$$

We define a function $I(u) = f^2(u) - 2F(u)f'(u)$, where as before $F(u) = \int_0^u f(t)\, dt$. Assume there is a $\beta > \alpha$, such that

$$I(\beta) = f^2(\beta) - 2F(\beta)f'(\beta) \geq 0. \tag{2.32}$$

The following lemma has originated from P. Korman, Y. Li and T. Ouyang [30].

Lemma 2.12 *Assume that $f(u)$ satisfies the conditions (2.30), (2.31) and (2.32). Let (λ, u) be any critical point of (2.1), such that $u(0) \geq \beta$, and let $w(x)$ be the corresponding solution of the linearized problem (2.2). Then*

$$\int_0^1 f''(u(x))u_x(x)w^2(x)\, dx > 0. \tag{2.33}$$

Proof: We shall derive a convenient expression for the integral in (2.33). Differentiate (2.2)

$$w_x'' + \lambda f'(u)w_x + \lambda f''(u)u_x w = 0. \tag{2.34}$$

Multiplying the equation (2.34) by w, the equation (2.2) by w_x, integrating and subtracting, we express

$$\lambda \int_0^1 f''(u)u_x w^2\, dx = {w'}^2(1) - \lambda w^2(0)f'(u(0)). \tag{2.35}$$

By differentiation, we verify that $u''(x)w(x) - u'(x)w'(x)$ is constant for all x, and hence

$$u''(x)w(x) - u'(x)w'(x) = -\lambda w(0)f(u(0)) \text{ for all } x \in [-1,1]. \tag{2.36}$$

Evaluating this expression at $x = 1$, we obtain

$$w'(1) = \frac{\lambda w(0)f(u(0))}{u'(1)}. \tag{2.37}$$

Multiplying (2.1) by u', and integrating over $(0,1)$, we have

$$u'^2(1) = 2\lambda F(u(0)). \tag{2.38}$$

Using (2.38) and (2.37) in (2.35), we finally express

$$\lambda \int_0^1 f''(u)u_x w^2 \, dx = \frac{w^2(0)}{2F(\rho)} I(\rho), \tag{2.39}$$

where we denote $\rho = u(0)$. By our assumption, $I(\beta) \geq 0$. Since

$$I'(\rho) = -2F(\rho)f''(\rho) > 0 \quad \text{for } \rho \geq \beta,$$

we conclude that $I(\rho) > I(\beta) \geq 0$, and the lemma follows. ♢

The following lemma contains the crucial trick, which has originated from P. Korman, Y. Li and T. Ouyang [30]. It says that for convex-concave problems only turns to the right are possible in the (λ, α) plane, once the maximum value of the solution, $u(0)$, has reached a certain level.

Lemma 2.13 *In the conditions of the preceeding Lemma 2.12, assume again that $u(0) \geq \beta$, and $w(x)$ the corresponding solution of the linearized problem (2.2). Then*

$$\int_0^1 f''(u(x))w^3(x) \, dx < 0. \tag{2.40}$$

Proof: Let $(\lambda, u(x))$ be a critical point of (2.1). Since $u(0) \geq \beta > \alpha$, it follows that the function $f''(u(x))$ changes sign exactly once on $(0,1)$, say at x_0. Then we have

$$f''(u(x)) < 0 \text{ for } x \in (0, x_0), \quad f''(u(x)) > 0 \text{ for } x \in (x_0, 1). \tag{2.41}$$

We have proved in Theorem 2.3 that the functions $w(x)$ and $-u'(x)$ intersect exactly once on $(0,1)$. By scaling $w(x)$ we may assume that they intersect at x_0. ($w(x)$ is a solution of a linear problem, and hence it is defined up to a constant multiple.) In view of Lemma 2.12, we then have

$$\int_0^1 f''(u(x))w^3(x)\,dx < \int_0^1 f''(u(x))w^2(-u_x)\,dx < 0,$$

since by (2.41) the integrand on the right is pointwise greater than the one the left. $\diamondsuit$

The same approach can be used to prove the following more general theorem, which was implicit in P. Korman, Y. Li and T. Ouyang [30], see also T. Ouyang and J. Shi [50].

Theorem 2.4 *([30])*
(i) Assume that $f(0) \geq 0$, $f''(u) < 0$ for $0 < u < u_0$, $f''(u) > 0$ for $u > u_0$. Then only turns to the left are possible on the solution curve.
(ii) Assume that $f(0) \leq 0$, $f''(u) > 0$ for $0 < u < u_0$, $f''(u) < 0$ for $u > u_0$. Then only turns to the right are possible on the solution curve.

(Of course, in both cases we conclude existence of at most two positive solutions, with the maximum values lying in the first positive hump of $f(u)$.)

We are now ready for the main result of this section, see P. Korman and Y. Li [29].

Theorem 2.5 *Assume that $f(u)$ satisfies the conditions (2.30) and (2.31), and moreover,*

$$\lim_{u\to\infty} \frac{f(u)}{u} = 0. \tag{2.42}$$

With $h(u) \equiv 2F(u) - uf(u)$, assume that

$$h(\alpha) < 0. \tag{2.43}$$

Then the solution set of (2.1) consists of one curve, which is exactly S-shaped, i.e. it starts at $\lambda = 0$, $u = 0$, it makes exactly two turns, and then it continues for all $\lambda < \infty$, without any more turns.

Proof: By the implicit function theorem there is a curve of positive solutions of (2.1), starting at $\lambda = 0$, $u = 0$. As in Theorem 2.2, this curve continues for increasing λ, until a possible singular solution (λ_0, u_0) is reached, at which point the Crandall-Rabinowitz Theorem 1.2 applies. By (2.19) it follows that only turns to the left are possible if $u(0) < \alpha$, since $f(u)$ is convex for $u \in (0, \alpha)$. Until the first critical point (λ_0, u_0) is reached, the solutions are stable. Indeed, the solution curve starts at $(\lambda = 0, u = 0)$, which is a stable solution (the principal eigenvalue of the corresponding linearized problem $= \frac{\pi^2}{2}$), while any change of stability requires a passage through a singular point. By the Theorem 2.3 when $u(0) = \alpha$ the solution is unstable. Hence a singular solution was reached before that, and since only turns to the left are possible when $u(x) < \alpha$, it follows that *exactly one* turn has occured, and at $u(0) = \alpha$ the solution curve travells to the left.

We now show that the solution curve keeps travelling to the left, until $u(0)$ increases to the level when only turns to the right are possible. For that we take a close look at the function $h(u) = 2F(u) - uf(u)$. Since

$$h'(u) = f(u) - uf'(u), \quad h''(u) = -uf''(u),$$

it follows that the function $h'(u)$ is decreasing on $(0, \alpha)$ and increasing on (α, ∞). We have $h'(0) = f(0) > 0$, and so $h'(u)$ can have at most two roots. We claim that it has exactly two roots, u_1 and u_2 with $h'(u)$ being positive on $(0, u_1) \cup (u_2, \infty)$, and negative on (u_1, u_2). Indeed, existence of the first root is clear, since $h(0) = 0$ and $h(\alpha) < 0$. As for the second root u_2, if it did not exist, we would have

$$uf'(u) > f(u) \ \text{ for all } u > \alpha. \tag{2.44}$$

Integrating (2.44),

$$f(u) > \frac{f(\alpha)}{\alpha} u \ \text{ for all } u > \alpha,$$

contradicting the assumption (2.42). So that the function $h(u)$ starts with $h(0) = 0$, it is increasing on $(0, u_1)$, decreasing on (u_1, u_2), with absolute minimum at u_2, and then it increases on the interval (u_2, ∞). By Theorem 2.3 the solution curve keeps traveling to the left, while $u(0) \in (\alpha, u_2)$.

Picture 1 : **The function** $h(u)$

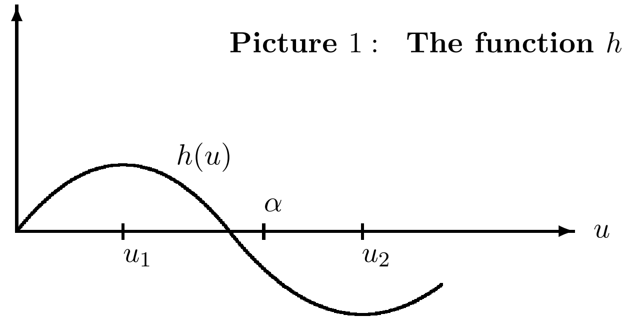

Figure 2: An S-shaped solution curve

We claim that for $u(0) > u_2$ the Lemma 2.13 applies. For that we need to check that for $\beta = u_2$ the condition (2.32) holds. Indeed, since $h(u_2) < 0$, we have $f(u_2)u_2 > 2F(u_2)$. Hence

$$I(u_2) = f^2(u_2) - 2F(u_2)f'(u_2) > f^2(u_2) - u_2 f(u_2)f'(u_2) = 0,$$

and the claim follows (observe that $f'(u_2) = \frac{f(u_2)}{u_2} > 0$). By Lemma 2.13, only turns to the right are possible when $u(0) > u_2$.

Let us now put it all together. We have a curve of solutions, which starts at $(\lambda = 0, u = 0)$. As we travel on this curve, $u(0)$ is always increasing. By the time we reach $u(0) = \alpha$ level, the solution curve has made exactly one turn to the left. When $\alpha < u(0) < u_2$ the solution curve travels to the left. When $u(0) > u_2$, the solution curve cannot travel to the left indefinitely, since it is easy to see that solutions are bounded for bounded λ. Hence, the curve must turn to the right. Since for $u(0) > u_2$ only turns to the right are possible, exactly one such turn occurs. It follows that the solution curve is exactly S-shaped. $\diamondsuit$

In the Figure 2 we give an example of an S-shaped solution curve. Notice that *Mathematica* has drawn the vertical axis around $\lambda = 3$. Also observe that an actual S-shaped solution curve is way different from what most people would draw by hand.

2.4 Cubic-like nonlinearities

We again consider the problem

$$u'' + \lambda f(u) = 0 \quad x \in (-1,1), \;\; u(-1) = u(1) = 0, \tag{2.45}$$

where $f(u)$ behaves like a cubic with three distinct roots, with a model example $f(u) = (u-a)(u-b)(c-u)$. Namely, we assume that the function

$f(u) \in C^2(\bar{R}_+)$ has three non-negative roots at $0 \le a < b < c$, and

$$f(u) > 0 \ \text{ on } [0,a) \cup (b,c), \quad f(u) < 0 \ \text{ on } (a,b) \cup (c,\infty), \qquad (2.46)$$
$$\textstyle\int_a^c f(u)\,du > 0,$$

Moreover, we assume there is an $\alpha > 0$, so that

$$f''(u) > 0 \ \text{ for } 0 \le u < \alpha, \quad f''(u) < 0 \ \text{ for } u > \alpha. \qquad (2.47)$$

This problem was originally studied using time-maps, see J. Smoller and A. Wasserman [58] and S.-H. Wang [60], [61]. In P. Korman, Y. Li and T. Ouyang [30] the bifurcation approach was applied. The case of $a = 0$ turned out to be easier for both time-maps and bifurcation approaches, while in case $a > 0$ some restriction on a (a bound from above) was necessary for both approaches.

We shall do the case $a = 0$ first, after two simple lemmas. (I.e. $f(u)$ is modeled on $f(u) = u(u-b)(c-u)$.) Let $\beta \in (b,c)$ be the unique point satisfying

$$f'(\beta) = \frac{f(\beta)}{\beta}. \qquad (2.48)$$

(I.e. the point where the straight line through the origin is tangent to the graph of $y = f(u)$.) Clearly, $\beta > \alpha$. The following lemma shows that no turns of the solution curve are possible until the maximum value of the solution reaches a certain level.

Lemma 2.14 *Assume that $f(u) \in C^2$ satisfies the conditions (2.46) and (2.47). If $u(x)$ is a critical solution of (2.45) then*

$$u(0) > \beta. \qquad (2.49)$$

Proof: We claim that

$$f'(u) > \frac{f(u)}{u} \ \text{ for } 0 < u < \beta. \qquad (2.50)$$

Indeed, denote $p(u) = uf'(u) - f(u)$. Then $p(0) = p(\beta) = 0$, and $p'(u) = uf''(u)$, which implies that $p(u)$ is increasing on $(0,\alpha)$ and decreasing on (α,β). Then (2.50) follows, and hence by Lemma 2.6 the linearized equation has only the trivial solution, in case $u(0) \le \beta$. $\diamondsuit$

Lemma 2.15 *Assume that $f(u) \in C^2$ satisfies the conditions (2.46) and (2.47), with $a = 0$. Assume $u(x)$ is a critical solution of (2.45), and let $w(x)$ be solution of the corresponding linearized problem (2.2). Then*

$$\int_0^1 f''(u) w^3 \, dx < 0. \tag{2.51}$$

Proof: We begin by showing that

$$\int_0^1 f''(u) u_x^2 w \, dx = 0. \tag{2.52}$$

We have (using the equations (2.45) and (2.2))

$$\left(u''w' - u'w''\right)' = \lambda f''(u) u_x^2 w.$$

Integrating over $(0,1)$, and using that $w''(1) = -\lambda f'(u(1))w(1) = 0$ and $u''(1) = -\lambda f(u(1)) = -\lambda f(0) = 0$, we conclude (2.52) (it is here that we use that $f(0) = 0$, i.e. $a = 0$).

We now proceed similarly to Lemma 2.13. Similarly to that lemma, we show that the functions $w(x)$ and $-u'(x)$ intersect exactly once on $(0,1)$. Observe that by the Lemma 2.14, we have $u(0) > \beta > \alpha$ at any critical solution $u(x)$. Let $\xi \in (0,1)$ be the point where $u(\xi) = \alpha$. By scaling $w(x)$ we may assume that $w(x)$ and $-u'(x)$ intersect at ξ. Then on the interval $(0,\xi)$, where $f''(u(x))$ is negative, we have $u_x^2 < w^2$, while on the interval $(\xi,1)$, where $f''(u(x))$ is positive, we have $u_x^2 > w^2$. We then have, in view of (2.52),

$$\int_0^1 f''(u) w^3 \, dx < \int_0^1 f''(u) u_x^2 w \, dx = 0,$$

since the integral on the left is pointwise smaller than the one on the right. $\diamondsuit$

The following theorem is from P. Korman, Y. Li and T. Ouyang [30].

Theorem 2.6 *Assume that $f(u) \in C^2$ satisfies the conditions (2.46) and (2.47), with $a = 0$. Then there is a critical λ_0 such that for $\lambda < \lambda_0$ the problem (2.45) has no positive solutions, it has exactly one positive solution at $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a single solution curve, which for $\lambda > \lambda_0$ has two branches $0 < u^-(x,\lambda) < u^+(x,\lambda)$, with $u^+(x,\lambda)$ strictly monotone increasing in λ, and $\lim_{\lambda \to \infty} u^+(x,\lambda) = c$. On the lower branch, $u^-(0,\lambda)$ is monotone decreasing, $\lim_{\lambda \to \infty} u^-(x,\lambda) = 0$ for all $x \neq 0$, while $u^-(0,\lambda) > b$ for all λ. We also have $\lim_{\lambda \to \infty} u^-(0,\lambda) = \theta$, where θ is defined by the relation $\int_0^\theta f(u)\,du = 0$.*

Figure 3: A parabola-like solution curve, when $a = 0$

Proof: If $\lambda c > \frac{\pi^2}{4}$, then existence of positive solutions follows by monotone iterations. Indeed, $\phi = c$ is a supersolution of (2.45), while $\psi = \epsilon \cos \frac{\pi}{2} x$ is a subsolution of the same problem, if ϵ is chosen sufficiently small ($\lambda = \frac{\pi^2}{4}$ and $\phi_1 = \cos \frac{\pi}{2} x$ give, of course, the principal eigenpair of the Laplacian on $(-1, 1)$). We now continue the positive solution (any one) for decreasing λ. At regular points we use the implicit function theorem for continuation, while the singular point(s) will be discussed below. We cannot continue this curve indefinitely for decreasing λ, since it has no place to go. Indeed, solutions are bounded by c, and so the right hand side of the equation (2.45) goes to zero, and hence $u(x) \to 0$ as $\lambda \to 0$. But that is impossible, since $f(u)$ is negative near $u = 0$, while at the point of maximum $u''(0) \leq 0$. Hence at some critical $\lambda = \lambda_0$ and $u = u_0$ the solution curve cannot be continued further for decreasing λ.

As before, we show that the Crandall-Rabinowitz Theorem 1.2 applies at (λ_0, u_0). According to the Lemma 2.15 a turn to the right must occur at this, and any other critical point. Hence, exactly one turn happens, and the solution curve has exactly two branches.

The properties of the solution branches are easy to prove, see [30]. $\diamondsuit$

In Figure 3 we give an example for the above theorem. Notice that *Mathematica* has chosen the point $(6, 2)$ as the point where the axes intersect.

Next we turn to the case when $a > 0$. I.e. we assume that $f(u)$ satisfies the conditions (2.46) and (2.47), with the cubic $f(u) = (u-a)(u-b)(c-u)$ being our model example. Similarly to the above, we denote by $\beta \in (b, c)$ the unique point satisfying

$$f'(\beta) = \frac{f(\beta)}{\beta - a}. \tag{2.53}$$

(I.e. the point where the straight line through the point $(a, 0)$ is tangent to the graph of $y = f(u)$.) Clearly, $\beta > \alpha$. The proof of the following lemma is similar to that of Lemma 2.14, and so we omit it (see [33]).

Lemma 2.16 *Assume that $f(u) \in C^2$ satisfies the conditions (2.46) and (2.47). If $u(x)$ is a critical solution of (2.45) then*

$$u(0) > \beta. \tag{2.54}$$

We define a constant $\tau \in (b, c)$ by $f'(\tau) = 0$, i.e. τ is the second root of $f'(u)$. We also recall the function $I(u) = f^2(u) - 2F(u)f'(u)$, defined previously.

Lemma 2.17 *Assume that $f(u) \in C^2$ satisfies the conditions (2.46) and (2.47). Assume that either*

$$\int_a^\tau f(u)\, du \le 0, \tag{2.55}$$

or

$$I(\beta) \ge 0. \tag{2.56}$$

If $u(x)$) is a critical solution of (2.45), and $w(x)$ is a solution of the corresponding linearized problem, then

$$\int_0^1 f''(u(x))u'(x)w^2(x)\, dx < 0. \tag{2.57}$$

Proof: For any solution of (2.45) we have $\int_a^{u(0)} f(u)\, du > 0$ (just multiply the equation by u' and integrate between $x = 0$ and the point $x = \xi$, such that $u(\xi) = a$.) So that if (2.55) holds, then $u(0) > \tau$, i.e. $f'(u(0)) < 0$. Then (2.57) follows from the formula (2.35) for the integral (2.57). In case the condition (2.56) holds, the proof proceeds the same way as in Lemma 2.12. ♢

Remark We can replace the condition (2.55) by requiring that $u(0) > \tau$.

The following result was essentially proved in P. Korman, Y. Li and T. Ouyang [33].

Theorem 2.7 *Assume that $f(u) \in C^2$ satisfies the conditions (2.46) and (2.47). Assume either the condition (2.56) is satisfied, or else assume that any solution of the problem (2.45), with $u(0) \in (\beta, \tau)$ is non-critical. Then there exists a critical λ_0, such that the problem (5.27) has exactly one positive solution for $0 < \lambda < \lambda_0$, exactly two positive solutions at $\lambda = \lambda_0$, and exactly three positive solutions for $\lambda_0 < \lambda < \infty$. Moreover, all solutions lie on two smooth solution curves. One of the curves, referred to as the lower curve, starts at $(\lambda = 0, u = 0)$, it is increasing in λ, and $\lim_{\lambda \to \infty} u(x, \lambda) = a$ for $x \in (-1, 1)$. The upper curve is a parabola-like curve with exactly one turn to the right.*

Figure 4: A two-piece solution curve for a cubic, with $a > 0$

Proof: The properties of the lower curve are easy to prove. According to the implicit function theorem there is a curve of positive solutions, starting at $\lambda = 0$ and $u = 0$. Since $f'(u) < 0$ when $u < a$, it follows by Lemma 2.7 that solutions are non-degenerate, and hence they can be continued for all $\lambda > 0$. It is easy to see that the solutions on this curve are increasing in λ, and $\lim_{\lambda \to \infty} u(x, \lambda) = a$ for all $x \in (-1, 1)$, see [30].

Turning to the upper curve, recall that critical solutions are possible only if $u(0) > \beta$. If condition (2.56) holds then we have (2.57). The same way as in Lemma 2.12 we show that at any critical point

$$\int_0^1 f''(u(x)) w^3(x)\, dx < 0, \tag{2.58}$$

which means that only turns to the right are possible on the upper curve. Similarly to Theorem 2.6 for the $a = 0$ case, we show existence of solutions on the upper curve, and that the upper curve has to turn. Hence, exactly one turn occurs on the upper curve, and its other properties are proved similarly to the $a = 0$ case. In the other case, when (2.55) holds, we know that no critical points are possible, until $u(0) > \tau$. But then (2.57) holds, which implies (2.58), and we proceed the same way as in the first case. $\diamondsuit$

The above result shows that either one gets "lucky" at the level $u(0) = \beta$, i.e. the condition (2.56) holds, and the above Theorem 2.7 applies, or else the interval (β, τ) is "dangerous", i.e. we need to rule out the possibility of any turns when $\beta < u(0) < \tau$ (since we cannot tell their direction). For that computer assisted proofs are feasible. In fact, in a recent paper P. Korman, Y.Li and T. Ouyang [33] have given three independent computer assisted proofs in case of a cubic. We describe their result next. Let $f(u) = (u - a)(u - b)(c - u)$, with $0 < a < b < c$. For the problem (2.45) to have a positive solution it is necessary that $\int_a^c f(u)\, du > 0$, i.e.

$$b < \frac{a + c}{2}. \tag{2.59}$$

It was shown on [33] that under the necessary condition (2.59) the above Theorem 2.7 applies, providing an exact multiplicity result for the general cubic. In the next section we present a new tool, used in [33] to give one of the computer assisted proofs.

2.5 Computing the location of bifurcation

Assume that for the problem

$$u''(x) + f(u(x)) = 0, \;\; x \in (-1,1), \;\; u(-1) = u(1) = 0 \tag{2.60}$$

bifurcation occurs at $u(0) = \alpha$, i.e. the corresponding linearized problem

$$w''(x) + f'(u(x))w(x) = 0, \;\; x \in (-1,1), \;\; w(-1) = w(1) = 0 \tag{2.61}$$

admits a non-trivial solution. The following result of P. Korman, Y. Li and T. Ouyang [33] provides a way to determine all possible α's at which bifurcation may occur, i.e. the corresponding solution of (2.60) is singular.

Theorem 2.8 *A positive solution of the problem (2.60) with the maximal value $\alpha = u(0)$ is singular if and only if*

$$G(\alpha) \equiv F(\alpha)^{1/2} \int_0^\alpha \frac{f(\alpha) - f(\tau)}{\left[F(\alpha) - F(\tau)\right]^{3/2}} \, d\tau - 2 = 0. \tag{2.62}$$

Proof: We need to show that the problem (2.61) has a non-trivial solution. By direct verification the function $w(x) = -u'(x) \int_x^1 \frac{1}{u'^2(t)} \, dt$ satisfies the equation in (2.61). Also $w(1) = 0$. If we also have

$$w'(0) = 0, \tag{2.63}$$

then since $u(x)$ is an even function, the function $w(x)$ is also even (by uniqueness for initial value problems), and hence $w(-1) = 0$, which gives us a non-trivial solution of (2.61). Conversely, every non-trivial solution of (2.61) is an even function, and hence (2.63) is satisfied.

Using the equation in (2.60), we compute

$$w'(x) = f(u(x)) \int_x^1 \frac{1}{u'^2(t)} \, dt + \frac{1}{u'(x)} \, .$$

Since the energy $\frac{u'^2}{2}(x) + F(u(x))$ is constant,

$$\frac{u'^2}{2}(x) + F(u(x)) = F(u(0)) = F(\alpha).$$

On the interval $(0,1)$ we express

$$u'(x) = -\sqrt{2}\sqrt{F(\alpha) - F(u(x))}. \tag{2.64}$$

We use this formula in the integral $\int_x^1 \frac{1}{u'(t)^2}\,dt$, and then we make a change of variables $t \to s$, by letting $s = u(t)$. We have

$$2^{3/2}\int_x^1 \frac{1}{u'^2(t)}\,dt = -\int_x^1 \frac{u'(t)\,dt}{[F(\alpha)-F(u(t))]^{3/2}} = -\int_{u(x)}^0 \frac{1}{[F(\alpha)-F(s)]^{3/2}}\,ds. \tag{2.65}$$

Using the formulas (2.64) and (2.65), we express

$$2^{3/2}w'(x) = \int_0^{u(x)} \frac{f(u(x))}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau - \frac{2}{[F(\alpha)-F(u(x))]^{1/2}}. \tag{2.66}$$

If we try to set here $x=0$, then both terms on the right are infinite. Instead, we observe that

$$\begin{aligned} -\frac{2}{[F(\alpha)-F(u)]^{1/2}} &= -\int_0^u \frac{d}{d\tau}\frac{2}{[F(\alpha)-F(\tau)]^{1/2}}\,d\tau - \frac{2}{F(\alpha)^{1/2}} \\ &= -\int_0^u \frac{f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau - \frac{2}{F(\alpha)^{1/2}}. \end{aligned} \tag{2.67}$$

Using (2.67) in (2.66), we obtain

$$2^{3/2}w'(x) = \int_0^{u(x)} \frac{f(u(x))-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau - \frac{2}{F(\alpha)^{1/2}}. \tag{2.68}$$

The integral on the right is now non-singular, as we let $x \to 0$. At $x=0$ we see that (2.63) is equivalent to (2.62). ♢

In case of a cubic $f(u) = (u-a)(u-b)(c-u)$, P. Korman, Y. Li and T. Ouyang [33] have used the formula (2.62) to give a computer assisted proof that there are no turning points in the dangerous region, $u(0) \in (\beta, \tau)$, thus establishing the exact multiplicity result (Theorem 2.7) from the previous section.

2.6 Computing the direction of bifurcation

We have seen that at a critical solution $u(x)$ of (2.60) the integral $I = \int_0^1 f''(u)w^3\,dx$ governs the direction of bifurcation. Also, it is known that in case $I \neq 0$ a critical solution $u(x)$ is *non-degenerate*, i.e. it persists when the equation is perturbed slightly (i.e. the turning points persist under perturbations), see e.g. [32]. The following result from P. Korman, Y. Li and T. Ouyang [33] allows one to compute the integral I as a function of $\alpha = u(0)$.

Theorem 2.9 *At any critical solution $u(x)$ of (2.60), with $u(0) = \alpha$,*

$$I = c \int_0^\alpha f''(u) \left(\int_u^\alpha f(s)\, ds \right) \left(\int_0^u \frac{ds}{\left(\int_s^\alpha f(t)\, dt\right)^{3/2}} \right)^3 du, \tag{2.69}$$

where $c = \frac{1}{4\sqrt{2}} u'^3(1) w'^3(1) > 0$.

This formula is rather involved, but using *Mathematica* it can be evaluated numerically. In a future paper, with Y. Li and T. Ouyang, we use this result to handle equations modeled on polynomials of arbitrary power.

2.7 Pitchfork bifurcation and symmetry breaking

So far for the problem

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0 \tag{2.70}$$

we have considered the cases when $f(0) \geq 0$. As we have observed earlier, this condition implies that $|u_x(\pm 1, \lambda)| \neq 0$ for any positive solution $u(x, \lambda)$. Since we also know that $u_x(x, \lambda) < 0$ for all $x \in (0, 1)$, there is no way for a positive solution to become sign-changing, as we vary λ (no interior roots, or zero slope at the boundary are possible). The situation changes drastically in case

$$f(0) < 0. \tag{2.71}$$

A solution may develop a zero slope, and become sign-changing. In fact, a pitchfork bifurcation usually happens. In addition to the sign-changing symmetric solution, two symmetry-breaking solution emerge, see P. Korman [20].

Let us consider the problem

$$u''(x) + u^{2k}(x) - \lambda = 0, \;\; \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0, \tag{2.72}$$

which we will relate to the problems of type (2.70) shortly. Here $k \geq 1$ is an integer. For $k = 1$ this problem was exhaustively analyzed in J.C. Scovel's Ph.D. thesis [55], and in H. P. McKean and J.C. Scovel [44]. They used explicit integration via elliptic functions, which means that their method does not work for $k > 1$. It turned out that the solution set of (2.72) for $k = 1$ consists of infinitely many identically looking curves. Each curve is a parabola like curve, with pitchfork bifurcation on one of the branches. (I.e. there is exactly one turn, and exactly one point of pitchfork bifurcation on each curve, see Picture 2.) V. Anuradha and R. Shivaji [6] have studied a related problem. Using the quadrature technique, they showed existence of infinitely many points of bifurcation. In [20] P. Korman had used bifurcation theory to approach the problem (2.72), and in particular the case of $f(u) = u^{2k}$, with $k > 1$. We were able to generalize some, but not all, of the results of H. P. McKean and J.C. Scovel [44].

It is well known that at $\lambda = 0$ there exists a unique positive solution of (2.72). This solution is known to be non-degenerate, so that we can continue it for small $\lambda > 0$. Setting $u(x) = \mu v(x)$, with μ determined by the relation $\mu^{2k} = \lambda$, we convert the problem (2.72) into (a particular case of the problem (2.70))

$$(2.73)\quad v''(x) + \lambda(v^{2k}(x) - 1) = 0, \ \text{ for } -1 < x < 1, \ \ v(-1) = v(1) = 0,$$

where λ is a new parameter (equal to μ^{2k-1}). With the parameter now in front of the nonlinearity, the Lemma 2.8 applies, and hence we can always continue both positive and sign-changing solutions of (2.73) (and also of (2.72). Observe that the curve of positive solutions does not turn for $\lambda > 0$ (for $g(v) \equiv v^{2k} - 1$, we have $vg'(v) > g(v)$ for all $v > 0$). By Theorem 2.1 this curve of positive solutions cannot be continued for all $\lambda > 0$ (the function $g(v) = v^{2k} - 1$ has no "stable" roots, i.e. roots where derivative is negative). By the Sturm's comparison theorem, it is easy to see that positive solutions cannot become unbounded at a finite λ. Hence, solutions on this curve must eventually stop being positive, and the only way this can happen is that $u'(\pm 1) = 0$ at some λ_0 (in view of the symmetry of positive solutions).

We now outline the pitchfork bifurcation analysis for the general problem (2.70), and more details can be found in P. Korman [20]. So suppose the problem (2.70) has a curve of positive solutions $u(x, \lambda)$, so that for $\lambda < \lambda_0$ we have $u_x(1, \lambda) < 0$, while at $\lambda = \lambda_0$ we have $u_x(1, \lambda_0) = 0$. The function $u_x(x, \lambda_0)$ is then the solution of the corresponding linearized problem ($u_0 = u(x, \lambda_0)$)

$$(2.74)\quad w'' + \lambda_0 f'(u_0) w = 0 \quad \text{for } -1 < x < 1, \ \ w(-1) = w(1) = 0.$$

The null-space of the linearized problem is one-dimensional (by Lemma 2.5), and it is spanned by the odd function $u_x(x, \lambda_0)$. If we now restrict to the space of even functions, the null-space will be empty, and hence by the implicit function theorem the solution curve $u(x, \lambda)$ continues for $\lambda > \lambda_0$, as sign-changing symmetric (even) solutions. We can compute the tangential direction for this curve at $\lambda = \lambda_0$:

$$u_\lambda(x, \lambda_0) = x u_x(x, \lambda_0). \tag{2.75}$$

Indeed, the function $u_\lambda(x, \lambda_0) - x u_x(x, \lambda_0)$ is an even function, solving the linearized problem (2.74). Hence it must be zero, justifying (2.75). If we let $v = u - u(x, \lambda)$, where $u(x, \lambda)$ is the curve of sign-changing symmetric solutions, then for $\lambda > \lambda_0$ we have a trivial solution $v = 0$. We showed in [20] that the conditions of the Crandall-Rabinowitz theorem on bifurcation from the trivial solutions are satisfied at $\lambda = \lambda_0$, giving rise to a parabola-like curve of symmetry breaking solutions, see also M. Ramaswamy [51]. Their tangential direction is given by $u_x(x, \lambda_0)$.

One of the reasons we were not able to fully recover the beautiful results of McKean and Scovel [44], is that we could not tell the direction of the pitchfork bifurcation: which way the symmetry breaking solutions bifurcate, toward $\lambda > \lambda_0$ or $\lambda < \lambda_0$? Recently X. Hou, P. Korman and Y. Li [17] has given a computer assisted way (again computer assisted!) to settle this question. Here is their result.

Theorem 2.10 *Consider the problem (2.72), with $1 \le k \le 720$. Let λ_0 be the point of pitchfork bifurcation. (The value of λ_0 was explicitly computed in [17].) Then there is a negative $\bar\lambda = \bar\lambda(k) < 0$, so that the problem (2.72) has exactly two positive solutions for $\bar\lambda < \lambda < 0$, it has exactly one positive and one negative solution on $(0, \lambda_0)$. Moreover, there is a $\lambda_1(k) > \lambda_0$, so that the problem (2.72) has four solutions on (λ_0, λ_1), one negative (and symmetric), one sign-changing and symmetric (with $u(0) > 0$), and two asymmetric solutions.*

In Picture 2 we present a picture of pitchfork bifurcation from [17] (produced by X. Hou). We draw $u'(-1)$ as a function of λ. In that figure solid lines denote positive and negative solutions, the dashed line denotes sign-changing symmetric solutions, and the doted lines stand for the symmetry breaking solutions.

2.8 Sign-changing solutions

We consider sign-changing solutions of the two point problem

$$u'' + f(u) = 0 \quad \text{for } x \in (0,1), \;\; u(0) = u(1) = 0. \tag{2.76}$$

Picture 2 : **Pitchfork bifurcation**

Notice that we pose the problem over the interval $(0,1)$, since sign-changing solutions need not be symmetric. Also we do not have λ in front of $f(u)$ (one can think that λ is observed into $f(u)$). Corresponding linearized problem is

$$(2.77) \qquad w'' + f'(u)w = 0 \quad \text{for } x \in (0,1), \;\; w(0) = w(1) = 0.$$

The following result is from P. Korman and T. Ouyang [39].

Theorem 2.11 *Let $f \in C^2(R)$, and assume that either one of the following two inequalities holds*

$$(2.78) \qquad \frac{f(u)}{u} - f'(u) > 0 \;\; (< 0) \;\; \textit{for almost every } u \in R.$$

Then any solution of the problem (2.76), satisfying $u'(0) \neq 0$, is non-singular (i.e. (2.77) admits only the trivial solution).

Proof: Assume on the contrary that the problem (2.77) admits a non-trivial solution $w(x)$.
Step 1. We show that the number of roots of u and w inside $(0,1)$ differs by one. Assume for definiteness that $f'(u) > \frac{f(u)}{u}$ for almost every $u \in R$. If we regard (2.76) as a linear equation $u'' + \frac{f(u)}{u}u = 0$, then by the Sturm comparison theorem the function $w(x)$ has a root between any two roots of $u(x)$. Since both functions vanish at the endpoints, $x = 0$ and $x = 1$, it follows that w has one more interior root than u.

Step 2. We will show that u and w have the same number of interior roots. This will result in a contradiction, proving the theorem. We denote by n_u the number of interior roots of u, and use the same notation for other functions. The functions w and u' satisfy the same linear equation, and hence their roots are interlaced. Since w vanishes at the endpoints and u' does not, it follows that $n_{u'} = n_w + 1$. Since $n_{u'} = n_u + 1$, it follows that $n_u = n_w$. $\diamondsuit$

Corresponding to the solution $u(x)$ of (2.76) we may consider an eigenvalue problem

$$\varphi'' + f'(u)\varphi + \mu\varphi = 0 \quad \text{for } x \in (0,1), \;\; \varphi(0) = \varphi(1) = 0. \tag{2.79}$$

The eigenvalues of (2.79) form a sequence $\mu_1 < \mu_2 < \ldots < \mu_n < \ldots$, tending to infinity. The number of negative eigenvalues is called the *Morse index* of $u(x)$. The following theorem is from P. Korman and T. Ouyang [39].

Theorem 2.12 *Let $u(x)$ any solution of the problem (2.76), with k interior roots and satisfying $u'(0) \neq 0$. Then the Morse index of $u(x)$ is either k or $k+1$. Moreover, the Morse index equals k if the first inequality in (2.78) holds, and it equals $k+1$ if the second inequality in (2.78) holds.*

Similar results for balls in R^n have been given in J. Shi and J. Wang [57].

2.9 The Neumann problem

Consider the Neumann problem

$$u'' + \lambda f(u) = 0 \quad \text{for } 0 < x < 1, \;\; u'(0) = u'(1) = 0. \tag{2.80}$$

We are interested in the solution branches bifurcating off constant solutions. By translation we may assume the constant solution to be zero, i.e. we assume that

$$f(0) = 0, \tag{2.81}$$

and that $f(u) \in C^2(a^-, a^+)$ for some $-\infty \leq a^- < 0 < a^+ \leq \infty$. We assume that

$$uf(u) > 0 \;\text{ on } (a^-, a^+). \tag{2.82}$$

We consider solutions of (2.80) such that $u(x) \in (a^-, a^+)$. It suffices to consider only the *increasing* solutions of (2.80), i.e. $u'(x) > 0$ on $(0,1)$, since other solutions can be produced from them by reflection, pasting and scaling. Clearly we have $u(0) < 0 < u(1)$, since $x = 0$ and $x = 1$ are points of minimum and maximum respectively.

The corresponding linearized problem is

$$w'' + \lambda f'(u)w = 0 \quad \text{for } 0 < x < 1, \;\; w'(0) = w'(1) = 0. \tag{2.83}$$

If this problem has only the trivial solution, then the solution branches bifurcating from zero do not turn. A simple condition for this to happen goes back to Z. Opial [47], see also R. Schaaf [53]. Namely, we assume that either one of the following two inequalities holds

$$\frac{f(u)}{u} - f'(u) > 0 \;\; (< 0) \;\text{ for every } u \in (a^-, a^+) \setminus \{0\}. \tag{2.84}$$

It is an elementary exercise to show that (2.84) will follow if either one of the following two inequalities holds

$$uf''(u) > 0 \;\; (< 0) \;\text{ for every } u \in (a^-, a^+) \setminus \{0\}. \tag{2.85}$$

The following result is of course known, see [47] and [53], although previously it was stated in different terms (involving monotonicity of time maps), and proved by different methods.

Theorem 2.13 *Assume that the conditions (2.81), (2.82) and (2.84) (or (2.85)) hold, and $u(x) \in (a^-, a^+)$ for all $x \in (0,1)$. Then the linearized problem (2.83) admits only the trivial solution.*

Proof: Assume on the contrary that $w(x)$ is a non-trivial solution of (2.83). Observe that $u'(x)$ satisfies the same equation (2.83), $u'(x) > 0$ for $x \in (0,1)$ and $u'(0) = u'(1) = 0$. It follows by the Sturm comparison theorem that $w(x)$ has exactly one root on $(0,1)$; we call it η, i.e. $w(\eta) = 0$. We may assume (by scaling) that $w(0) < 0$ and $w(1) > 0$, and hence $w'(\eta) > 0$. Let ξ denote the unique root of the increasing solution $u(x)$, i.e. $u(\xi) = 0$ and $u'(\xi) > 0$. Writing the equation (2.80) in the form $u'' + \lambda \frac{f(u)}{u} u = 0$, and combining it with (2.83), we have

$$(u'w - uw')' + \lambda \left[\frac{f(u)}{u} - f'(u) \right] uw = 0. \tag{2.86}$$

We now consider two cases.

Case 1. $\xi \le \eta$. Assume that the first inequality holds in (2.84), i.e. the quantity in the square bracket in (2.86) is positive. We integrate (2.86) over the interval $(\eta, 1)$, where both $u(x)$ and $w(x)$ are positive

$$u(\eta)w'(\eta) + \lambda \int_\eta^1 \left[\frac{f(u)}{u} - f'(u) \right] uw \, dx = 0.$$

We have a contradiction, since both terms on the left are positive.

If the second inequality holds in (2.84), i.e. the quantity in the square bracket in (2.86) is negative, we integrate (2.86) over the interval $(0, \xi)$, where both $u(x)$ and $w(x)$ are negative

$$u(\xi)w'(\xi) + \lambda \int_0^\xi \left[\frac{f(u)}{u} - f'(u) \right] uw \, dx = 0.$$

Again, we have a contradiction, since both terms on the left are negative.

Case 2. $\xi \geq \eta$. If the first inequality holds in (2.84), we integrate over $(0,\eta)$, where both $u(x)$ and $w(x)$ are negative

$$-u(\eta)w'(\eta) + \lambda \int_0^\eta \left[\frac{f(u)}{u} - f'(u)\right] uw\, dx = 0.$$

Both terms on the left are positive, a contradiction. If the second inequality holds in (2.84), we integrate over $(\xi,1)$, where both $u(x)$ and $w(x)$ are positive

$$-u'(\xi)w(\xi) + \lambda \int_\xi^1 \left[\frac{f(u)}{u} - f'(u)\right] uw\, dx = 0.$$

Both terms on the left are negative, again we have a contradiction. $\diamondsuit$

Beyond this simple theorem, we know of only two results on the Neumann problem. The first one is due to R. Schaaf [53]. It dealt with monotonicity of time maps, here we rephrase it in terms of non-degeneracy of solutions.

Theorem 2.14 *([53]) Assume that the function $f(u)$ is either an $A - B$ or C function on the interval (a^-, a^+). Then the linearized problem (2.83) admits only the trivial solution.*

The other one is from P. Korman [23].

Theorem 2.15 *Assume that $f(u)$ satisfies $f'(u) > 0$ and $f'''(u) < 0$ on the interval $(0, a^+)$, and $f''(u) > 0$ on (a^-, a^+). Then the linearized problem (2.83) admits only the trivial solution.*

The last result is not very satisfactory. Its only advantage is that no third order assumptions on $f(u)$ are made on $(a^-, 0)$, while on $(0, a^+)$ such functions are of class $A - B$. R. Schaaf's result is better.

According to the condition (2.85), we can handle the cases when $f(u)$ changes concavity at its root $u = 0$. But what if it keeps the same concavity? We wish to pose the following problem.

Problem Assume that $f(0) = 0$, and

$$f''(u) > 0 \quad \text{for } u \in (a^-, a^+).$$

Is it true that any increasing solution of the Neumann problem (2.80), with values in (a^-, a^+), is non-degenerate (i.e. (2.83) has only the trivial solution)?

Using bifurcation approach, we can also treat some non-autonomous problems. For example,

$$u'' + \lambda b(x) f(u) = 0 \quad \text{for } 0 < x < 1, \;\; u'(0) = u'(1) = 0, \tag{2.87}$$

with the given function $b(x)$ being positive and continuous. The corresponding linearized problem is now

$$w'' + \lambda b(x) f'(u) w = 0 \quad \text{for } 0 < x < 1, \;\; w'(0) = w'(1) = 0. \tag{2.88}$$

The formula (2.86) still holds here (with an extra factor of $b(x)$ in front of the square bracket), and hence the arguments of the above theorem can be used unchanged. In particular, we conclude as above that any non-trivial solution of (2.88) cannot vanish exactly once. Unlike the autonomous problem, we cannot yet conclude that $w(x)$ is zero, since we cannot automatically exclude the possibilities that $w(x)$ has no roots, or at least two roots. We need to introduce another condition:

$$f'(u) \geq 0 \;\text{ for every } u \in (a^-, a^+). \tag{2.89}$$

Theorem 2.16 *Assume that $b(x)$ is positive and continuous on $[0, 1]$, and $f(u) \in C^1[a^-, a^+]$ satisfies the conditions (2.81), (2.82), (2.89), and the first inequality holds in (2.84). Let $u(x)$ be an increasing solution of the Neumann problem (2.80), satisfying $u(x) \in (a^-, a^+)$ for all $x \in (0, 1)$. Then the linearized problem (2.88) admits only the trivial solution.*

Proof: As we mentioned above, the arguments used in proof of Theorem 2.13 apply here as well. In case the first inequality holds in (2.84), we have proved in Theorem 2.13 that $w(x)$ cannot vanish on either side of ξ, the root of $u(x)$. Hence $w(x)$ keeps the same sign over $(0, 1)$. But then integrating the linearized equation (2.88),

$$\int_0^1 b(x) f'(u) w \, dx = 0,$$

which is a contradiction, since the integrand is of one sign. $\diamondsuit$

As an example, the function $f(u) = u - u^3$ satisfies the conditions of this theorem on the interval $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

2.10 Similarity of the solution branches

We saw in the previous two sections that under the same condition (2.84) we could prove non-degeneracy for both sign-changing solutions, and for the Neumann problem. It turns out that a curve of solutions with an odd number of sign changes is always similar to curves of solutions of Neumann problems. (I.e. both curves have the same number of critical points, with the same direction of turns.) Let us fix the notation, before we state the result. We consider the problem

$$u'' + \lambda f(u) = 0, \quad \text{on } (0,1), \tag{2.90}$$

subject to either Dirichlet

$$u(0) = u(1) = 0, \tag{2.91}$$

or Neumann

$$u'(0) = u'(1) = 0 \tag{2.92}$$

boundary conditions. We shall consider the solutions as the positive parameter λ varies, and refer to the solution curves as either Dirichlet or Neumann branches, depending on the boundary conditions used. Recall that by Lemma 2.1 any solution of the equation (2.90) is symmetric with respect to any of its critical points. This implies, in particular, that either minimum or maximum occurs at any critical point. It follows that any solution of Neumann problem is determined by its values on any subinterval $I \subset (0,1)$, whose end-points are two consecutive critical points of $u(x)$. We can then obtain the solution on the entire interval $(0,1)$ through reflections and translations. We refer to I, and any other interval uniquely determining the solution through reflections and translations, as a *determining interval.* The interval I, joining two consecutive critical points of $u(x)$, is also a determining interval for the Dirichlet problem. Another determining interval for Dirichlet problem is (ξ, η), where $0 \le \xi < \theta < \eta \le 1$ are three consecutive roots of $u(x)$. This interval contains both positive and negative humps (and all positive (negative) humps are translations of one another).

As we vary λ the number of roots on Dirichlet branches, as well as the number of monotonicity changes on Neumann branches, remain constant. Indeed, by Lemma 2.2 solutions of (2.90) cannot have points of positive maximum and negative minimum, and there is no other mechanism by which extra roots (or monotonicity changes) may be created.

The natural way to distinguish the Dirichlet branches is by the number of interior roots, and the Neumann branches can be identified by the number of changes of monotonicity (both properties remain constant on the solution curves). Any solution of the Dirichlet problem with at least one interior root contains a solution of the Neumann problem on a subinterval of $(0,1)$. Indeed, just consider the solution between two consecutive critical points. In order for solutions of the Neumann problem to contain in turn a solution of the Dirichlet problem, we need to impose some conditions on $f(u)$. Namely, we assume that

$$f(0) = 0, \tag{2.93}$$

and there exist two constants $-\infty \le m < 0 < M \le \infty$ so that
(f1) $f(u) > 0 \quad$ for $u \in (0, M)$,
(f2) $f(u) < 0 \quad$ for $u \in (m, 0)$.

Lemma 2.18 *Under the conditions (2.93), (f1) and (f2) any solution of the Neumann problem for (2.90), satisfying*

$$m < u(x) < M \quad \textit{for all } x \tag{2.94}$$

has a root between any two critical points.

Proof: Follows immediately, by multiplying the equation (2.90) by u', and integrating between any two consecutive critical points. $\diamondsuit$

Definition. We call two solution branches of (2.90) to be *similar* if for any solution on the either branch there is a determining interval so that by stretching of x, or by reflection $x \to 2a - x$, for some $a \in (0,1)$, we obtain a solution from the other branch on a (different) determining interval.
Clearly, if solution branches are similar then the corresponding solution curves in (λ, u) "plane" have the same shape.

The following result was proved in this form by P. Korman [24], although it can also be found in R. Schaaf [51].

Theorem 2.17 *All Neumann branches of (2.90) are similar, and if $f(u)$ satisfies the conditions (2.93), (f1) and (f2), while all solutions satisfy (2.94), then the Neumann branches are similar to the Dirichlet ones with an odd number of interior roots (and these Dirichlet branches are also all similar).*

Proof: We begin with Neumann branches. If a Neumann solution changes monotonicity twice, then its increasing part is a reflection of its decreasing part with respect to $x = \frac{1}{2}$. If a Neumann solution changes monotonicity n times, then all critical points occur at i/n, $i = 1, \dots, n-1$, and the graphs of solution on all intervals where it is increasing (decreasing) are translations of one another. Since an interval connecting any two critical points is a determining interval, the equivalence of the Neumann branches follows (via rescaling). $\diamondsuit$

If a Dirichlet solution has $2k-1$ interior roots, it has k identical positive humps and k identical negative humps. Assume for definiteness that solution starts with a negative hump, followed by a positive one, and so on. If ξ is the first point of (negative) minimum of $u(x)$, then the first interior root occurs at 2ξ. If $2\xi + \eta$ is the point of the first (positive) maximum, then the second interior root occurs at $2\xi + 2\eta$. The last critical point, a positive maximum, occurs at $1-\eta$. Observe that $k(2\xi+2\eta) = 1$, i.e. $\xi+\eta = \frac{1}{2k}$. So while both ξ and η vary with λ, $u(x)$ solves the Neumann problem on the interval $(\xi, 1-\eta)$, and this interval has a *fixed* length of

$$1 - \eta - \xi = \frac{2k-1}{2k}.$$

So that any Dirichlet solution curve "carries" inside it a solution of a Neumann problem on a fixed interval (which can be made to be $(0,1)$ by rescaling), and hence the Dirichlet branch cannot have any more complexity (like extra turns) than any Neumann branch.

Conversely, consider the Neumann problem with $2k+1$ changes of monotonicity. Assume for definiteness that $u(0) < 0$. Then $u(1) > 0$. Assume that $\xi = \xi(\lambda)$ is the smallest interior root, and $1-\eta$ is the largest one, $\eta = \eta(\lambda)$. On the interval $(0,1)$ we then have $2k+1$ negative half-humps, each of width ξ, and $2k+1$ positive ones, each of width η. So that $\xi + \eta = \frac{1}{2k+1}$. On the interval $(\xi, 1-\eta)$ we have a solution of the Dirichlet problem with $2k-1$ interior roots, and the length of this interval is

$$1 - \eta - \xi = \frac{2k}{2k+1},$$

which does not vary with λ. So that any Neumann branch "carries" inside it a solution of a Dirichlet problem on a fixed interval, and hence the Neumann branch cannot have any more complexity than the corresponding Dirichlet branch with an odd number of interior zeroes.

Finally, the Dirichlet branches with odd number of interior zeroes are all similar, since any two such branches are similar to a pair of Neumann branches, but Neumann branches are all similar. $\diamondsuit$

The Dirichlet branches with even number of interior zeroes may behave differently, as the following example due to R. Schaaf [53] shows.

Example ([53]) For the problem

$$u'' + \lambda(e^u - 1) = 0 \quad \text{on } (0,1), \;\; u(0) = u(1) = 0$$

the branch bifurcating from the principal eigenvalue does not turn, while all other branches have exactly one turn.

3 A class of symmetric nonlinearities

For the autonomous equation (2.1) both phase-plane analysis and bifurcation theory apply. If we allow explicit dependence of the nonlinearity on x, i.e. consider

$$(3.1) \qquad u'' + \lambda f(x,u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0,$$

then the problem becomes much more complicated. For example, solutions of the corresponding linearized problem need not be of one sign. In the papers P. Korman and T. Ouyang a class of $f(x,u)$ has been identified, for which the theory of positive solutions is very similar to that for the autonomous case, see e.g. [34], [35] and [36]. Further results in this direction have been given in P. Korman, Y. Li and T. Ouyang [30], and P. Korman and J. Shi [40]. Namely, we assume that $f \in C^2$ satisfies

$$(3.2) \qquad f(-x,u) = f(x,u) \quad \text{for all } -1 < x < 1, \text{ and } u > 0,$$

$$(3.3) \qquad f_x(x,u) \le 0 \quad \text{for all } 0 < x < 1, \text{ and } u > 0.$$

Under the above conditions any positive solution of (3.1) is an even function, with $u'(x) < 0$ for all $x \in (0,1]$, see B. Gidas, W.-M. Ni and L. Nirenberg [15]. (For the one-dimensional problem (3.1) a different proof of the symmetry of solutions is given in P. Korman [18]. It is a little simpler than the moving plane method of [15], and it allows to relax somewhat the condition (3.3).) As before the linearized problem

$$(3.4) \qquad w'' + \lambda f_u(x,u)w = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0$$

will be important for the multiplicity results.

Lemma 3.1 *([34]) Under the conditions (3.2), and (3.3) any non-trivial solution of (3.4) is of one sign. Moreover, $w(x)$ is an even function, and it spans the null set of (3.4).*

Proof: Assume that $w(x)$ has a root on $[0,1)$ (the case when $w(x)$ vanishes on $(-1,0]$ is similar). We may assume (taking $-w$ if necessary) that there is a subinterval (x_1, x_2), $0 \le x_1 < x_2 \le 1$, so that $w(x) > 0$ on (x_1, x_2), and $w(x_1) = w(x_2) = 0$. Integrating the relation $[u'w' - u''w]' = \lambda f_x w$ over (x_1, x_2),

$$u'(x_2)w'(x_2) - u'(x_1)w'(x_1) = \lambda \int_{x_1}^{x_2} f_x w \, dx.$$

We have a contradiction, since the quantity on the left is positive, while the one on the right is non-positive.

The null set of (3.4) is one dimensional, since it can be parameterized by $w'(1)$. To prove that $w(x)$ is even, observe that $w(-x)$ is also a solution of (3.1), and hence $w(-x) = cw(x)$ for some constant c (since the null set of (3.4) is one dimensional). Evaluating this at $x = 0$, we conclude that $c = 1$ (since $w(0) > 0$), and the claim follows. ♢

The next lemma shows that the Crandall-Rabinowitz Theorem 1.2 applies at any critical solution.

Lemma 3.2 *Under the conditions (3.2), and (3.3) let $u(x)$ be a critical solution of (3.1), and $w(x)$ a solution of the corresponding linearized problem. Then we have*

$$\int_0^1 f(x,u) w \, dx > \frac{1}{2\lambda} u'(1)w'(1) > 0. \tag{3.5}$$

Proof: By the preceeding lemma we may assume that $w(x) > 0$. We then have

$$(u''w - u'w')' = -\lambda f_x w > 0 \text{ for } x > 0.$$

So that the function $u''w - u'w'$ is increasing on $(0,1)$, and then

$$u''w - u'w' < -u'(1)w'(1) \text{ for } x > 0.$$

Integrating this over $(0,1)$, and expressing u'' from the equation (3.1), we conclude (3.5). ♢

The following result from P. Korman and J. Shi [40] is an extension of Lemma 2.3. Unlike the autonomous case, several conditions are now needed.

Theorem 3.1 *(See [40]) In addition to (3.3) and (3.4) assume that*

$$f(x,u) > 0 \quad \textit{for all } -1 < x < 1, \textit{ and } u > 0. \tag{3.6}$$

Then the set of positive solutions of (3.1) can be parameterized by their maximum values $u(0)$. (I.e. $u(0)$ uniquely determines the pair $(\lambda, u(x))$.)

Proof: Assume on the contrary $v(x)$ is another solution of (3.1), corresponding to some parameter $\mu \geq \lambda$, but $u(0) = v(0)$. The case of $\mu = \lambda$ is not possible in view of uniqueness of initial value problems, so assume that $\mu > \lambda$. Then $v(x)$ is a supersolution of (3.1), i.e.

$$v'' + \lambda f(x, v) < 0 \quad \text{for } -1 < x < 1, \quad v(-1) = v(1) = 0. \tag{3.7}$$

Since $v''(0) < u''(0)$, it follows that $v(x) < u(x)$ for $x > 0$ small. Let $0 < \xi \leq 1$ be the first point where the graphs of $u(x)$ and $v(x)$ intersect (i.e. $v(x) < u(x)$ on $(0, \xi)$). We now multiply the equation (3.1) by u', and integrate over $(0, \xi)$. Denoting by $x_2(u)$ the inverse function of $u(x)$ on $(0, \xi)$, we have

$$\frac{1}{2} u'^2(\xi) + \int_{u(0)}^{u(\xi)} f(x_2(u), u) \, du = 0. \tag{3.8}$$

Similarly denoting by $x_1(u)$ the inverse function of $v(x)$ on $(0, \xi)$, we have from (3.7)

$$\frac{1}{2} v'^2(\xi) + \int_{u(0)}^{u(\xi)} f(x_1(u), u) \, du > 0. \tag{3.9}$$

Subtracting (3.9) from (3.8), noticing that $x_2(u) > x_1(u)$ for all $u \in (u(\xi), u(0))$, and using the condition (3.3), we have

$$\frac{1}{2} \left[u'^2(\xi) - v'^2(\xi) \right] + \int_{u(\xi)}^{u(0)} \left[f(x_1(u), u) - f(x_2(u), u) \right] du < 0. \tag{3.10}$$

Since both terms on the left are positive, we obtain a contradiction. $\diamondsuit$

Next we consider positive solutions of the boundary value problem

$$u'' + \lambda b(x) f(u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0. \tag{3.11}$$

We assume that $b(x) \in C^1[-1, 1]$ satisfies $b(x) > 0$ for $x \in [-1, 1]$, and $b(x) = b(-x)$, $b'(x) < 0$ for $x \in (0, 1)$. We also assume that $f(u) > 0$, so that this problem belongs to the class discussed above. For any solutions $u(x)$ let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. we assume that $w(x) > 0$ satisfies

$$\begin{gathered} w'' + \lambda b(x) f'(u) w + \mu w = 0 \quad \text{for } -1 < x < 1, \\ w(-1) = w(1) = 0. \end{gathered} \tag{3.12}$$

The following theorem is taken from P. Korman and J. Shi [40].

Theorem 3.2 *Assume $f \in C^2[0,\infty)$, $f(u) > 0$, $f'(u) > 0$ and $f''(u) > 0$ for all $u > 0$, and for some $\alpha > 0$ the condition (2.43) is satisfied. Then the solution of (3.11) with $u(0) = \alpha$ is unstable if it exists.*

Proof: In the proof of Theorem 2.3, (2.25) and (2.26) are still true. Assume now that $u(x)$ is stable, i.e. $\mu \geq 0$ in (3.12). Then $w(x)$ is a positive solution of the problem

$$w'' + g(x,w) = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0, \tag{3.13}$$

with $g(x,w) = \lambda b(x) f'(u(x))w + \mu w$. Since $g(x,w)$ is even in x, and

$$g_x = \lambda b'(x) f'(u) w + \lambda b(x) f''(u) u' w < 0 \quad \text{on } (0,1),$$

the theorem of B. Gidas, W.-M. Ni and L. Nirenberg [15] applies to (3.13). It follows that $w(x)$ is an even function with $w'(x) < 0$ on $(0,1)$. Recall that $w(x)$ is determined up to a constant multiple. Since $w(x)$ is decreasing, while $-u'(x)$ is increasing on $(0,1)$, by scaling $w(x)$ we can achieve (2.28). Using (2.25), (2.28), and also (2.26), we have (2.29).

Since $b(x) > 0$, $b'(x) < 0$ in $(0,1)$ using (2.25) and (2.29), we have

$$\begin{aligned}
& \textstyle\int_0^1 b(x)\,[f(u) - uf'(u)]\,w(x)\,dx \\
= \quad & \textstyle\int_0^{x_0} b(x)\,[f(u) - uf'(u)]\,w(x)\,dx \\
& \textstyle + \int_{x_0}^1 b(x)\,[f(u) - uf'(u)]\,w(x)\,dx \\
< \quad & \textstyle\int_0^{x_0} b(x_0)\,[f(u) - uf'(u)]\,w(x)\,dx \\
& \textstyle + \int_{x_0}^1 b(x_0)\,[f(u) - uf'(u)]\,w(x)\,dx \\
= \quad & \textstyle b(x_0) \int_0^1 [f(u) - uf'(u)]\,w(x)\,dx \leq 0.
\end{aligned} \tag{3.14}$$

On the other hand, multiplying the equation (3.12) by u, the equation (3.1) by w, subtracting and integrating over $(0,1)$, we have

$$\int_0^1 b(x)\,[f(u) - uf'(u)]\,w(x)\,dx = \frac{\mu}{\lambda} \int_0^1 uw\,dx \geq 0. \tag{3.15}$$

We reach a contradiction by combining (3.14) and (3.15). $\diamondsuit$

As an application we have the following exact multiplicity result from P. Korman and J. Shi [40]. It extends the corresponding result in [34] by not restricting the behavior of $f(u)$ at infinity. The Theorem 3.1 above allows us to conclude the uniqueness of the solution curve.

Theorem 3.3 *We assume that $b(x) \in C^1[-1,1]$ satisfies $b(x) > 0$ for $x \in [-1,1]$, and $b(x) = b(-x)$, $b'(x) < 0$ for $x \in (0,1)$. Assume $f \in C^2[0,\infty)$, $f(u) > 0$, $f'(u) > 0$ and $f''(u) > 0$ for all $u > 0$, while $h(\alpha) \le 0$ for some $\alpha > 0$. Then there exist two constants $0 \le \bar{\lambda} < \lambda_0$, so that the problem (3.11) has no solution for $\lambda > \lambda_0$, exactly two solutions for $\bar{\lambda} < \lambda < \lambda_0$, and in case $\bar{\lambda} > 0$ it has exactly one solution for $0 < \lambda < \bar{\lambda}$. Moreover, all solutions lie on a unique smooth solution curve. If we moreover assume that $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, then $\bar{\lambda} = 0$.*

Example. The theorem 3.3 applies (with $\bar{\lambda} = 0$) to an example from combustion theory

$$u'' + \lambda b(x) e^u = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0,$$

where $b(x)$ satisfies the above conditions.

P. Korman and T. Ouyang [34] have considered a class of *indefinite* problems

$$(3.16)\; u''(x) + \lambda u(x) + h(x)u^p(x) = 0, \;\; -1 < x < 1, \;\; u(-1) = u(1) = 0.$$

Here $p > 1$, and λ a real parameter. The given function $h(x)$ is assumed to be even, and it is allowed to change sign on $(-1,1)$. By using bifurcation analysis, as above, as well as earlier work of T. Ouyang [48], [49], it was possible to give an exhaustive description of the set of positive solutions of (3.16).

We denote by $\phi_1 = \cos\frac{\pi}{2}x$, the principal eigenfunction of $-u''$ on $(-1,1)$, corresponding to the principal eigenvalue $\lambda_1 = \frac{\pi^2}{4}$. We assume that $h(x) \in C^1(-1,1) \cap C^0[-1,1]$ is an even function, and moreover

$$h(0) > 0, \quad \text{and } \; h'(x) < 0 \;\text{ for } x \in (0,1), \tag{3.17}$$

$$\int_{-1}^{1} h(x)\phi_1^{p+1}(x)\,dx < 0. \tag{3.18}$$

(Notice that the last assumption implies that $h(x)$ changes sign.)

Theorem 3.4 *([34]) Assume that the conditions (3.17) and (3.18) hold for the problem (3.16). Then there is a critical λ_0, $\lambda_0 > \lambda_1$, so that for $-\infty < \lambda \le \lambda_1$ the problem (3.16) has exactly one positive solution, it has exactly two positive solutions for $\lambda_1 < \lambda < \lambda_0$, exactly one at $\lambda = \lambda_0$, and no positive solutions for $\lambda > \lambda_0$. Moreover, all positive solutions lie on a unique* *continuous in λ curve, which bifurcates from zero at $\lambda = \lambda_1$ (to the right), it continues without any turns to $\lambda = \lambda_0$, at which it turns to the left, and then continues without any more turns for all $-\infty < \lambda \le \lambda_0$. We also have* $\max_x u(x) \to \infty$ *as* $\lambda \to -\infty$.

As far as we know, this is still the only known exact multiplicity result for indefinite problems.

Let us mention next the cubic problems

$$u''+\lambda(u-a(x))(u-b(x))(c(x)-u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0, \tag{3.19}$$

with given even functions $0 \le a(x) \le b(x) \le c(x)$. As we discussed above, for constant a, b and c, the exact multiplicity question has been settled only recently by P. Korman, Y. Li and T. Ouyang [33], via a computer assisted proof. One can expect that under some conditions the same global picture holds for variable coefficients. This was established for several special cases. P. Korman, Y. Li and T. Ouyang [30] have given an exact multiplicity result in case $a = b = 0$. P. Korman and T. Ouyang [36] had done the same in case $a = 0$, and P. Korman and T. Ouyang [38] had given an exact multiplicity result in case when $a > 0$ is a constant.

Next we indicate an extension. Consider a problem with a variable diffusion coefficient

$$(a(x)u')' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0. \tag{3.20}$$

We assume that a given function $a(x) \in C^1[-1,1]$ is even, and it satisfies

$$a(x) > 0 \;\text{ and }\; xa'(x) \le 0, \;\text{ for } x \in [-1,1]. \tag{3.21}$$

We perform a change of variables $x \to s$, given by

$$s = \int_0^x \frac{d\,t}{a(t)}.$$

If we denote by $s_0 = \displaystyle\int_0^1 \frac{d\,t}{a(t)}$, then this transformation gives a one-to-one map of the interval $(-1,1)$ onto $(-s_0, s_0)$. Moreover, $s > 0$ (< 0) iff $x > 0$ (< 0). The problem (3.20) transform into

$$u_{ss} + \lambda a(x(s))f(u) = 0 \quad \text{for } -s_0 < x < s_0, \;\; u(-s_0) = u(s_0) = 0. \tag{3.22}$$

Observe that the function $s = s(x)$ is odd, and hence its inverse $x = x(s)$ is also odd, and then $a(x(s))$ is even. In view of (3.21)

$$\frac{d}{ds}a(x(s)) = a'(x(s))a(x(s)) \leq 0 \ \ (\geq 0) \ \text{ if } s > 0 \ (< 0).$$

If we now assume that $f(u) > 0$ for $u > 0$, then the problem (3.22) satisfies the conditions (3.2) and (3.3). Hence, we can translate our results, in particular the Theorem 3.3, to the problem (3.20).

4 General nonlinearities

Without the symmetry assumptions on $f(x,u)$ the problem is much harder. We restrict to a subclass of such problems, i.e. we now consider positive solutions of the boundary value problem

$$u'' + \lambda\alpha(x)f(u) = 0 \quad \text{for } a < x < b, \quad u(a) = u(b) = 0, \tag{4.1}$$

on an arbitrary interval (a,b). We assume that $f(u)$ and $\alpha(x)$ are positive functions of class C^2, i.e.

$$f(u) > 0 \ \text{ for } u > 0, \ \ \alpha(x) > 0 \ \text{ for } x \in [a,b]. \tag{4.2}$$

As before, it will be crucial for bifurcation analysis to prove positivity for the corresponding linearized problem

$$w'' + \lambda\alpha(x)f'(u)w = 0 \quad \text{for } a < x < b, \quad w(a) = w(b) = 0. \tag{4.3}$$

The following result was proved in P. Korman and T. Ouyang [38], although our exposition here is a little different.

Lemma 4.1 *In addition to the conditions (4.2), assume that*

$$\frac{3}{2}\frac{\alpha'^2}{\alpha} - \alpha'' < 0 \quad \textit{for all } x \in (a,b). \tag{4.4}$$

If the linearized problem (4.3) admits a non-trivial solution, then we may assume that $w(x) > 0$ on (a,b).

Proof: Let $z(x) = g(x)u'(x)$, with $g(x)$ to be chosen shortly. Then $z(x)$ satisfies the equation

$$z'' + \lambda\alpha(x)f'(u)z = g''(x)u'(x) - \lambda\left(2g'(x)\alpha(x) + \alpha'(x)g(x)\right)f.$$

We now chose $g(x) = \alpha(x)^{-1/2}$. Then $2g'(x)\alpha(x) + \alpha'(x)g(x) = 0$, while

$$g''(x) < 0 \quad \text{for } a < x < b, \tag{4.5}$$

in view of the condition (4.4).

Notice that any positive solution of (4.1) is a concave function, and hence it has only one critical point, the point of global maximum. Let x_0 be the point of maximum of $u(x)$. We have

$$z'' + \lambda\alpha(x)f'(u)z = g''(x)u'(x), \tag{4.6}$$

with the right hand side negative on (a, x_0) and positive on (x_0, b). This will make it impossible for $w(x)$ to vanish inside (a, b). Indeed, if we assume that $w(x)$ vanishes on say (x_0, b), we could find two consecutive roots of $w(x)$, $x_0 \le x_1 < x_2 \le b$ so that $w(x_1) = w(x_2) = 0$, while $w(x) > 0$ on (x_1, x_2). We now multiply the equation (4.6) by $w(x)$, the equation (4.3) by $z(x)$, subtract and integrate over (x_1, x_2), obtaining

$$-g(x_2)u'(x_2)w'(x_2) + g(x_1)u'(x_1)w'(x_1) = \int_{x_1}^{x_2} g''(x)u'(x)w(x)\,dx.$$

We have a contradiction, since the quantity on the left is negative, and the integral on the right is positive. $\diamondsuit$

Remarks

1. Recall the Schwarzian derivative from Complex Analysis and Dynamical Systems
$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$
If one denotes $A(x) = \int \alpha(x)\,dx$, then our condition (4.4) says that the Schwarzian derivative of $A(x)$ is positive.

2. Semilinear equations on an annulus in R^n, $n > 2$, can be reduced by a standard change of variables to the problem (4.1), with $\alpha(x) = x^{-2k}$ and $k = 1 + \frac{1}{n-2}$, see e.g. [19]. One sees that our condition (4.4) just misses this kind of functions. In [19] positivity of $w(x)$ was proved under an extra assumption that the annulus is "thin".

We shall present a new result on positivity of $w(x)$, after we prove a simple lemma.

Lemma 4.2 *Assuming the conditions (4.2), let x_0 be the unique point of maximum of the positive solution of (4.1). Assume that*

$$\alpha'(x) < 0 \quad \text{on } (x_0, b). \tag{4.7}$$

If the corresponding linearized problem (4.3) admits a non-trivial solution $w(x)$, then this solution cannot vanish inside (x_0, b).

Proof: Assuming the contrary, let γ be the largest root of $w(x)$ on (x_0, b), and assume that $w(x) > 0$ on (γ, b). (The number of roots of $w(x)$ inside (a, b) is at most finite, as follows by the Sturm's comparison theorem, since both functions $f'(u(x))$ and $\alpha(x)$ are bounded on $[a, b]$, and λ is fixed. Hence, there is a largest root γ.) Differentiate the equation (4.1)

$$u_x'' + \lambda\alpha(x)f'(u)u_u + \lambda\alpha'(x)f(u) = 0. \tag{4.8}$$

Multiplying the equation (4.8) by $w(x)$, the equation (4.3) by $u'(x)$, subtracting and integrating, we have

$$-u'(b)w'(b) + u'(\gamma)w'(\gamma) + \lambda\int_\gamma^b \alpha'(x)f(u(x))w(x)\,dx = 0.$$

This results in a contradiction, since all terms on the left are negative. $\diamondsuit$

Remark If $\alpha'(x) > 0$ on (a, x_0), then a similar proof shows that $w(x)$ cannot vanish inside (a, x_0).

Lemma 4.3 *Assume the conditions (4.2) hold, and in addition assume that*

$$\alpha'(x) < 0 \quad \text{on } (a, b), \tag{4.9}$$

and

$$2\alpha(x) + x\alpha'(x) > 0 \quad \text{on } (a, b). \tag{4.10}$$

If the linearized problem (4.3) admits a non-trivial solution, then we may assume that $w(x) > 0$ on (a, b).

Proof: Let x_0 be the unique point of maximum of the solution $u(x)$. By the previous Lemma 4.2 it follows that $w(x)$ cannot vanish on (x_0, b). Assuming that $w(x)$ vanishes on $(a, x_0]$, let $\gamma \in (a, x_0]$ be the first root of $w(x)$, and we may assume that $w(x) > 0$ on (a, γ). We consider the function $\zeta(x) = x\left[u'(x)w'(x) + \lambda\alpha(x)f(u(x))w(x)\right] - u'(x)w(x)$, introduced by M. Tang [59]. One computes

$$\zeta'(x) = \lambda\left[2\alpha(x) + x\alpha'(x)\right]f(u)w. \tag{4.11}$$

Integrating over (a, γ),

$$\gamma u'(\gamma)w'(\gamma) - au'(a)w'(a) = \lambda \int_a^\gamma \left[2\alpha(x) + x\alpha'(x)\right] f(u)w \, dx.$$

We have a contradiction, since the quantity on the left is negative, while the integral on the right is positive. $\diamondsuit$

Remark Our condition (4.10) again just misses the case of an annulus.

Positivity of $w(x)$ can be used to prove uniqueness and exact multiplicity results. For example, we can prove the following theorem.

Theorem 4.1 *For the problem (4.1) assume that the conditions (4.2) hold, and that either the condition (4.4) holds, or the conditions (4.9) and (4.10) hold. In addition assume that $f''(u) > 0$ for all $u > 0$, and $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$. Then there is a critical $\lambda_0 > 0$, so that the problem (4.1) has exactly two positive solutions for $0 < \lambda < \lambda_0$, exactly one positive solution at $\lambda = \lambda_0$, and no positive solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a unique smooth solution curve, which starts at $(\lambda = 0, u = 0)$, bends back at $\lambda = \lambda_0$, and tends to infinity as $\lambda \to 0$.*

Proof: The proof is similar to that of the Theorem 2.2, except for proving the uniqueness of the solution curve (since the maximum value of the solution no longer identifies that solution). However, if another solution curve existed, one of its ends would have to go through the point $(\lambda = 0, u = 0)$, contradicting the uniqueness of solutions near regular points, which follows by the implicit function theorem. (Since $w > 0$ at the turning point, one of the branches is increasing in λ, i.e. it is decreasing for decreasing λ. By the arguments of [34], or [35], the monotonicity is preserved along the branch, and hence this branch must go into the origin, as $\lambda \to 0$.) $\diamondsuit$

Example The theorem applies to the problem

$$u'' + \lambda\alpha(x)e^u = 0, \quad a < x < b, \quad u(a) = u(b) = 0,$$

if $\alpha(x) > 0$ satisfies either the condition (4.4), or the conditions (4.9) and (4.10).

5 Time Maps

5.1 There are several different formulas for the time map

Let $u = u(t)$ be solution of the initial value problem,

$$u'' + f(u) = 0, \quad u(0) = 0, \quad u'(0) = p.$$

Using ballistic analogy, we can interpret this as "shooting" from the ground level, at an angle $p > 0$. Let $T/2$ denote the time it takes for the projectile to reach its maximum amplitude α, $\alpha = \alpha(p)$. By symmetry of positive solutions, $T = T(p)$ is then the time when the projectile falls back to the ground, the *time map*. Since the energy is constant (as before, $F(u) = \int_0^u f(t)\,dt$)

$$\frac{1}{2}\left(\frac{du}{dt}\right)^2 + F(u(t)) = F(\alpha) = \frac{1}{2}p^2.$$

Solving this for $\frac{dt}{du}$, and integrating

$$T/2 = \frac{1}{\sqrt{2}}\int_0^{\alpha} \frac{du}{\sqrt{F(\alpha) - F(u)}}\,, \tag{5.1}$$

which lets us compute $T = T(\alpha)$ (or $T = T(p)$, since $\alpha = \alpha(p)$). This formula has been used extensively for a long time, see e.g. W.S. Loud [43], T. Laetsch [41], K.J. Brown et al [9], J. Smoller and A. Wasserman [58], S.-H. Wang [60], [61], I. Addou [1], [2], I. Addou and S.-H. Wang [3], S.-H. Wang and T.S. Yeh [64], and J. Cheng [10], [11]. It is not easy to use this formula. The integral is improper at $u = \alpha$, so that one needs a regularizing substitution before differentiating in α. One regularizing substitution is $u = \alpha \sin\theta$, which gives

$$T/2 = \frac{\alpha}{\sqrt{2}}\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{F(\alpha) - F(\alpha\sin\theta)}}\,. \tag{5.2}$$

The integrand is now bounded. The formula (5.2) can be used for numerical computations, as well as for proving theorems. For more information we refer the reader to the above mentioned papers, particularly to the recent papers of S.-H. Wang and his coworkers.

5.2 Time map formula through global linearization

We are interested in positive solutions of the two point problem for $u = u(t)$

$$u'' + f(u) = 0, \quad 0 < t < T, \;\; u(0) = u(T) = 0. \tag{5.3}$$

We do not consider the end point T to be fixed, but rather depending on $p = u'(0)$ (or on the maximum value of the solution α, $\alpha = u(T/2)$). To obtain the formula for $T = T(p)$, we begin by transforming (5.3) into the system form

$$\begin{aligned} u' &= y \\ y' &= -f(u), \end{aligned} \tag{5.4}$$

together with the initial conditions

$$u(0) = 0, \ y(0) = p. \tag{5.5}$$

Let $F(u) = \int_0^u f(t)\,dt$. In the linear case when $f(u) = u$, we have $F(u) = \frac{1}{2}u^2$. We now define the function $g(x)$, for $x \geq 0$, by

$$F(g(x)) = \frac{1}{2}x^2. \tag{5.6}$$

In other words, $g(x) = F^{-1}(\frac{1}{2}x^2)$, and the inverse function F^{-1} is defined, provided we assume throughout this section that $f(u) \in C^2\,(0,a) \cap C[0,a]$ for some $0 < a \leq \infty$, and

$$f(u) > 0 \quad \text{for } u \in (0,a). \tag{5.7}$$

We assume also that

$$\text{either} \ \ f(0) > 0, \quad \text{or} \quad f(0) = 0 \ \text{ and } \ f'(0) > 0. \tag{5.8}$$

Differentiate (5.6)

$$f(g(x))g'(x) = x. \tag{5.9}$$

In (5.4) we let $u = g(x)$, then multiply the second equation by $g'(x)$, and use (5.9)

$$\begin{gathered} g'(x)x' = y \\ g'(x)y' = -f(g(x))g'(x) = -x. \end{gathered} \tag{5.10}$$

We now change the independent variable in (5.10), $t \to \theta$, by solving

$$\frac{d\,t}{d\,\theta} = g'(x(t)), \ \ t(0) = 0. \tag{5.11}$$

Then the system (5.4) is linearized, and the problem (5.4), (5.5) transforms into

$$\begin{gathered} \tfrac{d\,x}{d\,\theta} = y \\ \tfrac{d\,y}{d\,\theta} = -x \\ x(0) = 0, \ y(0) = p. \end{gathered} \tag{5.12}$$

Solution of (5.12) is

$$x = p\sin\theta, \quad y = p\cos\theta.$$

Using this in (5.11) and integrating, we have the formula for the time map

$$T = \int_0^\pi g'(p \sin\theta)\, d\theta. \tag{5.13}$$

This formula was derived by R. Schaaf [53], and was used by her to obtain a number of uniqueness and multiplicity results.

Separating variables in (5.11) and integrating

$$\int_0^T \frac{f(g(x(t)))}{x(t)}\, dt = \pi, \tag{5.14}$$

where we have used (5.9) to express g'. From the definition of $g(x)$ we have

$$x = \sqrt{2F(g(x))} = \sqrt{2F(u)},$$

and hence we can rewrite (5.14) as

$$\int_0^T \frac{f(u(t))}{\sqrt{F(u(t))}}\, dt = \sqrt{2}\pi. \tag{5.15}$$

This formula was derived in a different way by P. Korman and Y. Li [28], where the quantity on the left was referred to as "generalized average" of the solution of (5.3). The reason why this term was chosen is that in case $f(u) = u^3$, this formula gives the average value of the solution: $\int_0^T u(t)\, dt = \frac{\pi}{\sqrt{2}}$.

Remarks

1. We needed the positivity of $f(u)$ so that the inverse function F^{-1} is defined, however there is no need to distinguish between $f(0) = 0$ and $f(0) > 0$ cases for both formulas (5.13) and (5.15). In case $f(0) > 0$ the integral in (5.15) (and in (5.14)) is improper at both end points, however since $u'(0) \neq 0$ and $u'(T) \neq 0$, the integral converges. (For small t, $u(t) \sim u'(0)t$, $F(u(t)) \sim f(0)u(t) \sim f(0)u'(0)t$.)

2. Similarly, in the derivation of the time map formula (5.13), we run into an improper integral in case $f(0) > 0$. Indeed, when solving for $\theta = \theta(t)$ in (5.11), we have $\theta = \int_0^t \frac{ds}{g'(x(s))}$, which is an improper integral at $s = 0$. However, as we have just seen, it is a convergent integral. Hence the time map formula in (5.13) is valid in both cases $f(0) = 0$, and $f(0) > 0$.

3. Let us collect the properties of the function $g(x)$. We have $g(0) = 0$, $g'(x) = \frac{\sqrt{2F(g(x))}}{f(g(x))} > 0$ for $x > 0$. We also have $g'(0) = 0$ in case $f(0) > 0$, and, by L'Hopital's rule as was observed in R. Schaaf [53], $g'(0) = \frac{1}{\sqrt{f'(0)}}$ in case $f(0) = 0$. Observe that $g'(0)$ is defined, thanks to the condition (5.8).

It is sometimes more convenient to express $T = T(\alpha)$, where α is the maximum value of the solution, $\alpha = u(T/2)$. Since the energy $\frac{1}{2}{u'}^2(x) + F(u(x))$ is constant, it follows that $p = \sqrt{2F(\alpha)}$, and hence

$$T/2 = \int_{\pi/2}^{\pi} g'(p \sin\theta)\, d\theta = \int_0^{\pi/2} g'\left(\sqrt{2F(\alpha)} \cos\theta\right)\, d\theta. \tag{5.16}$$

$T/2$ is, of course, the time it takes the solution to travel from its maximum to zero.

Example Consider the problem ($u = u(x)$)

$$u'' + \lambda e^u = 0, \;\; x \in (0,1), \;\; u(0) = u(1) = 0. \tag{5.17}$$

By a change of independent variable we convert it to

$$u'' + e^u = 0, \;\; x \in (0,1), \;\; u(0) = u(T) = 0, \tag{5.18}$$

where $T = \sqrt{\lambda}$. Here $f(u) = e^u$, $F(u) = e^u - 1$, and $g(x) = \ln\left(\frac{1}{2}x^2 + 1\right)$. The integral in (5.16) is then relatively simple, and in fact *Mathematica* gives

$$\sqrt{\lambda}/2 = \sqrt{\frac{2}{e^\alpha}} ArcTanh[\sqrt{\frac{e^\alpha - 1}{e^\alpha}}]. \tag{5.19}$$

Plotting this formula (with λ along the horizontal axis and α along the vertical one), we obtain the same bifurcation diagram as obtained by standard integration. Computation this way is considerably faster than by integration, and we also observe that here $f(0) > 0$. The formula (5.19) can also be obtained by explicit integration of the problem (5.17).

We can proceed similarly for the general case

$$u'' + \lambda f(u) = 0, \;\; x \in (0,1), \;\; u(0) = u(1) = 0. \tag{5.20}$$

We do not have a simple formula for $g(x)$ anymore, however from (5.16) (see also (5.9)) we obtain (as before, $T = \sqrt{\lambda}$)

$$\sqrt{\lambda}/2 = \int_0^{\pi/2} \frac{\sqrt{2F(\alpha)} \cos\theta}{f\left(F^{-1}\left(F(\alpha)\cos^2\theta\right)\right)}\, d\theta. \tag{5.21}$$

This formula will provide probably one of the most efficient ways to compute the bifurcation diagrams, once the evaluation of the inverse function F^{-1} is numerically implemented.

Assume now there is an $a > 0$, so that $f(a) = 0$, and $f(u) > 0$ for $u > a$, while no assumptions on the sign of $f(u)$ are made when $u \in (0, a)$. If we denote by $T_1/2$ the time it takes the solution to travel from its maximum to $u(x) = a$, then

$$T_1/2 = \int_{\theta_0}^{\pi/2} g'(\sqrt{2F(\alpha)}\cos\theta)\, d\theta, \quad \text{where } \theta_0 = sin^{-1}\sqrt{\tfrac{F(a)}{F(\alpha)}}. \tag{5.22}$$

The following theorem we proved in [27].

Theorem 5.1 *Assume that for some* $0 < a < b \leq \infty$ *we have*

$$f(u) > 0, \quad \textit{for } a < u < b, \tag{5.23}$$

$$f'(u)\int_a^u f(t)\, dt - \frac{1}{2}f^2(u) > 0, \quad \textit{for } a < u < b. \tag{5.24}$$

(Observe that we implicitly assume that $f(a) = 0$*.) Then the problem*

$$u'' + f(u) = 0, \quad x \in (0,1), \quad u(0) = u(1) = 0$$

has at most one positive solution, with $a < \alpha = u(1/2) < b$*.*

What is remarkable here is than no assumptions whatsoever are made on $f(u)$ when $u \in (0, a)$. We used generalized averages to prove this result, but it should be possible to obtain it from the formula (5.22) too. In fact, there is a similar result in R. Schaaf's book [53] (a little less general than the above theorem). Observe that (5.24) will follow if $f(a) = 0$ and $f''(u) > 0$ for $a < u < b$. A more general result for p-Laplacian case has been given recently by J. Cheng [10].

We now show how the time map formula gives rise to uniqueness and multiplicity results for the Dirichlet problem. Compute

$$\frac{dT}{dp} = \int_0^\pi g''(p\sin\theta)\sin\theta\, d\theta, \tag{5.25}$$

where $g''(x)$ is computed from (5.9) (written in the form $g'(x) = \frac{\sqrt{2F(u)}}{f(u)}$, $u = g(x)$)

$$g''(x) = \frac{f^2 - 2Ff'}{f^3}(u), \quad \text{with } u = g(x). \tag{5.26}$$

If the time map $T(p)$ is monotone, then clearly the positive solution of the Dirichlet problem (5.3) for any *fixed* T is unique. Hence, we have uniqueness of solutions if either

$$I(u) \equiv f'(u)F(u) - \frac{1}{2}f^2(u) > 0 \quad \text{for almost all } u > 0, \tag{5.27}$$

or the opposite inequality holds. This condition was derived by R. Schaaf [53], and it also follows from the generalized averages in [28].

We observe next that this condition does not add anything to the standard uniqueness condition

$$uf'(u) - f(u) \ \text{ does not change sign for } u > 0. \tag{5.28}$$

I.e. (5.28) holds whenever (5.27) does, and so the condition (5.28) is both simpler and more general.

Indeed, we begin by observing

$$\frac{d^2}{du^2}\left(\sqrt{F(u)}\right) = \frac{I(u)}{2F^{3/2}(u)} \equiv J(u),$$

where $J(u)$ has the same sign as $I(u)$. Integrating between some $a > 0$ and $u > 0$,

$$\frac{d}{du}(\sqrt{F(u)}) = \int_a^u J(\xi)\,d\xi + c > 0, \tag{5.29}$$

where $c = \frac{f(a)}{2\sqrt{F(a)}} > 0$. Integrating (5.29),

$$\sqrt{F(u)} = cu + c_1 + \int_a^u (u - \xi)J(\xi)\,d\xi, \tag{5.30}$$

where $c_1 = -\int_0^a \xi J(\xi)\,d\xi$. From (5.30) we find $F(u)$, and then $f(u)$ and $f'(u)$ by differentiation. We then have

$$uf'(u) - f(u) = 2\sqrt{F(u)}J(u)u + 2\int_0^u \xi J(\xi)\,d\xi\left(\int_a^u J(\xi)\,d\xi + c\right).$$

In view of (5.29), the quantity in the bracket is positive, and it follows that if $J(u)$ is positive (negative), so is $uf'(u) - f(u)$.

We now consider the problem

$$u'' + \lambda f(u) = 0, \ \ x \in (0,1), \ \ u(0) = u(1) = 0, \tag{5.31}$$

depending on a positive parameter λ. As before, we can convert it to the problem (5.3), with $T = \sqrt{\lambda}$. If we can show that $T''(p) > 0$ (or $T''(p) < 0$) for all $p > 0$ it will follow that for any λ there is at most two p's with $T(p) = \sqrt{\lambda}$, i.e. at most two solutions of (5.31). Since

$$T''(p) = \int_0^\pi g'''(p \sin\theta) \sin^2\theta \, d\theta,$$

it suffices to show that the function $g'''(u)$ keeps the same sign. By the formula (1-1-15) in R. Schaaf [53]

$$g'''(u) = -g'(x) \frac{3f'(u)\left(f^2(u) - 2F(u)f'(u)\right) + 2F(u)f(u)f''(u)}{f^4(u)}, \quad \text{with } u = g(x),$$

which led her to the following condition: if

$$\text{(5.32)} \qquad 3f'(u)\left(f^2(u) - 2F(u)f'(u)\right) + 2F(u)f(u)f''(u) > 0$$
$$(\text{or } < 0 \ ), \quad \text{for all } u > 0$$

then the problem (5.31) has at most two positive solutions. Since the condition (5.32) is not easy to verify, R. Schaaf [53] went on to develop her $A - B$ and C conditions, which are sufficient for (5.32) to hold.

Condition (5.32) says that $g'(u)$ is either convex or concave. Working with the generalized averages, P. Korman and Y. Li [28] have shown that the same result is true if $\frac{1}{g'(u)}$ is convex (also in the case $\frac{1}{g'(u)}$ concave, but this possibility is included in the case when $g'(u)$ is concave). This led them to the following condition: if

$$\text{(5.33)} \quad \frac{1}{2}f''(u)F^2(u) + \frac{3}{8}f^3(u) - \frac{3}{4}f(u)f'(u)F(u) > 0, \ \text{ for all } u > 0$$

then the problem (5.31) has at most two positive solutions. Observe that this condition is different from (5.32). Conditions (5.32) and (5.33) work in both cases $f(0) = 0$ and $f(0) > 0$. Also, computer algebra can help in verifying these conditions.

Example The function $f(u) = 2 + e^{-u} \sin u$ satisfies (5.33). This function changes concavity infinitely many times. A straightforward computation, using *Mathematica*, shows that for this function the left hand side of (5.33) is positive, tending to 10.125 as $u \to \infty$. Hence, the problem (5.31) with this $f(u)$ has at most two positive solutions for any $\lambda > 0$.

Remarks

1. The time map formula can be also developed for the p-Laplacian case. Actually, even more general case is developed in Section 2.5 of R. Schaaf's book [53].

2. Finally, we mention why we constantly stress that all results about the time map hold in both cases $f(0) = 0$ and $f(0) > 0$. The important book by R. Schaaf [53] treats the $f(0) = 0$ case in Chapter 1, while the case $f(0) > 0$ (and also the case $f(0) < 0$) is postponed to Chapter 3. Some readers might form an incorrect impression that the book covers only the $f(0) = 0$ case (as in the MathSciences Review of that book).

5.3 Variational formula for the time map

In addition to the two formulas for the time map, discussed above, a curious variational formula has been discovered by R. Benguria and M.C. Depassier, see [8], which has also references to their earlier papers. If $u(t)$ is a solution of

$$u'' + \lambda f(u) = 0, \quad 0 < t < 1, \;\; u(0) = \alpha, \;\; u'(0) = u(1) = 0, \tag{5.34}$$

it is shown by R. Benguria and M.C. Depassier that

$$\lambda = \max_{g \in D} \frac{1}{2} \frac{\left(\int_0^\alpha g'(y)^{1/3}\, dy \right)^3}{\int_0^\alpha f(y) g(y)\, dy},$$

where $D = \{g | g \in C^1(0, \alpha), g' > 0, g(0) = 0\}$. By rescaling, this formula is of course equivalent to a time map formula. It was used in [8] to obtain lower and upper bounds for time maps.

5.4 A non-local problem

Using the generalized inverses, we now give a complete description of the solution set of a non-local problem. We begin with a simple observation. It is well known that for any $L > 0$ the problem (here $u = u(x)$)

$$u'' + u^3 = 0, \;\; 0 < x < L, \quad u(0) = u(L) = 0$$

has a unique positive solution, and a unique negative solution. If we now take a positive solution on the interval $(0, L/k)$, followed by the negative solution on $(L/k, 2L/k)$, and so on, then we obtain a solution with $k - 1$ sign changes, for any positive integer k.

We now consider a non-local problem, where instead of a second boundary condition we prescribe the average value of the solution on some fixed interval $(0, L)$

$$u'' + u^3 = 0, \quad 0 < x < L, \tag{5.35}$$
$$u(0) = 0,$$
$$\int_0^L u(s)\, ds = \alpha,$$

where α is a prescribed constant. We are interested in both positive, negative and sign-changing solutions, i.e. we shall talk of solutions with k sign changes, where $k \geq 0$. Without loss of generality we may assume $\alpha \geq 0$ (otherwise, consider $v = -u$). If $\alpha = 0$, it is clear that there exists exactly two solution of (5.35) with k sign changes, for any odd $k \geq 1$. Indeed, a solution of the equation in (5.35) with $u(0) = u(L) = 0$ having an odd number of roots inside $(0, L)$, and its negative, provide the desired solutions of (5.35). So that we may assume $\alpha > 0$.

Theorem 5.2 *([27]) For any $0 < \alpha < \frac{\pi}{\sqrt{2}}$ there exists exactly one solution of (5.35) with k sign changes, for any $k \geq 0$. For $\alpha = \frac{\pi}{\sqrt{2}}$ there exists exactly one solution with k sign changes, for any even $k \geq 0$, and no solutions if k is odd. For any $\alpha > \frac{\pi}{\sqrt{2}}$ the problem (5.35) has no solutions.*

Proof: The problem "scales right". Setting $x = bt$, and $u = \frac{1}{b}v$, we see that $v = v(t)$ satisfies

$$v'' + v^3 = 0, \quad 0 < t < \tfrac{L}{b}, \tag{5.36}$$
$$v(0) = 0,$$
$$\int_0^{\frac{L}{b}} v(s)\, ds = \alpha.$$

Comparing with (5.35), we see that only the length of the interval has changed. Hence we have a one-to-one map between the solution sets on any two intervals. So consider a solution $U(x)$ of the equation $u'' + u^3 = 0$, with $u(0) = 0$, which has k sign changes, whose roots are $x = 1, 2, \ldots$, and such that $U(x) > 0$ on $(0, 1)$, $U(x) < 0$ on $(1, 2)$, and so on. According to the formula (5.15), the integral of $U(x)$ over any of its positive humps is equal to $\frac{\pi}{\sqrt{2}}$, while the integral of $U(x)$ over any of its negative humps is $-\frac{\pi}{\sqrt{2}}$. Imagine cutting this solution with a sliding vertical line $x = \xi$. By continuity, for any $\alpha \in (0, \frac{\pi}{\sqrt{2}}]$ we can find a unique $\xi \in (0, 1]$ so that $U(x)$ is positive solution of (5.35) on the interval $(0, \xi)$. We then map this solution to the original interval $(0, L)$ by the above transformation. Similarly, for any $\alpha \in (0, \frac{\pi}{\sqrt{2}})$ we can find a unique $\xi \in (1,2)$ so that we have a solution of (5.35) on the interval $(0,\xi)$, with exactly one sign change. We then map $U(x)$ to the original interval, as before. Similarly we construct solutions with arbitrarily many sign changes.

By (5.15), no solution is possible in case $\alpha > \frac{\pi}{\sqrt{2}}$. $\diamondsuit$

6 Numerical Computation of Solutions

Good analytical understanding of a problem goes hand in hand with efficient numerical calculation of its solution. We know that for positive solutions the maximum value $u(0) = \alpha$ uniquely determines the solution pair $(\lambda, u(x))$ of the problem

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0, \tag{6.1}$$

see Lemma 2.3 above. We also know that the parameter λ in (6.1) can be "scaled out", i.e. $v(x) \equiv u(\frac{1}{\sqrt{\lambda}}x)$ solves the equation $v'' + f(v) = 0$, while $v(0) = u(0) = \alpha$, and $v'(0) = u'(0) = 0$. The root of $v(x)$ is $r = \sqrt{\lambda}$. We therefore solve the initial value problem

$$v'' + f(v) = 0, \quad v(0) = \alpha, \;\; v'(0) = 0, \tag{6.2}$$

and find its first positive root r. Then $\lambda = r^2$ by the above remarks. This way for each α we can find the corresponding λ. After we choose sufficiently many α_n and compute the corresponding λ_n, we can plot the pairs (λ_n, α_n), obtaining a bifurcation diagram in (λ, α) plane. We stress that the resulting two-dimensional bifurcation curve gives a faithful representation of the solution set of (6.1), since the value $u(0) = \alpha$ uniquely determines the solution pair $(\lambda, u(x))$. The program for solving (6.1) is essentially one short loop, involving the **NDSolve** command in *Mathematica*. It can be found at the author's web-page: http://math.uc.edu/∼kormanp/.

An equally good way to do numerical computations is by direct integration. For the problem (6.2) we have $r = T/2$, where as before r is the first positive root, and T is the time map. I.e. $\lambda = T^2/4$. Using the formula (5.2) for the time map, we have

$$\lambda = \frac{1}{2}\alpha^2 \left(\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\int_{\alpha\sin\theta}^{\alpha} f(u)\,du}}\, d\theta \right)^2 . \tag{6.3}$$

The *Mathematica* program based on (6.3) is so short and simple, that we include its listing here. It solves the problem (6.1) for $f(u) = e^{5u/(5+u)}$, and produces an S-shaped bifurcation curve, in agreement with our results. (Our program is solving the Dirichlet problem on the interval $(0, 1)$, rather than $(-1, 1)$, which accounts for the extra factor of 4.)

We see absolutely no need to ever use finite differences (or finite elements) for the problem (6.1). If we divide the interval $(0, 1)$ into n pieces, with step $h = 1/n$ and subdivision points $x_i = ih$, and denote by u_i the numerical approximation of $u(x_i)$, the finite difference approximation of (6.1) is

$$\text{(6.4)} \quad \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \lambda f(u_i) = 0, \ \ 1 \le i \le n-1, \ \ u_0 = u_n = 0.$$

This is a system of *nonlinear* algebraic equations, more complicated in every way than the original problem (6.1). In particular, this system often has more solutions than the corresponding differential equation (6.1). The existence of the extra solutions (not corresponding to the solutions of (6.1)) has been recognized for a while, and a term *spurious solutions* has been used. For example in case $f(u) = e^u$ the solution curve of (6.1) has exactly one turn (as we proved before), while the solution curve of (6.4) has three turns, see P. Korman [25]. Increasing the number of subdivision points n does not remove the two spurious turns, it just moves them closer to $\lambda = 0$. Actually the spurious turns are avoided in the opposite direction, when $n \le 6$. We found this hard to prove, even when $n = 2$. When studying the problem (6.4), we can no longer rely on the familiar tools from differential equations. Even in the case $f(u) = u^k$ the analysis of the problem (6.4) is very involved, see E. L. Allgower [4].

For the general problem (1.1) (with $f = f(x, u)$) we suggest using the predictor-corrector method. If solution $u(x, \lambda)$ is known, one approximates

$$\text{(6.5)} \qquad u(x, \lambda + \Delta\lambda) \simeq u(x, \lambda) + u_\lambda(x, \lambda)\Delta\lambda,$$

and then a very accurate approximation of $u(x, \lambda + \Delta\lambda)$ can be usually obtained in around 4 steps of Newton's iteration, with the initial guess given by (6.5). This way we can continue the solution in λ. To find $u_\lambda(x,\lambda)$ one solves a linear problem

(6.6) $u_\lambda'' + \lambda f'(u)u_\lambda + f(u) = 0 \quad \text{for } -1 < x < 1, \;\; u_\lambda(-1) = u_\lambda(1) = 0.$

To solve (6.6) one uses finite differences. (There are no spurious solutions for linear problems!) The resulting tri-diagonal system is easily solved by Gaussian elimination.

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